Stime di convergenza per le serie di Lie "prossime all'identità" - Definizioni di spazi "complessificati", norme di Fourier pesate e stime per derivate e parentesi di Poisson

Riconsideriamo il problema generale della dinamica: $H(\underline{p}, \underline{q}; \varepsilon) = H_0(\underline{p}) + \sum_{s=1}^{+\infty} \varepsilon^s H_s(\underline{p}, \underline{q})$ con $(\underline{p}, \underline{q}) \in \underbrace{\mathcal{G}}_{\in \mathbb{R}^n} \times \mathbb{T}^n$ e ε "piccolo parametro".

Serie di Lie come trasf. canoniche prossime all'identità:

$$\exp(\varepsilon L_X)\,\cdot := \sum_{j=0}^{+\infty} \frac{\varepsilon^j}{j!} L_X^j\,\cdot = \cdot + \underbrace{\sum_{j\geq 1}^{+\infty} \frac{\varepsilon^j}{j!} L_X^j\,\cdot}_{= O(\varepsilon)}$$

Proprietà fondamentali.

Proposizione: $\exp(\varepsilon L_X)\,\cdot$ è un operatore lineare che preserva il prodotto

$\Big($cioè $\exp(\varepsilon L_X)(f\cdot g) = [\exp(\varepsilon L_X) f]\cdot[\exp(\varepsilon L_X) g]\Big)$

e le parentesi di Poisson (cioè $\exp(\varepsilon L_X)\{f,g\} = \{\exp(\varepsilon L_X) f, \exp(\varepsilon L_X) g\}$)

Vale quindi anche il

Teorema [di scambio]: Sia f una funzione dinamica allora

$$f(\underline{p},\underline{q})\Big|_{\substack{\underline{p}=\exp(\varepsilon L_\chi)\underline{P}\\ \underline{q}=\exp(\varepsilon L_\chi)\underline{Q}}} = \exp(\varepsilon L_\chi) f(\underline{p},\underline{q})\Big|_{(\underline{p},\underline{q})=(\underline{P},\underline{Q})}$$

Esempi: 1 traslazione delle azioni $\chi(\underline{q}) = \underline{\xi}\cdot\underline{q}$, $\underline{\xi} \in \mathbb{R}^n$, costante

$$\exp(\varepsilon L_{\chi(\underline{q})})\underline{q} = \underline{q} + \sum_{j=1}^{+\infty} \frac{\varepsilon^j}{j!} L_\chi^{j-1} \cancel{\{\underline{q}, \underline{\xi}\cdot\underline{q}\}} = \underline{q}$$

$$\forall i=1,\dots,n \quad \exp\left(\varepsilon L_{X(q)}\right) p_i = p_i + \sum_{j=1}^{+\infty} \frac{\varepsilon^j}{j!} L_X^{j-1} \underbrace{\{p_i, \xi \cdot q\}}_{\text{costante}} =$$

$$= p_i - \varepsilon \xi_i + \cancel{\sum_{j=2}^{+\infty} \frac{\varepsilon^j}{j!} L_X^{j-2}\, \text{costante}}$$

$$\Rightarrow \exp\left(\varepsilon L_{X(q)}\right) p = p - \varepsilon \xi$$

2) $X = X(q)$ funzione trigonometrica

$$\Rightarrow \exp\left(\varepsilon L_{X(q)}\right) q = q \quad , \quad \exp\left(\varepsilon L_{X(q)}\right) p = p + \varepsilon \underbrace{\{p, X(q)\}}_{\text{funz. solo di } q}$$

dove $\{p_i, X(q)\} = \frac{-\partial}{\partial q_i} X(q)$

Per il calcolo delle serie di Lie di funz. dinami_ che è utile il diagramma del triangolo di Lie.

Vogliamo calcolare $f_0(p,q)+\varepsilon f_1(p,q)+\dots = \exp(\varepsilon L_\chi)\sum_{s=0}^{+\infty}\varepsilon^s f_s(p,q)$

	$\exp(\varepsilon L_\chi)$	$\exp(\varepsilon L_\chi)$	$\exp(\varepsilon L_\chi)$	
g_0	f_0			$O(\varepsilon^0)$
εg_1	$\varepsilon L_\chi f_0$	εf_1		$O(\varepsilon^1)$
$\varepsilon^2 g_2$	$\frac{\varepsilon^2}{2} L_\chi^2 f_0$	$\varepsilon^2 L_\chi f_1$	$\varepsilon^2 f_2$	$O(\varepsilon^2)$
$\varepsilon^s g_s$	$\frac{\varepsilon^s}{s!} L_\chi^s f_0$	$\frac{\varepsilon^s}{(s-1)!} L_\chi^{s-1} f_1$	$\dots\ \varepsilon^s f_s$	$O(\varepsilon^s)$

$$\Rightarrow \varepsilon^s g_s = \varepsilon^s \sum_{j=0}^{s} \frac{L_X^j}{j!} f_{s-j}$$

