SPECIAL MATCHINGS AND COXETER GROUPS

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Abstract. In the recent paper [F. Brenti, F. Caselli and M. Marietti, Special matchings and permutations in Bruhat orders, Adv. Applied Math., in press] it is proved that the special matchings of permutations generate a Coxeter group. In this paper we generalize this result to a class of Coxeter groups which includes many Weyl and affine Weyl groups. Our proofs are simpler, and shorter, than those in [loc. cit.]

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1. Introduction

Special matchings are a class of perfect matchings of the Hasse diagram of a partially ordered set which are analogous to the matchings given by multiplication by Coxeter generators in a Coxeter group partially ordered by Bruhat order. In particular, the special matchings of the lower Bruhat intervals of a Coxeter group play a crucial role in the computation of the Kazhdan-Lusztig and R-polynomials of W and in the proof of the combinatorial invariance conjecture for these polynomials (see, e.g. [1]) in the important special case of lower intervals ([2, 3]). They also play a role in the construction of representations of Coxeter groups and Hecke algebras ([3, §8], [4], [8]). Our purpose in this paper is to continue the study of the natural question of when the special matchings of a poset generate a Coxeter group. We show that this is true for all lower intervals of the Coxeter systems whose Dynkin diagram is a simply laced forest. This generalizes the recent result in [5] for the symmetric groups, and also answers in the affirmative a question posed in [3] for this class of groups. Our proofs are different and simpler than the ones in [5].

2. Notation and Background

In this section we collect some definitions, notations and results that will be used in the rest of this work. We let $\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, \ldots\}$, and for $n \in \mathbb{N}$ we let $[n] \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). We follow [10, Chap. 3] for notation and terminology concerning partially ordered sets. In particular, given a partially ordered set P and $x, y \in P$ we say that x *covers* y if $x \geq y$ and there is no $z \in P \setminus \{x, y\}$ such that $y \leq z \leq x$. We then write $y \lhd x$. We say that P is a poset with $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $\hat{0} \leq x$ for all $x \in P$. The *Hasse diagram* of P is the graph $H(P) = (P, E)$ where $E \stackrel{\text{def}}{=} \{\{x, y\} : x, y \in P$ and either $x \lhd y$ or $y \lhd x\}$.

We follow [7] for general Coxeter groups notation and terminology. In particular, a *Coxeter system* is a pair (W, S) where W is a group and $S \subset W$ is a set of generators for W subject only to relations of the form

$$(ss')^{m(s,s')} = 1,$$

where $m(s, s) = 1$ (i.e. any element of S is an involution), and $m(s, s') = m(s', s) \geq 2$, for $s \neq s' \in S$. We attribute the value ∞ to $m(s, s')$ if ss' has infinite order. The Coxeter graph of (W, S) is a multigraph obtained in the following way: take S as the set of vertices, then join a pair of distinct vertices s, t by $m(s, t) - 2$ edges. The cardinality of S is usually called the *rank* of W. Given a Coxeter system (W, S) and $w \in W$ we denote by $\ell(w)$ the length of w, with respect to S. We let

$$D_R(w) \stackrel{\text{def}}{=} \{s \in S : \ \ell(ws) < \ell(w)\},$$

and

$$D_L(w) \stackrel{\text{def}}{=} \{s \in S : \ \ell(sw) < \ell(w)\} = D_R(w^{-1}).$$

We call $D_R(w)$ and $D_L(w)$ respectively the right and the left *descent set* of w. We denote by e the identity of W.

By a *subword* of a word $s_1 s_2 \cdots s_q$ we mean a word of the form $s_{i_1} s_{i_2} \cdots s_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq q$. We will always assume that a Coxeter group W is partially ordered by *Bruhat order*. Recall (see, e.g., [7, §5.9]) that $u \leq v$ if and only if any reduced expression for v has a subword that is a reduced expression for u. Given $u, v \in W$ we let $[u, v] \stackrel{\text{def}}{=} \{x \in W : u \leq x \leq v\}$. We consider $[u, v]$ as a poset with the partial order induced by the Bruhat order on W. We call $[u, v]$ a *dihedral interval* of length r if it is isomorphic as a poset to the Coxeter group of rank 2 generated by s_1 and s_2, with $m(s_1, s_2) = r$.

