Complementi di "meccanica celeste"

– Metodo del potenziale efficace

Proposizione: nei moti centrali, la forza $\underbrace{\underline{F} = f(\|\underline{x}\|)\frac{\underline{x}}{\|\underline{x}\|}}$ ammette potenziale, cioè $U = U(\|\underline{x}\|)$ t.c. –

$$U(\rho) = -\int d\rho\, f(\rho)\ .$$

Dim.: $\underline{F} = -\operatorname{grad} U$

Si intende che $\rho = \|\underline{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$$\underline{F} \cdot \underline{e}_j \overset{?}{=} -\frac{\partial U}{\partial x_j}, \quad \text{dove } \underline{e}_j = (\delta_{1j}, \delta_{2j}, \delta_{3j})$$

delta di Kronecker

Calcoliamo

$$\frac{\partial U}{\partial x_j} = \frac{dU}{d\rho} \cdot \frac{\partial \rho}{\partial x_j} = -f(\rho) \cdot \frac{\partial \sqrt{x_1^2 + x_2^2 + x_3^2}}{\partial x_j}$$

$$= -f(\rho) \cdot \frac{1}{\cancel{2}\sqrt{x_1^2+x_2^2+x_3^2}} \cdot \cancel{2} x_j$$

$$= -f(\|\underline{x}\|) \cdot \frac{x_j}{\|\underline{x}\|} = -\underline{F} \cdot \underline{e}_j$$

$$\Rightarrow \begin{pmatrix} \frac{\partial U}{\partial x_1} \\ \frac{\partial U}{\partial x_2} \\ \frac{\partial U}{\partial x_3} \end{pmatrix} = -f(\|\underline{x}\|) \cdot \begin{pmatrix} x_1/\|\underline{x}\| \\ x_2/\|\underline{x}\| \\ x_3/\|\underline{x}\| \end{pmatrix} = -f(\|\underline{x}\|) \cdot \frac{\underline{x}}{\|\underline{x}\|} = -\underline{F}$$

C.V.D.

Proposizione: per i moti centrali si conserva la funzione energia

$$\mathcal{E}(\underline{x},\dot{\underline{x}}) = \frac{1}{2} m \dot{\underline{x}} \cdot \dot{\underline{x}} + U(\|\underline{x}\|)$$

Dim.: $\frac{d}{dt}\mathcal{E} = m\dot{\underline{x}} \cdot \frac{d}{dt}\dot{\underline{x}} + \frac{dU}{d\rho} \cdot \frac{d\rho}{dt} =$

$$= m\dot{\underline{x}} \cdot \ddot{\underline{x}} - f(\|\underline{x}\|) \cdot \frac{d\sqrt{x_1^2+x_2^2+x_3^2}}{dt} =$$

$$= m\dot{\underline{x}} \cdot \ddot{\underline{x}} - f(\|\underline{x}\|) \cdot \frac{2(x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3)}{2\sqrt{x_1^2+x_2^2+x_3^2}}$$

$$= m\dot{\underline{x}} \cdot \ddot{\underline{x}} - f(\|\underline{x}\|)\frac{\underline{x}}{\|\underline{x}\|} \cdot \dot{\underline{x}} =$$

$$= \dot{\underline{x}} \cdot \left(m\ddot{\underline{x}} - f(\|\underline{x}\|)\frac{\underline{x}}{\|\underline{x}\|}\right) = \dot{\underline{x}} \cdot (m\ddot{\underline{x}} - \underline{F}) = 0,$$

Si conserva $\frac{1}{2} m \dot{\underline{x}} \cdot \dot{\underline{x}} + U(\|\underline{x}\|) = \overline{E}$.

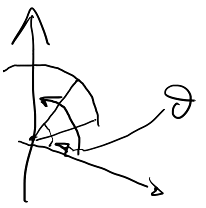
Si ricordi che si conserva $\underline{J} = m\underline{x} \wedge \dot{\underline{x}}$ e che il moto si svolge nel piano $\perp \underline{J} \neq \underline{0}$.