— . —

Formula di Cauchy per le derivate di funz. analitiche $f(z)$ dove $z \in B_\rho(0)$ dove f è analitica con $\sup_{z \in B_\rho(0)} |f(z)| < \infty$, allora sussiste

$$\frac{d^j f}{dz^j} \to f^{(j)}(z) = \frac{j!}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(z-\zeta)^{j+1}} d\zeta \quad \text{si prende } \gamma = \rho e^{i\vartheta} =: \zeta$$

γ parametrizzata da ζ

$$d\zeta = \rho \, d\vartheta$$

$$\Rightarrow f^{(j)}(0) = \frac{j!}{2\pi i} \oint \frac{f(\rho e^{i\vartheta})}{(-\rho e^{i\vartheta})^{j+1}} \rho \, d\vartheta$$

$$\Rightarrow |f^{(j)}(0)| \le \frac{j!}{2\pi}\oint_\gamma \frac{\sup_{B_\rho(0)}|f|}{\rho^{j+\cancel{1}}}\,\cancel{\rho}\, d\theta = \frac{j!}{\cancel{2\pi}}\,\frac{\sup_{B_\rho(0)}|f|\cdot \cancel{2\pi}}{\rho^j}$$

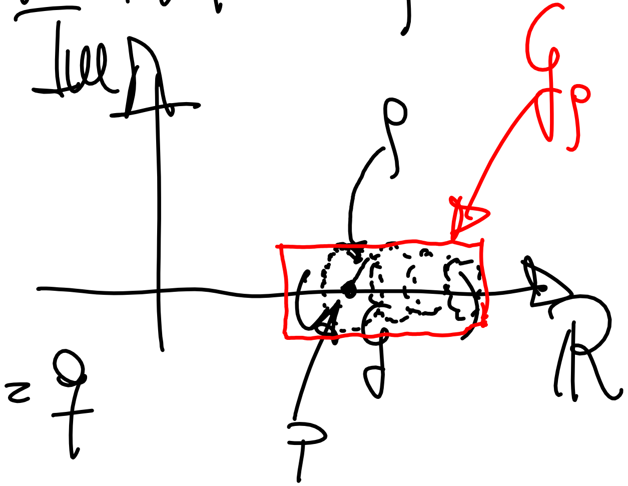
$$\le \frac{j!\, \sup_{z\in B_\rho(0)}|f(z)|}{\rho^j}$$

Inizialmente abbiamo sempre $(p,q)\in \underbrace{G}_{\text{aperto } \subseteq \mathbb{R}^n}\times \mathbb{T}^n$

Sia $G_\rho = \{ \underline{z}\in\mathbb{C}^n : \exists\, p\in G \text{ t.c. } \max_j |z_j - p_j| \le \rho\}$

Sia $\mathbb{T}^n_\sigma = \{q\in\mathbb{C}^n : \mathrm{Re}\, q\in\mathbb{T}^n, \max_j|\mathrm{Im}\, q_j|<\sigma\}$

$q=\{q_1,\dots,q_n\}\in\mathbb{T}^n_\sigma \Rightarrow (q_1+m_1 2\pi,\dots,q_n+m_n 2\pi)=q$



Def.: Si dice norma di Fourier pesata su $D_{\rho,\sigma} = G_\rho \times \mathbb{T}^n_\sigma$, la seguente

$$\|f\|_{(\rho,\sigma)} := \sum_{\underline{k}\in\mathbb{Z}^n} \sup_{p\in G_\rho} |c_{\underline{k}}(p)| \cdot e^{|\underline{k}|\sigma}$$

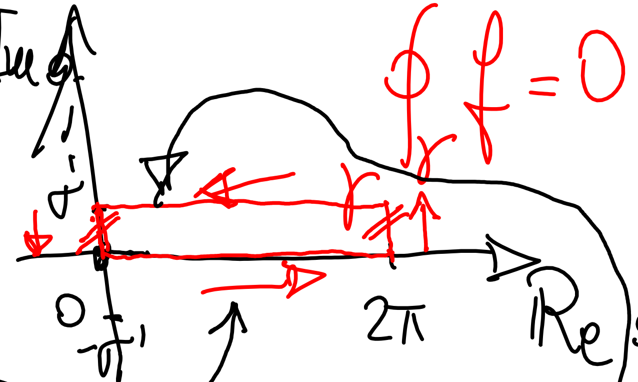
$= |k_1| + \dots + |k_n|$

dove $f(p,q) = \sum_{\underline{k}\in\mathbb{Z}^n} c_{\underline{k}}(p) e^{i\underline{k}\cdot q}$