For $J \subseteq S$, let W_J be the parabolic subgroup of W generated by the set J, and W^J be the set of minimal length representatives for left cosets:

$$W^J \stackrel{\text{def}}{=} \{w \in W \,:\, D_R(w) \subseteq S \setminus J\}.$$

It is well known that multiplication gives a bijection $W^J \times W_J \to W$. More precisely, we have the following result, whose proof can be found, e.g., in [7].

Proposition 2.1. *Let $J \subseteq S$. Then*

1. *Every $w \in W$ has a unique factorization $w = w^J w_J$ such that $w^J \in W^J$ and $w_J \in W_J$.*
2. *For this factorization, $\ell(w) = \ell(w^J) + \ell(w_J)$.*

Recall that a matching of a (simple, loopless) graph $G = (V, E)$ is an involution $M : V \to V$ such that $\{M(v), v\} \in E$ for all $v \in V$. Let P be a finite graded poset. Following [2] we say that a matching M of the Hasse diagram of P is a *special matching* of P if

$$u \lhd v \Longrightarrow M(u) \leq M(v),$$

for all $u, v \in P$ such that $M(u) \neq v$. We denote the set of special matchings of P by $SM(P)$.

We recall the following results which are proved, respectively, in [3] and in [6].

Lemma 2.2. *Let P be a graded poset, $M \in SM(P)$, and $x, y \in P$, $x \leq y$, be such that $M(x) \rhd x$ and $M(y) \lhd y$. Then M restricts to a special matching of $[x, y]$.*

Lemma 2.3. *Let P_i, $i \in [n]$, be graded posets with $\hat{0}$, $P = P_1 \times \cdots \times P_n$ and $M \in SM(P)$. Then there exist (a unique) $k \in [n]$ and (a unique) $M_k \in SM(P_k)$ such that*

$$M(p_1, \ldots, p_n) = (p_1, \ldots, p_{k-1}, M_k(p_k), p_{k+1}, \ldots, p_n)$$

for all $(p_1, \ldots, p_n) \in P$.

Clearly Lemma 2.3 can be inverted: for any $k \in [n]$, every special matching M_k of P_k gives rise to a special matching M of $P_1 \times \ldots \times P_n$ by letting $M(p_1, \ldots, p_n) = (p_1, \ldots, p_{k-1}, M_k(p_k), p_{k+1}, \ldots, p_n)$.

3. SLT-Coxeter systems

Let P be a graded poset. We denote by $\mathfrak{S}(P)$ the symmetric group of P, and by $\langle SM(P)\rangle$ the subgroup of $\mathfrak{S}(P)$ generated by the special matchings of P. Moreover, for all $M, N \in SM(P)$, we let $m(M,N) \stackrel{\text{def}}{=} o(M \circ N)$, the order of the composition $M \circ N$ as an element of the group $\mathfrak{S}(P)$. These assignments define a Coxeter system $(\hat{W}_P, SM(P))$. We let

$$\begin{array}{cccc} \Phi : & \hat{W}_P & \longrightarrow & \mathfrak{S}(P) \\ & M_1 \cdots M_k & \mapsto & (u \mapsto M_1 \circ \cdots \circ M_k(u)). \end{array}$$

By the definition of $\hat{W}_P$, the map Φ is clearly well-defined, it is a group homomorphism, and its image is the group $\langle SM(P)\rangle$.

Proposition 3.1. *Let P_i, $i \in [n]$, be graded posets with $\hat{0}$, $P = P_1 \times \cdots \times P_n$ (direct product of posets). Then $\langle SM(P)\rangle \cong \langle SM(P_1)\rangle \times \cdots \times \langle SM(P_n)\rangle$ and $\widehat{W}_P \cong \widehat{W}_{P_1} \times \ldots \times \widehat{W}_{P_n}$ (direct product of groups).*

Proof. It is clearly enough to prove the statement for $n = 2$. Let M be a special matching of P. By Lemma 2.3, one of the following two cases occurs:

1. the restriction of M to $P_1 \times \{\hat{0}\}$ gives a special matching of P_1;
2. the restriction of M to $\{\hat{0}\} \times P_2$ gives a special matching of P_2.