Riscriviamo le suddette leggi di conservazione in coord. polari.

$$\frac{1}{2} m (\dot{\rho}\underline{e}_\rho + \rho\dot{\theta}\underline{e}_\theta) \cdot (\dot{\rho}\underline{e}_\rho + \rho\dot{\theta}\underline{e}_\theta) + U(\rho) = \overline{E}$$

$$\Rightarrow \begin{cases} \frac{1}{2} m (\dot{\rho}^2 + \rho^2\dot{\theta}^2) + U(\rho) = \overline{E} \\ m\rho^2\dot{\theta} = J \end{cases}$$

Osserviamo che $\boxed{\dot{\vartheta} = J/(m\rho^2)}$



$$\Rightarrow \frac{1}{2}m\left(\dot{\rho}^2 + \frac{\cancel{\rho^2} \cdot J^2}{m^2 \rho^{\cancel{4}2}}\right) + U(\rho) = E$$

$$\Rightarrow \frac{1}{2}m\dot{\rho}^2 + \frac{J^2}{2m\rho^2} + U(\rho) = E$$

pot. centrifugo

$$\Rightarrow \boxed{\underbrace{\frac{1}{2}m\dot{\rho}^2}_{} + \underbrace{U_{eff}(\rho)}_{} = E}, \text{ dove } U_{eff}(\rho) = \frac{J^2}{2m\rho^2} + U(\rho)$$

Legge di conservazione dell'energia per il solo moto radiale (problema M.U.P.)

$$\Rightarrow t - t_0 = \pm \int_{\rho_0} \frac{dw}{\sqrt{\frac{2}{m}\left(E - U_{eff}(w)\right)}}$$

$$\dot{\rho} = \pm\sqrt{\frac{2}{m}\left(E - U_{eff}\right)}, \text{ si osservi che } \dot{\rho} = \frac{d\rho}{d\vartheta} \cdot \dot{\vartheta}$$

$$\Rightarrow \dot{\rho} = \frac{d\rho}{d\vartheta} \cdot J/m\rho^2$$

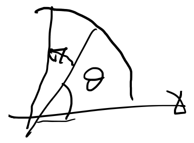
$$\frac{J}{m\rho^2}\frac{d\rho}{d\vartheta} = \pm\sqrt{\frac{2}{m}\left(E - U_{eff}(\rho)\right)}$$

$$\int d\vartheta = \pm \int \frac{J}{m\rho^2} \frac{d\rho}{\sqrt{\frac{2}{m}\left(E - U_{eff}(\rho)\right)}} \longrightarrow -d\left(\frac{1}{\rho}\right)$$

poniamo $u = 1/\rho$

$\rho_0 < \rho$

$$\Rightarrow \boxed{\vartheta - \vartheta_0 = \mp \int_{1/\rho}^{1/\rho_0} \frac{J\,du}{\sqrt{2m\left(E - U_{eff}(1/u)\right)}}}$$



pericentro

Soluzione per quadrature dell'eq. orbitale, ci fornisce $\rho = \rho(\vartheta)$.

$$\boxed{\Delta\theta = 2\int_{1/\rho_+}^{1/\rho_-} \frac{J\,du}{\sqrt{2m\left(E-U_{eff}(1/u)\right)}}}$$

pericentro

apocentro

formula per l'angolo tra 2 passaggi successivi al pericentro di un'orbita oscillatoria tra ρ_- e ρ_+, t.c. $U_{eff}(\rho_\pm)=E$.

$$\boxed{\Delta\theta = 2\int_{0}^{1/\rho_-} \frac{J\,du}{\sqrt{2m\left(E-U_{eff}(1/u)\right)}}}$$

formula dell'angolo di scattering per orbite illimitate, quindi con $\rho_+=+\infty$ e $U(\rho_-)=E$.

