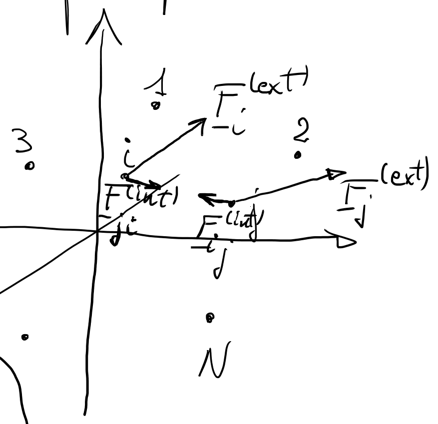
Mecc. Newtoniana per sistemi di N corpi puntiformi

$$m_i \ddot{\underline{x}}_i = \underline{F}_i \quad \forall\, i = 1, \dots, N$$



dove $\underline{F}_i = \underline{F}_i^{(ext)} + \sum_{j \neq i} \underline{F}_{ji}^{(int)}$

con $\underline{F}_{ji}^{(int)} = f_{ij}(\overline{P_iP_j}) \dfrac{\overrightarrow{P_jP_i}}{\overline{P_iP_j}}$, $f_{ij} : \mathbb{R}_+ \to \mathbb{R}$

$\forall\, j = 1, \dots, N,\ j \neq i$.

In particolare, $\boxed{\underline{F}_{ji}^{(int)} = -\underline{F}_{ij}^{(int)}}$ (III principio della dinamica)

e $\boxed{\underline{F}_{ji}^{(int)} \parallel \overrightarrow{P_jP_i}}$.

Def. (di baricentro): $\underline{x}_B = \dfrac{\sum_{j=1}^{N} m_j \underline{x}_j}{\sum_{j=1}^{N} m_j}$

Ponendo $M = \sum_{j=1}^{N} m_j$, $M \underline{x}_B = \sum_{j=1}^{N} m_j \underline{x}_j$.

$\Rightarrow \boxed{M \dot{\underline{x}}_B = \underline{P}} = \sum_{j=1}^{N} m_j \dot{\underline{x}}_j$

$\Rightarrow M \ddot{\underline{x}}_B = \sum_{j=1}^{N} m_j \ddot{\underline{x}}_j = \sum_{j=1}^{N} \underline{F}_j$

I^a eq. cardinale:

$$\dot{\underline{P}} = \underline{R}^{(ext)} := \sum_{j=1}^{N} \underline{F}_j^{(ext)}$$

<u>Verifica</u>: $$\dot{\underline{P}} = \sum_{j=1}^{N} \underline{F}_j = \sum_{j=1}^{N} \underline{F}_j^{(ext)} + \sum_{j=1}^{N} \sum_{i \neq j}^{N} \underline{F}_{ij}^{(int)}$$

$$= \sum_{j=1}^{N} \underline{F}_j^{(ext)} + \sum_{j=1}^{N} \sum_{i=1}^{j-1} \underbrace{\left(\underline{F}_{ij}^{(int)} + \underline{F}_{ji}^{(int)}\right)}$$

$$= \sum_{j=1}^{N} \underline{F}_j^{(ext)} + \underline{0} = \underline{R}^{(ext)} \quad \text{c.v.d.}$$

$$\underline{M}_Q := \sum_{j=1}^{N} \overrightarrow{QP_j} \wedge \underline{P}_j = \sum_{j=1}^{N} m_j \overrightarrow{QP_j} \wedge \underline{v}_j$$

$$\dot{\underline{M}}_Q = \frac{d}{dt}\left(\sum_{j=1}^{N} m_j \left(\overrightarrow{OP_j} - \overrightarrow{OQ_j}\right) \wedge \underline{v}_j\right) =$$

$$= \sum_{j=1}^{N} \left[m_j (\underline{v}_j - \underline{v}_Q) \wedge \underline{v}_j + m_j \overrightarrow{QP_j} \wedge \underline{a}_j \right]$$

$$= \sum_{j=1}^{N} \cancel{m_j \underline{v}_j \wedge \underline{v}_j} - \sum_{j=1}^{N} m_j \underline{v}_Q \wedge \underline{v}_j + \sum_{j=1}^{N} \overrightarrow{QP_j} \wedge m\underline{a}_j$$

$$= \underbrace{- \underline{v}_Q \wedge \sum_{j=1}^{N} m_j \underline{v}_j} + \underbrace{\sum_{j=1}^{N} \overrightarrow{QP_j} \wedge \underline{F}_j} = \underline{N}_Q - \underbrace{\underline{v}_Q \wedge \underline{P}}$$