The subgroup of $\widehat{W}_P$ generated by the special matchings for which 1. holds is isomorphic to $\widehat{W}_{P_1}$ and the subgroup generated by the special matchings for which 2. holds is isomorphic to $\widehat{W}_{P_2}$. Suppose that $M_1, M_2 \in SM(P)$ and that 1. and 2. hold, respectively, for M_1 and for M_2. Then clearly $M_1M_2 = M_2M_1$ since M_1 acts on the first component and M_2 acts on the second component. This proves the second assertion. Similarly, identifying $SM(P_1)$ and $SM(P_2)$ with subsets of $SM(P)$ (see the comments following Lemma 2.3), we have that every element of $\langle SM(P_1)\rangle$ commutes with every element of $\langle SM(P_2)\rangle$ and that $\langle SM(P_1)\rangle \cap \langle SM(P_2)\rangle = \{e\}$ so the result follows. $\square$

Let W be a Coxeter group. We say that $v \in W$ is *reducible* (in W) if there exists a partition $\{J_1, J_2\}$ of S, $v_1 \in W_{J_1}$ and $v_2 \in W_{J_2}$ such that $v = v_1v_2$. We say that v is *irreducible* if it is not reducible. Note that if v is irreducible then $s \leq v$ for all $s \in S$, and W is an irreducible Coxeter group.
Clearly, for any $v \in W$, there exists a partition $\{J_1, \ldots, J_n\}$ of S and irreducible elements $v_i \in W_{J_i}$ such that $v = v_1 \cdots v_n$. Such factorization is essentially unique, as shown in the next result.

Proposition 3.2. *Let $\{J_1, \ldots, J_n\}$ and $\{K_1, \ldots, K_m\}$ be two partitions of S and $v = v_1 \cdots v_n = u_1 \cdots u_m$, where $v_i \in W_{J_i}$ and $u_i \in W_{K_i}$ are irreducible elements for all i. Then $n = m$, $\{J_1, \ldots, J_n\} = \{K_1, \ldots, K_n\}$ and $\{v_1, \ldots, v_n\} = \{u_1, \ldots, u_n\}$.*

Proof. Let $i \in [n]$. We claim that K_i is either disjoint from or contained in J_1. Otherwise let $t \in K_i \setminus J_1$ and $s \in K_i \cap J_1$. If $m(t,s) > 2$ then $ts \not\leq u_i$ since $ts \not\leq v = v_1 \cdots v_n$. It follows that u_i can be written as the product of an element in $W_{K_i \cap J_1}$ and an element in

$W_{K_i \setminus J_1}$ which contradicts the irreducibility of u_i. An inductive procedure based on the same argument shows that if K_i is disjoint from $J_1, \ldots, J_h$ then it is either disjoint from or contained in J_{h+1}. So we conclude that $\{K_1, \ldots, K_m\}$ is a refinement of $\{J_1, \ldots, J_n\}$. By the symmetry of the argument we have that these two partitions of S must coincide. So let $J_i = K_h$. But $[e, v] \cap W_{J_i} = [e, v_i]$ and similarly $[e, v] \cap W_{K_h} = [e, u_h]$, and hence we may conclude that $v_i = u_h$. □

The center of a Coxeter group can be easily described (see [9, Section 2]).

Proposition 3.3. *Let W be an irreducible Coxeter group. If W is infinite or finite of type A_n ($n \geq 2$), D_k (k odd), E_6, $I_2(m)$ (m odd), then $Z(W) = \{e\}$. Otherwise $Z(W) = \{e, w_0\}$, where w_0 is the longest element of W.*

For brevity, we say that a Coxeter system (W, S) is an SLT-Coxeter system if it is simply laced (i.e. $m(s,t) \leq 3$ for all $s, t \in S$) and its Coxeter graph is a tree.

From now on W will always denote a fixed (but arbitrary) SLT-Coxeter group and v a fixed (but arbitrary) element of W. With no lack of generality, we may assume that $s \leq v$ for all $s \in S$.

Given $L \subseteq S$ and $s \in L$ we let

$$\begin{array}{rccc} M_{s,L}: & W & \longrightarrow & W \\ & u & \mapsto & u^L s u_L. \end{array}$$

In particular, $M_{s,\{s\}}$ and $M_{s,S}$ are the right and left multiplications by s, respectively. When no confusion arises, we denote by $M_{s,L}$ also any restriction of it. The next result is a special case of [3, Theorem 7.6]. Here and in what follows we let $SM_v \stackrel{\text{def}}{=} SM([e, v])$.