Esperimento di Rutherford (1911)

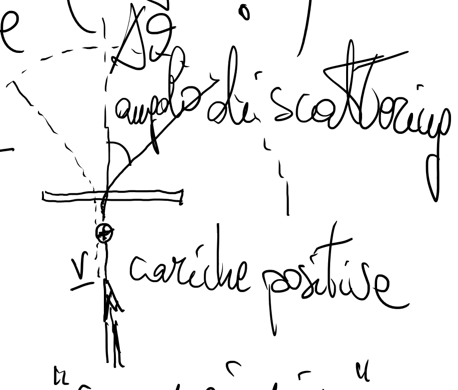
questa è una relazione l'una con l'altra

$$\Delta\theta = 2\int_{0}^{1/\rho_-} \frac{J\,du}{\sqrt{2m\left(E-\frac{Ju^2}{2m}-U(1/u)\right)}}$$

noto

massa dello ione

ignoto



Si trova che $\Delta\theta$ era consistenti con $U(\rho)=k/\rho$

Studio di $U_\alpha(\rho) = -\frac{k}{\rho^\alpha}$ con $\alpha \in (0,2)$

$U_1(\rho) = -\frac{k}{\rho} = -\int d\rho \, \frac{k}{\rho^2}$, pot. di Keplero.

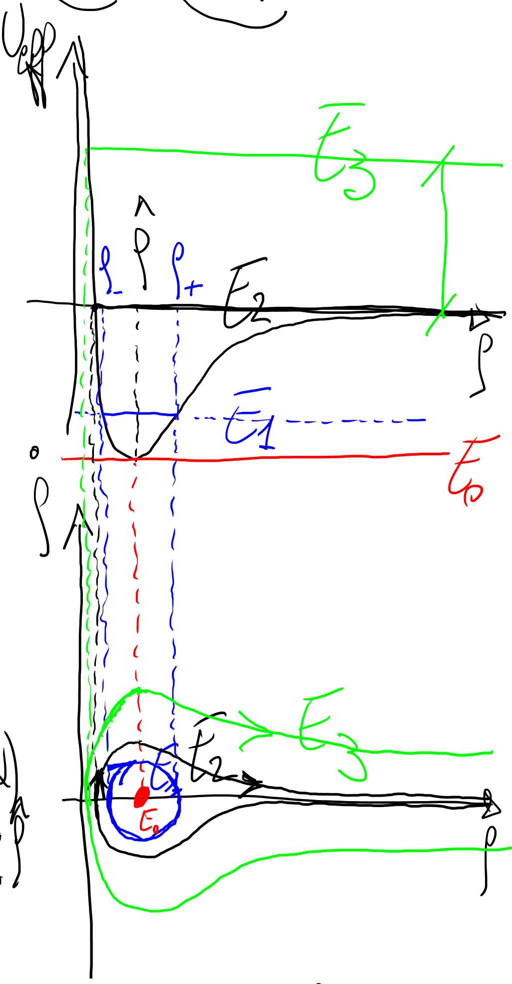
Oss. che $\lim_{\rho \to 0^+} U_{eff}(\rho) = \lim_{\rho \to 0^+} \underbrace{-\frac{k}{\rho^\alpha}} + \underbrace{\frac{J^2}{2m\rho^2}} = +\infty$

$\lim_{\rho \to +\infty} U_{eff}(\rho) = 0^-$

$$U'_{eff}(\rho) = +\frac{\alpha k}{\rho^{1+\alpha}} - \frac{J}{m\rho^3} = \frac{\alpha k m \rho^{2-\alpha} - J}{m\rho^3}$$

$U'_{eff}(\rho) > 0$ per $\rho > \left(\frac{J}{\alpha k m}\right)^{1/(2-\alpha)} =: \hat{\rho}$

$U'_{eff}(\rho) < 0$ per $0 < \rho < \left(\frac{J}{\alpha k m}\right)^{1/(2-\alpha)} =: \hat{\rho}$



Per $E = E_0 = U_{eff}(\hat{\rho})$ abbiamo 1 sol. di quiete in $\rho(t) = \hat{\rho}$ p.to di equil. stabile

Per $E = E_1 \in (U_{eff}(\hat{\rho}), 0)$ abbiamo 1 orbita periodica di oscillazione tra $\rho_\pm$ t.c. $U(\rho_\pm) = E_1$

Per $E = E_2 = 0$ abbiamo 1 orbita illimitata

Per $E = E_3 > 0$ " " " " .