Oss.: Sia f analitica su $D_{\rho,\sigma'}$ con $\sigma' > \sigma$ e $\sup_{(p,q)\in D_{\rho,\sigma'}} |f(p,q)| < +\infty$, allora $\|f\|_{(\rho,\sigma)} < +\infty$

e $\sup_{(p,q)\in D_{\rho,\sigma}} |f(p,q)| < \|f\|_{(\rho,\sigma)}$

Ricordiamo che $\boxed{c_{\underline{k}}(\underline{p}) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} d\underline{q}\, f(\underline{p},\underline{q})\, e^{-i\underline{k}\cdot\underline{q}}}$



$$\Rightarrow \forall q_j \quad \int dq_j\, f(\underline{p},\underline{q})\, e^{-i\underline{k}\cdot\underline{q}} = \int dq_j\, f(\underline{p},\underline{q})\, e^{-i\underline{k}\cdot\underline{q} - ik_j i\sigma'}$$

$$\forall j \Rightarrow \frac{1}{2\pi}\left|\int dq_j\, f(\underline{p},\underline{q})\, e^{-i\underline{k}\cdot\underline{q}}\right| \le \frac{1}{2\pi} \sup_{(\underline{p},\underline{q}) \in D_{\rho\sigma}} |f(\underline{p},\underline{q})| \left|\int dq_j\, e^{-|k_j|\sigma'}\right|$$

$$\le \sup_{(\underline{p},\underline{q}) \in D_{\rho,\sigma}} |f(\underline{p},\underline{q})|\, e^{-|k_j|\sigma'}$$

$$|c_{\underline{k}}(\underline{p})| = \frac{1}{(2\pi)^n}\left|\int dq_1 \dots dq_n\, f(\underline{p},\underline{q})\, e^{-i\underline{k}\cdot\underline{q}}\right| \le \sup_{(\underline{p},\underline{q}) \in D_{\rho,\sigma}} |f(\underline{p},\underline{q})|\, e^{-|\underline{k}|\sigma'}$$

$$\Rightarrow \|f\|_{(\rho,\sigma)} := \sum_{\underline{k}\in\mathbb{Z}^n} \sup_{p\in G_\rho} |c_{\underline{k}}(p)|\, e^{|\underline{k}|\sigma}$$

$$\leq \sum_{\underline{k}\in\mathbb{Z}^n} \sup_{(p,q)\in D_{\rho,\sigma'}} |f(p,q)|\, e^{|\underline{k}|(\sigma-\sigma')}$$

$$\leq +\infty$$

poiché, quando $\sigma<\sigma'$, $\Rightarrow \sum_{\underline{k}\in\mathbb{Z}^n} e^{|\underline{k}|(\sigma-\sigma')} < \infty$

Inoltre, siccome

$$\sup_{(p,q)\in D_{\rho,\sigma}} |f(p,q)| = \sup_{(p,q)\in D_{\rho,\sigma}} \left|\sum_{\underline{k}\in\mathbb{Z}^n} c_k(p)\, e^{-i\underline{k}\cdot q}\right|$$

$$\leq \sup_{(\underline{p},\underline{q})\in D_{\rho,\sigma}} \sum_{\underline{k}\in\mathbb{Z}^n} \left|c_{\underline{k}}(\underline{p})\right| e^{+|k_1||\mathrm{Im}\, q_1|+\ldots+|k_n||\mathrm{Im}\, q_n|}$$

$$\leq \sup_{(\underline{p},\underline{q})\in D_{\rho,\sigma}} \sum_{|\underline{k}|\in\mathbb{Z}^n} \left|c_{\underline{k}}(\underline{p})\right| e^{|k_1|\sigma+\ldots+|k_n|\sigma}$$

$$\leq \sum_{|\underline{k}|\in\mathbb{Z}^n} \sup_{\underline{p}\in G_\rho} \left|c_{\underline{k}}(\underline{p})\right| \cdot e^{|\underline{k}|\sigma} =: \|f\|_{(\rho,\sigma)}$$