Theorem 3.4. *Let $M \in SM_v$, $s = M(e)$, $J = \{r \in S : M(r) = sr\}$ and $K = \{r \in S : M(r) = rs\}$. Then $u^J \in W_K$ and*

$$M(u) = M_{s,J}(u)$$

for all $u \leq v$.

For $s, t \in S$, we say that t is a *neighbor* of s, and we write $t \in \mathrm{Ne}(s)$, if $m(s,t) = 3$ (i.e. if s and t are adjacent in the Coxeter graph of W). For $t \in \mathrm{Ne}(s)$ we denote by $\mathrm{Br}_s(t)$ the *branch of s through t*, i.e. the set consisting of s and of the simple transpositions $r \in S$ such that the (necessarily unique) minimal path connecting r to s in the Coxeter graph contains t. Hence two branches of s either coincide or intersect exactly in s. Let $L \subseteq S$ be a union of branches of s. We let

- $L' = L \cap \mathrm{Ne}(s)$
- $L'' = L \setminus (L' \cup \{s\})$
- $\bar{L} = \bigcup_{t \in \mathrm{Ne}(s) \setminus L'} \mathrm{Br}_s(t)$

Note that L is uniquely determined by L' and we have $L = L' \uplus L'' \uplus \{s\}$ and $S = L' \uplus L'' \uplus \{s\} \uplus \bar{L}' \uplus \bar{L}''$.

Our next result shows that the special matchings in Theorem 3.4 can be described in terms of branches.

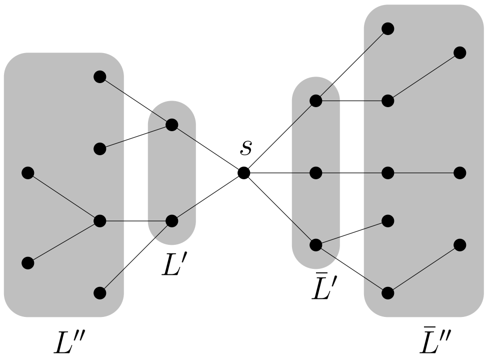

FIGURE 1. Example of union of branches L and its complement $\bar{L}$

Theorem 3.5. *Let $M \in SM_v$. Then there exists a unique $L \subseteq S$, which is a union of branches of $s = M(e)$, such that*

$$M(u) = M_{s,L}(u),$$

for all $u \leq v$. Furthermore $u^L \in W_{\bar{L}}$.

Proof. Let $J = \{r \in S : M(r) = sr\}$. Then, by Theorem 3.4, we have $M(u) = u^J s u_J$ for all $u \leq v$. Let $L' = \{r \in J : rs \neq sr\}$. This is a set of neighbors of s and so we can consider the union L of the corresponding branches

$$L = \bigcup_{t \in L'} \mathrm{Br}_s(t).$$

(The notation L' is clearly consistent with the previous one.)

We claim that $u^J s u_J = u^L s u_L$ for all $u \leq v$. By construction we have $J = L \cup \bar{L}''$. Since L and $\bar{L}''$ are not adjacent in the Coxeter graph of W, we can write $u_J = x_{\bar{L}''} x_L$, where $x_{\bar{L}''} \in W_{\bar{L}''}$ and $x_L \in W_L$.
Let $K = \{r \in S : M(r) = rs\}$. Then we have $K = \bar{L} \cup L''$ and hence, by Theorem 3.4, $u^J \in W_{\bar{L} \cup L''}$. So we have $u^J x_{\bar{L}''} \in W_{\bar{L} \cup L''}$ and we can factorize this element as $u^J x_{\bar{L}''} = y_{\bar{L}} y_{L''}$, with $y_{\bar{L}} \in W_{\bar{L}}$ and $y_{L''} \in W_{L''}$. Hence we have

$$u = y_{\bar{L}} y_{L''} x_L.$$

Since $L \cap \bar{L} = \{s\}$, we have $D_R(y_{\bar{L}}) \cap L \subseteq \{s\}$. It follows that

$$u^L = y_{\bar{L}} \varepsilon \text{ and } u_L = \varepsilon y_{L''} x_L,$$

where $\varepsilon = s$ if $s \in D_R(y_{\bar{L}})$, and $\varepsilon = e$ otherwise. So $u^L \in W_{\bar{L}}$ and

$$u^J s u_J = u^J s x_{\bar{L}''} x_L = u^J x_{\bar{L}''} s x_L = y_{\bar{L}} y_{L''} s x_L = y_{\bar{L}} s y_{L''} x_L = y_{\bar{L}} \varepsilon s \varepsilon y_{L''} x_L = u^L s u_L$$

as desired. The uniqueness of L follows immediately from the assumption on v made before Theorem 3.4. $\square$