Moti "a rosetta"

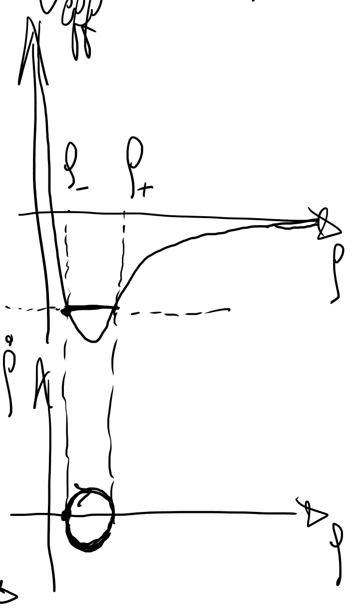
- $$\theta-\theta_0 = \int_{1/\rho}^{1/\rho_0} \frac{J\,du}{\sqrt{2m\left(E-U_{eff}(1/u)\right)}}$$ soluz. per quadrature dell'eq. orbitale $\rho=\rho(\theta)$

- $$J = m\rho^2\dot\theta \Rightarrow \dot\theta = \frac{J}{m\rho^2}$$

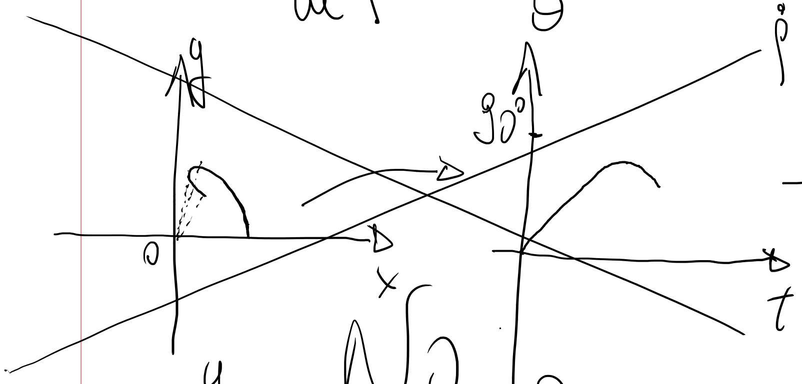
$$\theta-\theta_0 = \boxed{\int_{t_0}^{t} \frac{J\,d\tau}{m\rho^2(\tau)}}$$

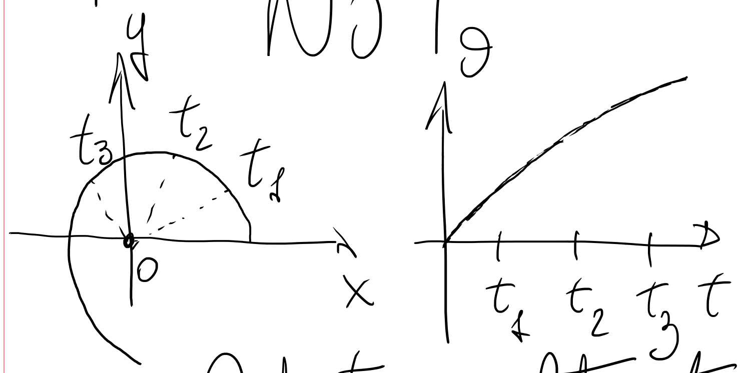
supponendo di aver già invertito $t=t(\rho)$ dato dal problema di M.U.P.:

$$t-t_0 = \pm\int_{\rho_0}^{\rho} \frac{dw}{\sqrt{\frac{2}{m}\left(E-U_{eff}(w)\right)}} \qquad \frac{1}{2}m\dot\rho^2 + U_{eff}(\rho) = E$$

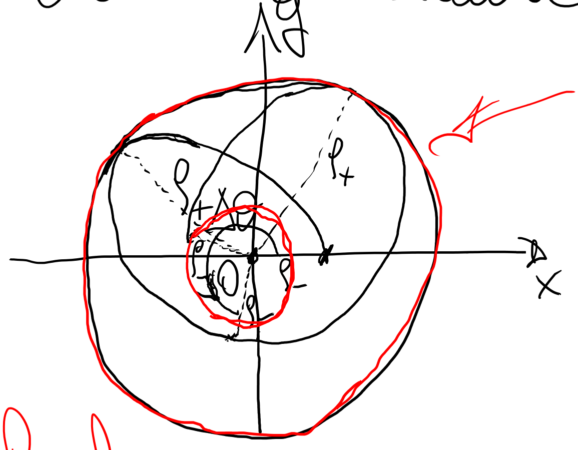


$$\dot\theta = \frac{J}{m\rho^2} > 0$$





Orbita risultante nel piano $\perp J$



Moto a rosetta compreso nella corona circolare di raggio max $=\rho_+$ e di raggio min. $=\rho_-$