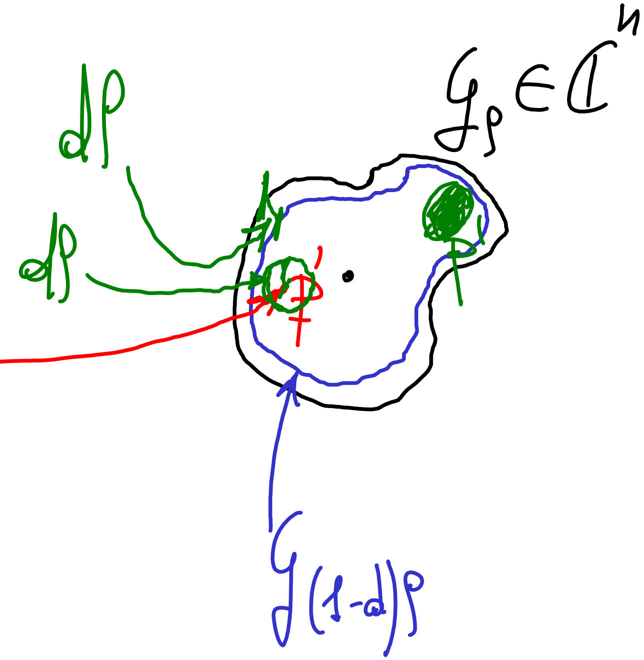
Prop.: Sia f analitica in $D_{(\rho,\sigma)}$ e t.c. $\|f\|_{(\rho,\sigma)}<+\infty$, allora $\forall\ j=1,\ldots,n$ e $\forall\ d\in(0,1)$ si ha che:

$$\left\|\frac{\partial f}{\partial p_j}\right\|_{(1-d)(\rho,\sigma)} \leq \frac{\|f\|_{(\rho,\sigma)}}{d\rho}, \quad \left\|\frac{\partial f}{\partial q_j}\right\|_{(1-d)(\rho,\sigma)} \leq \frac{\|f\|_{(\rho,\sigma)}}{e\, d\sigma}$$

Dim.: Si consideri

$$\left\| \frac{\partial f}{\partial P_j} \right\|_{(1-\delta)(\rho,\sigma)} := \sum_{k\in\mathbb{Z}^n} \boxed{\sup_{P\in \mathcal{G}_{(1-\delta)\rho}} \left| \frac{\partial c_k(P)}{\partial P_j}\right|} e^{|k|(1-\delta)\sigma}$$

$$\not\le \sup_{P'\in \mathcal{G}_{(1-\delta)\rho}} \left| \frac{\partial c_k(P')}{\partial P_j}\right|$$



$\forall\, P' \in \mathcal{G}_{(1-\delta)\rho}$ si ha che $B_{\delta\rho}(P') \in \mathcal{G}_\rho$

applichiamo la disug. di Cauchy per la derivata prima in P'

su una palla centrata in P' e di raggio $\delta\rho$

$$\Rightarrow \sup_{p' \in \mathcal{G}_{(1-d)\rho}} \left| \frac{\partial c_{\underline{k}}}{\partial p_j}(p') \right| \leq \sup_{p \in \mathcal{G}_\rho} \frac{|c_{\underline{k}}(p)|}{d\rho}$$

$$\Rightarrow \left\| \frac{\partial f}{\partial p_j} \right\|_{(\rho,\sigma)} = \sum_{\underline{k} \in \mathbb{Z}^n} \sup_{p \in \mathcal{G}_{(1-d)\rho}} \left| \frac{\partial c_{\underline{k}}}{\partial p_i}(p) \right| e^{(1-d)|\underline{k}|\sigma}$$

$$= \frac{1}{d\rho} \sum_{\underline{k} \in \mathbb{Z}^n} \sup_{p \in \mathcal{G}_\rho} |c_{\underline{k}}(p)| e^{(1-d)|\underline{k}|\sigma} \leq \frac{\|f\|_{(\rho,\sigma)}}{d\rho}$$

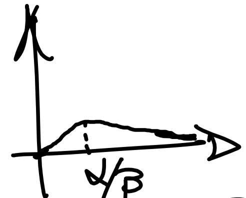
$$\forall j \quad \left\| \frac{\partial f}{\partial q_j} \right\|_{(1-d)(\rho,\sigma)} \leq \sum_{\underline{k} \in \mathbb{Z}^n} \sup_{p \in \mathcal{G}_{d\rho}} |k_j| |c_{\underline{k}}(p)| e^{(1-d)|\underline{k}|\sigma}$$