Si noti che se Q fisso, oppure Q = B, oppure $\underline{v}_Q \parallel \underline{P}$

$$\Rightarrow \dot{\underline{M}}_Q = \underline{N}_Q \quad \text{dove} \quad \underline{N}_Q := \sum_{j=1}^{N} \overrightarrow{QP_j} \wedge \underline{F}_j$$

II eq. cardinale: $\underline{N}_Q = \underline{N}_Q^{(ext)}$,

quindi $\dot{\underline{M}}_Q = \underline{N}_Q^{(ext)} - \underline{v}_Q \wedge \underline{P}$

<u>Verifica</u>: $\underline{N}_Q = \sum_{j=1}^{N} \overrightarrow{QP_j} \wedge \underline{F}_j =$

$$= \sum_{j=1}^{N} \overrightarrow{QP_j} \wedge \underline{F}_j^{(ext)} + \sum_{j=1}^{N} \overrightarrow{QP_j} \wedge \sum_{i \neq j} \underline{F}_{ij}^{(int)}$$

Concentriamoci su $\sum_{j=1}^{N} \overrightarrow{QP_j} \wedge \sum_{i \neq j} \underline{F}_{ij}^{(int)} =$

$$= \sum_{j=1}^{N} \overrightarrow{QP_j} \wedge \sum_{i=1}^{j-1} \underline{F}_{ij}^{(int)} + \sum_{j=1}^{N} \overrightarrow{QP_j} \wedge \sum_{i=j+1}^{N} \underline{F}_{ji}^{(int)}$$

$$= \sum_{j=1}^{N} \overrightarrow{QP_j} \wedge \sum_{i=1}^{j-1} \underline{F}_{ij}^{(int)} - \sum_{j=1}^{N} \overrightarrow{QP_j} \wedge \sum_{i=j+1}^{N} \underline{F}_{ij}^{(int)}$$

$$= \sum_{j=1}^{N} \overrightarrow{QP_j} \wedge \sum_{i=1}^{j-1} \underline{F}_{ij}^{(int)} - \sum_{i=1}^{N} \overrightarrow{QP_i} \wedge \sum_{j=1}^{i-1} \underline{F}_{ji}^{(int)}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{i-1} \left(\overrightarrow{QP_j} - \overrightarrow{QP_i} \right) \wedge \underline{F}_{ij}^{(int)} =$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{i-1} \overrightarrow{P_iP_j} \wedge \underline{F}_{ij}^{(int)} = \underline{0} .$$

$$\Rightarrow \underline{N}_Q = \underline{N}_Q^{(ext)}$$

c.v.d.

Se $\underline{F}_j^{(ext)} = \underline{0} \quad \forall j = 1, \dots, N$

(1) $\dot{\underline{P}} = \underline{0} \implies \underline{v}_B(t) = \underline{v}_B(0)$

(2) $\dot{\underline{\Pi}}_Q + \underline{v}_Q \wedge \underline{P} = 0$, se $\underline{v}_Q = 0$, $Q = B$, $\underline{v}_Q // \underline{P}$,

allora $\dot{\underline{\Pi}}_Q = 0$.