Vedasi la figura per $U = -1/\rho^{3/2}$ nelle note (è meglio!!).

Proposizione: in un moto "a rosetta" l'orbita nel piano è periodica (=chiusa) sse $\Delta\theta = \frac{m}{n} \cdot 2\pi$ con $m, n \in \mathbb{N}$ $\left(\frac{\Delta\theta}{2\pi} \in \mathbb{Q}\right)$.

Proposizione: in un moto "a rosetta" se $\frac{\Delta\theta}{2\pi} \in \mathbb{R} \setminus \mathbb{Q}$ allora l'orbita riempie densamente la corona circolare tra ρ_- e ρ_+.

Teorema (Bertrand): Ci sono 2 soli potenziali del tipo $U = U(\rho)$ tali per cui ogni orbita limitata è anche chiusa, essi sono: $U(\rho) = -k/\rho$ e $U(\rho) = \frac{1}{2} k \rho^2$.

Si risolva l'eq. che determina $\rho_\pm$ per il problema di Keplero:

$$\frac{1}{2} m \dot{\rho}^2 + U_{eff}(\rho) = E$$

$$\Rightarrow U_{eff}(\rho_\pm) = E \Rightarrow \frac{J^2}{2m\rho^2} - \frac{k}{\rho} = E$$

poniamo $u = 1/\rho$ $\Rightarrow \frac{J^2}{2m} u^2 - ku = E$

$$\Rightarrow u^2 - \frac{2mk}{J^2} u - \frac{2mE}{J^2} = 0$$

$$\Rightarrow u_\pm = \frac{mk}{J^2} \pm \sqrt{\frac{m^2k^2}{J^4} + \frac{2mE}{J^2}} = \frac{mk}{J^2}\left(1 \pm \sqrt{1+\frac{2J^2E}{mk^2}}\right)$$

$$\rho_\pm = \frac{p}{1\mp e} \Rightarrow u_\pm = \frac{1\pm e}{p}$$

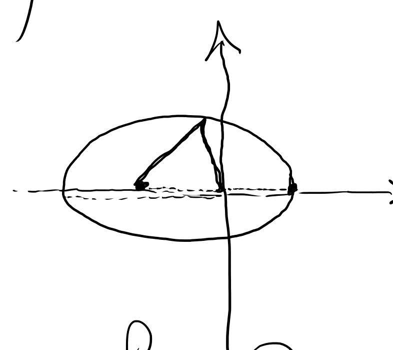
$$\Rightarrow \frac{1}{p} = \frac{mk}{J^2} \Rightarrow p = \frac{J^2}{mk}, \quad e = \sqrt{1+\frac{2J^2E}{mk}}$$

dall'ultima eq. $e=0 \Leftrightarrow E = -\frac{mk}{2J^2}$

(ellisse) $e\in(0,1) \Leftrightarrow E\in\left(-\frac{mk}{2J^2},0\right)$

(parabola) $e=1 \Leftrightarrow E=0$

(iperbole) $e>1 \Leftrightarrow E>0$



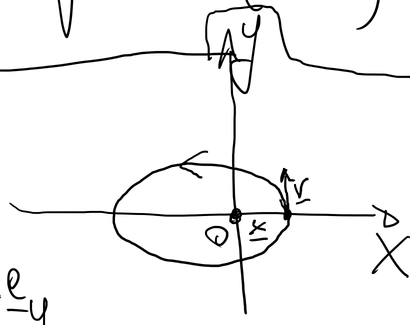
Osserviamo che $\rho_+ = \frac{p}{1-e}$, $\rho_- = \frac{p}{1+e}$

$$\rho_+ + \rho_- = 2a = \frac{p}{1-e}+\frac{p}{1+e} = \frac{2p}{1-e^2}$$

$$\Rightarrow p = a(1-e^2)$$

$$\Rightarrow a(1-e^2) = \frac{J^2}{mk} \Rightarrow \boxed{J=\sqrt{mka\,(1-e^2)}}$$