Siccome $x^\alpha e^{-\beta x} \leq \left(\frac{\alpha}{\beta}\right)^\alpha e^{-\cancel{\beta} \cdot \frac{\alpha}{\cancel{\beta}}} = \left(\frac{\alpha}{e\beta}\right)^\alpha$

con α, β, x positivi

$$\frac{d}{dx} x^\alpha e^{-\beta x} = \alpha x^{\alpha-1} e^{-\beta x} - \beta x^\alpha e^{-\beta x}$$

$$= 0 \Leftrightarrow \beta x = \alpha \Rightarrow x = \frac{\alpha}{\beta}$$



$\hookrightarrow$ allora abbiamo che $|k_j| e^{-\delta |k_j| \sigma} \leq \dfrac{1}{e \delta \sigma}$

$$\left\| \frac{\partial f}{\partial q_j} \right\|_{(1-\delta)(\rho,\sigma)} \leq \sum_{\underline{k} \in \mathbb{Z}^n} \frac{\sup_{p \in G_\rho} |c_{\underline{k}}(p)| e^{|k|\sigma}}{e \delta \sigma} =: \frac{1}{e \delta \sigma} \|f\|_{(\rho,\sigma)}$$

C.V.D.

Prop. Siano g, f analitiche in $D_{(\rho,\sigma)}$ e $D_{(1-\delta')(\rho,\sigma)}$ con $\|f\|_{(1-\delta')(\rho,\sigma)}$ e $\|g\|_{(\rho,\sigma)}$ finite, allora $\forall \delta, \delta'$ t.c. $\delta + \delta' < 1$

$$\|\{f, g\}\|_{(1-\delta'-\delta)(\rho,\sigma)} \leq 2 \frac{\|f\|_{(1-\delta')(\rho,\sigma)} \|g\|_{(\rho,\sigma)}}{e\, \delta \cdot (\delta+\delta') \rho \sigma}$$

Dim.: Siano $f(p,q)=\sum_{k\in\mathbb{Z}^n} c_k(p)\, e^{ik\cdot q}$,

$$g(p,q)=\sum_{k'\in\mathbb{Z}^n} c'_{k'}(p)\, e^{ik'\cdot q}$$ gli sviluppi in serie di Fourier di f, g

$$\left\|\{f,g\}\right\|_{(1-d'-d)} \leq \sum_{k,k'\in\mathbb{Z}^n}\sum_{j=1}^{n}\left(\frac{|k_j|}{(d'+d)\rho}+\frac{|k'_j|}{d\rho}\right)\sup_{p\in G_{(1-d')\rho}}|c_k(p)|\sup_{p\in G_\rho}|c'_{k'}(p)|\, e^{(1-d'-d)(|k|+|k'|)\sigma}$$

$$< \sum_{k'\in\mathbb{Z}^n}\sup_{p\in G_\rho}|c'_{k'}(p)|\, e^{|k'|\sigma}\sum_{k\in\mathbb{Z}^n}\sup_{p\in G_{(1-d')\rho}}|c_k(p)|\frac{|k|\, e^{-|k|d\sigma}}{(d'+d)\rho}\, e^{(1-d')|k|\sigma}$$

$$+\sum_{k\in\mathbb{Z}^n}\sup_{p\in G_{(1-d')\rho}}|c_k(p)|\, e^{|k|(1-d')\sigma}\sum_{k'\in\mathbb{Z}^n}\sup_{p\in G_\rho}|c'_{k'}(p)|\cdot\frac{|k'|\, e^{-|k'|(d+d')\sigma}}{d\rho}\, e^{|k'|\sigma}$$

applichiamo ai termini nei riquadri rossi la disuguaglianza sottolineata (in rosso) alla fine di 2 pagine sopra e così otteniamo le maggiorazioni dei coefficienti moltiplicativi riportati in rosso a fine delle 2 righe

$$< \sum_{k' \in \mathbb{Z}^n} \sup_{p \in \mathcal{G}_\rho} |c'_{k'}(p)| \, e^{|k'|\sigma} \sum_{k \in \mathbb{Z}^n} \sup_{p \in \mathcal{G}_{(1-d')\rho}} |c_k(p)| \, \frac{e^{(1-d')|k|\sigma}}{(d'+d)\rho} \cdot \frac{1}{ed\sigma}$$

$$+ \sum_{k \in \mathbb{Z}^n} \sup_{p \in \mathcal{G}_{(1-d')\rho}} |c_k(p)| \, e^{|k|(1-d')\sigma} \sum_{k' \in \mathbb{Z}^n} \sup_{p \in \mathcal{G}_\rho} |c'_{k'}(p)| \, \frac{e^{|k'|\sigma}}{d\rho} \cdot \frac{1}{e(d+d')\sigma}$$

$$< \frac{2}{e(d+d')d\rho\sigma} \|f\|_{(1-d')(\rho,\sigma)} \|g\|_{(\rho,\sigma)} \qquad \text{C.V.D.}$$