Teorema (di König): $T := \frac{1}{2} \sum_{j=1}^{N} m_j \underline{v}_j \cdot \underline{v}_j$

è t.c. $T = \frac{1}{2} \sum_{j=1}^{N} m_j (\underline{v}_j - \underline{v}_B) \cdot (\underline{v}_j - \underline{v}_B) + \frac{1}{2} M \underline{v}_B \cdot \underline{v}_B$

Verifica: $T = \frac{1}{2} \sum_{j=1}^{N} m_j \underline{v}_j \cdot \underline{v}_j =$

$$= \frac{1}{2} \sum_{j=1}^{N} m_j \left[(\underline{v}_j - \underline{v}_B) + \underline{v}_B\right] \cdot \left[(\underline{v}_j - \underline{v}_B) + \underline{v}_B\right] =$$

$$= \frac{1}{2} \sum_{j=1}^{N} m_j (\underline{v}_j - \underline{v}_B) \cdot (\underline{v}_j - \underline{v}_B) +$$

$$\frac{1}{2} \sum_{j=1}^{N} m_j \underline{v}_B \cdot \underline{v}_B + \underline{v}_B \cdot \sum_{j=1}^{N} m_j (\underline{v}_j - \underline{v}_B) =$$

$$= \frac{1}{2} \sum_{j=1}^{N} m_j (\underline{v}_j - \underline{v}_B) \cdot (\underline{v}_j - \underline{v}_B) + \frac{1}{2} M \underline{v}_B \cdot \underline{v}_B + \underline{v}_B \cdot \cancel{(\underline{P} - M \underline{v}_B)}$$

c.v.d.

Proposizione: $\underline{M}_Q = \underline{S} + \overrightarrow{QB} \wedge M \underline{v}_B$,
dove il momento angolare di "spin" è quello relativo al baricentro, cioè

$$\underline{S} = \sum_{j=1}^{N} m_j \overrightarrow{BP}_j \wedge (\underline{v}_j - \underline{v}_B)$$

Dim.: $\underline{M}_Q = \sum_{j=1}^{N} m_j \overrightarrow{QP}_j \wedge \left[(\underline{v}_j - \underline{v}_B) + \underline{v}_B\right] =$

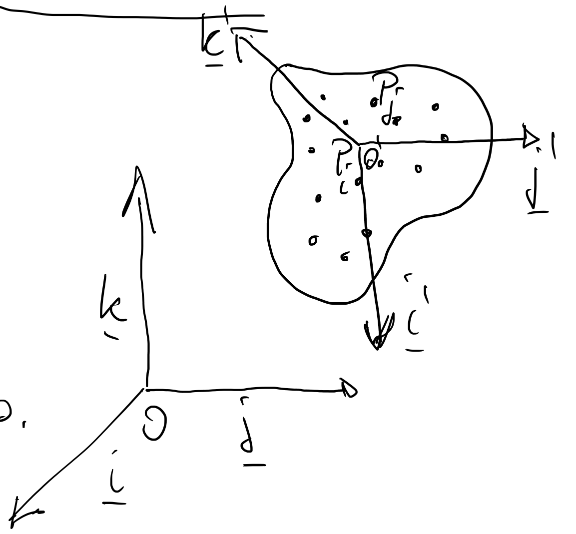
$$= \sum_{j=1}^{N} m_j \overrightarrow{QP}_j \wedge (\underline{v}_j - \underline{v}_B) + \left(\sum_{j=1}^{N} m_j \overrightarrow{QP}_j\right) \wedge \underline{v}_B =$$

$$= \sum_{j=1}^{N} \left[m_j \overrightarrow{QB} \wedge (\underline{v}_j - \underline{v}_B)\right] + \sum_{j=1}^{N} \left[m_j \overrightarrow{BP}_j \wedge (\underline{v}_j - \underline{v}_B)\right]$$

$$+ M \overrightarrow{QB} \wedge \underline{v}_B = \overrightarrow{QB} \wedge \left(\cancel{\sum_{j=1}^{N} m_j \underline{v}_j - M \underline{v}_B}\right) + \underline{S}$$

$$+ M \overrightarrow{QB} \wedge \underline{v}_B = \underline{S} + M \overrightarrow{QB} \wedge \underline{v}_B \quad \text{c.v.d.}$$

Corpi rigidi



$\forall\, i, j = 1, \dots, N$

$i \neq j \quad \overline{P_i P_j} = c_{ij}$ indip. dal tempo

Sia $(O', \underline{i}', \underline{j}', \underline{k}')$ una terna solidale al corpo rigido $\Rightarrow$ basta una roto-traslazione per determinare la posizione e la giacitura del corpo rigido a ogni istante di tempo.