$E = E(a,e)$?



$$\underline{J} = m\frac{p}{1+e}\underline{e}_x \wedge \underline{v} = ma(1-e)v\,\underline{e}_x\wedge\underline{e}_y$$

$$J\underline{e}_z = ma(1-e)v\,\underline{e}_z \Rightarrow v = \frac{J}{ma(1-e)}$$

$$E = \frac12 m\underline{v}\cdot\underline{v} - \frac{k}{a(1-e)} = \frac12\frac{J^2}{ma^2(1-e)^2} - \frac{k}{a(1-e)} =$$

$$= \frac12 \frac{1}{a^{\cancel{2}}} \frac{\cancel{m}k\cancel{a}\,\cancel{(1-e^2)}^{1+e}}{\cancel{m}(1-e)^{\cancel{2}}} - \frac{k}{a(1-e)} = \frac{k}{2a(1-e)}(1+e-2) =$$

$$= \frac{k}{2a(1-e)}(e-1) = -\frac{k}{2a} \Leftrightarrow \boxed{a = -\frac{k}{2E}}$$

$$\boxed{E = -k/(2a)}$$

Proposizione: il vettore di Laplace-Runge-Lenz

$$\underline{A} = m\underline{v}\wedge\underline{J} - mk\frac{\underline{x}}{\rho} \quad \text{è costante}$$

del moto per il problema di Keplero.

Si ricordi che $\rho = \sqrt{x_1^2+x_2^2+x_3^2}$ e che $\underline{a}\wedge(\underline{b}\wedge\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$

Dim. $$\dot{\underline{A}} = \underbrace{m\underline{a}}_{\underline{F}}\wedge\underline{J} + \cancel{m\underline{v}\wedge\dot{\underline{J}}} - mk\frac{\underline{v}}{\rho} + \frac{mk}{\cancel{2}}\frac{\underline{x}}{\rho^3}\cancel{2}\cdot(x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3)$$

$$= -\frac{k}{\rho^2}\frac{\underline{x}}{\rho}\wedge(m\underline{x}\times\underline{v}) - mk\frac{\underline{v}}{\rho} + \frac{mk}{\rho^3}(\underline{x}\cdot\underline{v})\underline{x}$$

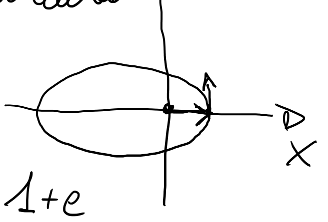
$$= \cancel{-\frac{mk}{\rho^3}(\underline{x}\cdot\underline{v})\underline{x}} + \cancel{\frac{mk\rho^2}{\rho^3}\underline{v}} - \cancel{\frac{mk\underline{v}}{\rho}} + \cancel{\frac{mk}{\rho^3}(\underline{x}\cdot\underline{v})\underline{x}} = \underline{0}$$

C.V.D.

Proposizione: per il problema di Keplero

$$\underline{A} = mk e\, \underline{e}_x$$ (si intende che l'asse x è diretto come il pericentro)

Dim.: Calcoliamo $\underline{A}$ al passaggio al pericentro



$$\boxed{\underline{A} = m\underline{v}\wedge\underline{J} - mk\frac{\underline{x}}{\rho}} =$$

$$= \cancel{m}\frac{J^2\underline{e}_y}{\cancel{m}a(1-e)}\wedge\underline{e}_z - mk\,\underline{e}_x = \frac{mk\cancel{a}\overbrace{\cancel{(1-e^2)}}^{1+e}}{\cancel{a}\cancel{(1-e)}}\underline{e}_x - mk\,\underline{e}_x$$

$$= \left[mk(\cancel{1}+e) - \cancel{mk}\right]\underline{e}_x = mke\,\underline{e}_x$$ C.V.D.