Bastano 6 valori per identificare la posizione spaziale del corpo rigido (3 coordinate di O', 3 parametri per identificare la rotazione che porta $\underline{i}', \underline{j}', \underline{k}'$ in $\underline{i}, \underline{j}, \underline{k}$).

$\Rightarrow$ 6 incognite (6 leggi del moto associate ai 6 valori che identificano la pos. del corpo rigido) e 6 eq.:

$$m\ddot{\underline{x}}_B = \underline{R}^{(ext)}, \quad \dot{\underline{M}}_B = \underline{N}_B^{(ext)}$$

dove $\underline{M}_B = \underline{S}$

Ci riconduciamo allo studio

della II eq. cardinale, nel caso del polo fisso Q oppure nel caso di B = Q con moto di B che è dato dalla soluzione della I eq. cardinale.

sist. di rif. mobile con origine in Q (p.to fisso)

$$\underline{v}_Q = \underline{v}_t + \cancel{\underline{v}_r} = \underline{v}_{O'} + \underline{\omega}\wedge\vec{O'P} = \cancel{\underline{v}_Q} + \underline{\omega}\wedge\vec{QP}$$

Consideriamo

$$\underline{\Pi}_Q = \sum_{j=1}^{N} m_j \vec{QP}_j \wedge \underline{v}_j =$$

$$= \sum_{j=1}^{N} m_j \vec{QP}_j \wedge \left(\underline{\omega}\wedge\vec{QP}_j\right) \qquad \boxed{\text{con Q p.to fisso!}}$$

Nel caso Q = B, abbiamo

dove $\underline{v}_Q = \underline{v}_t = \underline{v}_B + \underline{\omega}\wedge\vec{BP}$

$$\underline{\Pi}_B = \sum_{j=1}^{N} m_j \vec{BP}_j \wedge \left(\underline{v}_j - \underline{v}_B\right) =$$

$$= \sum_{j=1}^{N} m_j \vec{BP}_j \wedge \left(\cancel{\underline{v}_B} + \underline{\omega}\wedge\vec{BP}_j - \cancel{\underline{v}_B}\right)$$

$$= \sum_{j=1}^{N} m_j \vec{BP}_j \wedge \left(\underline{\omega}\wedge\vec{BP}_j\right)$$

Studiamo le proprietà dell'operatore d'inerzia $I : \mathbb{R}^3 \to \mathbb{R}^3$

t.c. $I(\underline{\omega}) = \underline{\Pi}_Q(\underline{\omega}) := \sum_{j=1}^{N} m_j \vec{QP}_j \wedge \left(\underline{\omega}\wedge\vec{QP}_j\right)$

rispetto a Q fisso oppure Q = baricentro B

Oss.: $T = \frac{1}{2}\sum_{j=1}^{N} m_j \underline{v}_j \cdot \underline{v}_j = \frac{1}{2}\sum_{j=1}^{n} m_j (\underline{\omega} \wedge \overrightarrow{QP}_j)\cdot(\underline{\omega} \wedge \overrightarrow{QP}_j)$

$$= \frac{1}{2}\sum_{j=1}^{N} m_j \left[\overrightarrow{QP}_j \wedge (\underline{\omega} \wedge \overrightarrow{QP}_j)\right]\cdot \underline{\omega}$$

$$= \frac{1}{2}\underline{\omega} \cdot \sum_{j=1}^{N} m_j \overrightarrow{QP}_j \wedge (\underline{\omega} \wedge \overrightarrow{QP}_j) = \frac{1}{2}\underline{\omega}\cdot J(\underline{\omega})$$

Nota: abbiamo applicato la regola del prod. misto: $\underline{a}\cdot(\underline{b}\wedge\underline{c}) = \underline{b}\cdot(\underline{c}\wedge\underline{a}) = \underline{c}\cdot(\underline{a}\wedge\underline{b})$ $\forall \underline{a},\underline{b},\underline{c}\in\mathbb{R}^3$

Proposizione (proprietà fondamentali dell'operatore d'inerzia): L'operatore d'inerzia J è lineare, simmetrico e def. non neg.

Dim. la linearità è banale.

Siano $\underline{u}, \underline{v} \in \mathbb{R}^3$ allora

$$\underline{u}\cdot J(\underline{v}) = \underline{u}\cdot \sum_{j=1}^{N} m_j \overrightarrow{QP}_j \wedge (\underline{v}\wedge\overrightarrow{QP}_j) =$$

$$= \sum_{j=1}^{N} m_j \underset{\underline{a}}{\underline{u}} \cdot \left(\underset{\underline{b}}{\overrightarrow{QP}_j} \wedge \underbrace{(\underline{v}\wedge\overrightarrow{QP}_j)}_{\underline{c}}\right) = \sum_{j=1}^{N} m_j (\underset{\underline{b}}{\underline{v}} \wedge \underset{\underline{c}}{\overrightarrow{QP}_j})\cdot \underbrace{(\underline{u}\wedge\overrightarrow{QP}_j)}_{\underline{a}}$$

del prod. misto $\underline{a}\cdot(\underline{b}\wedge\underline{c})$ del prod. misto $\underline{a}\cdot(\underline{b}\wedge\underline{c})$

$$= \sum_{j=1}^{N} m_j \left[\overrightarrow{QP}_j \wedge (\underline{u}\wedge\overrightarrow{QP}_j)\right]\cdot\underline{v} = \underline{v}\cdot J(\underline{u}),$$ quindi

J è simmetrico.

Siccome $\forall\, \underline{\omega}\in\mathbb{R}^3$, $\underline{\omega}\cdot J(\underline{\omega}) = 2T \geq 0$.

C.V.D.

Proposizione: Le componenti della matrice che rappresenta l'operatore d'inerzia in una terna $(O', \underline{i}', \underline{j}', \underline{k}')$ solidale al corpo rigido sono

$$\begin{pmatrix} \sum_{j=1}^{N} m_j (y_j^2+z_j^2) & -\sum_{j=1}^{N} m_j x_j y_j & -\sum_{j=1}^{N} m_j x_j z_j \\ -\sum_{j=1}^{N} m_j x_j y_j & \sum_{j=1}^{N} m_j (x_j^2+z_j^2) & -\sum_{j=1}^{N} m_j y_j z_j \\ -\sum_{j=1}^{N} m_j x_j z_j & -\sum_{j=1}^{N} m_j y_j z_j & \sum_{j=1}^{N} m_j (x_j^2+y_j^2) \end{pmatrix}$$

dove (x_j, y_j, z_j) sono le coordinate del punto j-esimo rispetto alla terna $(O', \underline{i}', \underline{j}', \underline{k}')$.

Dim.: $$\underline{M}_Q = \mathcal{I}(\underline{\omega}) = \sum_{j=1}^{N} m_j \overrightarrow{QP}_j \wedge (\underline{\omega} \wedge \overrightarrow{QP}_j)$$

Nota: si usa $\underline{a}\wedge(\underline{b}\wedge\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$

$$= \sum_{j=1}^{N} m_j \left[(\overrightarrow{QP}_j \cdot \overrightarrow{QP}_j)\underline{\omega} - (\overrightarrow{QP}_j \cdot \underline{\omega})\overrightarrow{QP}_j \right]$$

Proiettiamo, ad esempio, $\underline{M}_Q$ sul primo versore $\underline{i}'$:

$$\underline{M}_Q \cdot \underline{i}' = \sum_{j=1}^{N} m_j \left[(x_j^2 + y_j^2 + z_j^2)\omega_1 - (x_j\omega_1 + y_j\omega_2 + z_j\omega_3) x_j \right] =$$

$$= \sum_{j=1}^{N} m_j \left[\cancel{x_j^2\omega_1} - \cancel{x_j^2\omega_1} + (y_j^2+z_j^2)\omega_1 - x_j y_j \omega_2 - x_j z_j \omega_3 \right] =$$

$$= \left(\sum_{j=1}^{N} m_j (y_j^2+z_j^2) \, , \, -\sum_{j=1}^{N} m_j x_j y_j \, , \, -\sum_{j=1}^{N} m_j x_j z_j \right) \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

Questo giustifica la forma della prima riga della matrice d'inerzia, analogamente si dimostra la correttezza della 2ª e 3ª riga, calcolando $\underline{M}_Q \cdot \underline{j}'$ e $\underline{M}_Q \cdot \underline{k}'$.

C.V.D.

Def.: Si dice che $(\underline{O}', \underline{e}_1, \underline{e}_2, \underline{e}_3)$ è una terna principale d'inerzia se la corrispondente matrice è diagonale, cioè

$O' = Q$ quando Q è fisso

$$\begin{pmatrix} \sum_{j=1}^{N} m_j (y_j^2 + z_j^2) & 0 & 0 \\ 0 & \sum_{j=1}^{N} m_j (x_j^2 + z_j^2) & 0 \\ 0 & 0 & \sum_{j=1}^{N} m_j (x_j^2 + y_j^2) \end{pmatrix}$$

Corollario: Se il corpo rigido ha almeno tre p.ti non allineati tra loro allora l'op. d'inerzia è def. positivo

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Eq. di Eulero per il corpo rigido.

Oss.: $\underline{M}_Q = J(\underline{\omega}) = I_1 \omega_1 \underline{e}_1 + I_2 \omega_2 \underline{e}_2 + I_3 \omega_3 \underline{e}_3$

dove $\underline{e}_1, \underline{e}_2, \underline{e}_3$ è una terna principale d'inerzia

I_1, I_2, I_3 sono sulla diag. della matrice d'inerzia

e $\omega_1 = \underline{\omega} \cdot \underline{e}_1$, $\omega_2 = \underline{\omega} \cdot \underline{e}_2$, $\omega_3 = \underline{\omega} \cdot \underline{e}_3$

Dalla II eq. cardinale della dinamica

$$\dot{\underline{M}}_Q = \underline{N}^{(ext)}$$

Proiettiamola sugli assi $\underline{e}_1, \underline{e}_2, \underline{e}_3$:

$$\left[\frac{d}{dt}\left(\sum_{j=1}^{3} I_j \omega_j \underline{e}_j\right)\right] \cdot \underline{e}_i = \underline{N}^{(ext)} \cdot \underline{e}_i \quad \forall i = 1,2,3$$

$$\Rightarrow \left[\sum_{j=1}^{3}\left(I_j \dot{\omega}_j \underline{e}_j + I_j \underline{\omega}_j \dot{\underline{e}}_j\right)\right] \cdot \underline{e}_i = \underline{N}^{(ext)} \cdot \underline{e}_i ,$$

osserviamo $\dot{\underline{e}}_j = \underline{\omega} \wedge \underline{e}_j$, quindi

$$\Rightarrow I_i \dot{\omega}_i \underline{e}_i + \left[\sum_{j=1}^{3} I_j \omega_j \left(\underline{\omega} \wedge \underline{e}_j\right) \cdot \underline{e}_i\right] = \underline{N}^{(ext)} \cdot \underline{e}_i$$

per $i=1$, abbiamo

" $i=2$, abbiamo

" $i=3$, "

$$\boxed{\begin{cases} I_1 \dot{\omega}_1 - I_2 \omega_2 \omega_3 + I_3 \omega_3 \omega_2 = N_1^{(ext)} \\ I_2 \dot{\omega}_2 + I_1 \omega_1 \omega_3 - I_3 \omega_3 \omega_1 = N_2^{(ext)} \\ I_3 \dot{\omega}_3 - I_1 \omega_1 \omega_2 + I_2 \omega_2 \omega_1 = N_3^{(ext)} \end{cases}}$$

Eq. di Eulero per il corpo rigido