From now on, unless otherwise stated, whenever a special matching $M_{s,L}$ is considered, we always assume that L is a union of branches of s.

Lemma 3.6. *Let $M_{s,L_1}, M_{s,L_2} \in SM_v$. Then $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$.*

Proof. By contradiction, assume that there exist $t_1 \in L_1' \setminus L_2'$ and $t_2 \in L_2' \setminus L_1'$ and let $u = t_2 t_1$. We clearly have $u \leq v$, since $t_1 t_2 = t_2 t_1$, and $t_1, t_2 \leq v$. Then $u^{L_1} = t_2$ and $u_{L_1} = t_1$ and so we deduce that $M_{s,L_1}(u) = t_2 s t_1$. If we let $z = t_2 s t_1$, then $z^{L_2} = t_2 s t_1$. But, by Theorem 3.5, we also have $z^{L_2} \in W_{\overline{L_2}}$, and this is a contradiction since $t_2 \notin \overline{L_2}$. $\square$

Lemma 3.7. *Let $M_{s,L}, M_{t,K} \in SM_v$. Then $M_{s,L}$ and $M_{t,K}$ do not commute if and only if $m(s,t) = 3$, and either*

$$s \in K \text{ and } t \in L,$$

or

$$s \notin K \text{ and } t \notin L.$$

Proof. By [3, Lemma 4.3], to prove that two special matchings in a simply laced Coxeter group commute, it is enough to verify that they do on the identity element. We have

$$M_{t,K} M_{s,L}(e) = M_{t,K}(s) = \begin{cases} ts, & s \in K, \\ st, & s \notin K, \end{cases}$$

and

$$M_{s,L} M_{t,K}(e) = M_{s,L}(t) = \begin{cases} st, & t \in L, \\ ts, & t \notin L, \end{cases}$$

and the result clearly follows. $\square$

In the proofs of the following results we make use of the following observation: if $M_{s,L} \in SM_v$ then $M_{s,\bar{L}} \in SM_{v^{-1}}$, since $M_{s,\bar{L}}(u) = \Big(M_{s,L}(u^{-1})\Big)^{-1}$ for all $u \leq v^{-1}$.

Proposition 3.8. *Let $M_{s,L}, M_{t,K} \in SM_v$, with $m(s,t) = 3$.*
If $s \notin K$ and $t \notin L$ then $v = v_1 v_2$, with $v_1 \in W_{\bar{L} \cap \bar{K}}$ and $v_2 \in W_{L \cup K \setminus \{s,t\}}$.
If $s \in K$ and $t \in L$ then $v = v_1 v_2$, with $v_1 \in W_{\bar{L} \cup \bar{K} \setminus \{s,t\}}$ and $v_2 \in W_{L \cap K}$.

Proof. Suppose that $s \notin K$ and $t \notin L$. In this case we have $L \cap K = \emptyset$. Let $V = (s_1, \ldots, s_k)$, $s_i \in S$, be a reduced expression for v, i.e. $v = s_1 \cdots s_k$ and $l(v) = k$, and let $L(V) = \#\{(i,j) : i < j,\, s_i \in L \cup K \setminus \{s,t\}\, s_j \in \bar{L} \cap \bar{K}\}$. The result will be proved if we show that there exists a reduced expression V such that $L(V) = 0$. Suppose that $V = (s_1, \ldots, s_k)$ minimizes $L(V)$ among all reduced expressions of v. Let us show that there does not exist $i \in [k-1]$ such that $s_i \in L \cup K \setminus \{s,t\}$, $s_{i+1} \in \bar{L} \cap \bar{K}$. This implies that $L(V) = 0$ and we are done. We proceed by contradiction assuming that such an i exists. If s_i and s_{i+1} commute, then we find another reduced expression V' for v such that $L(V') = L(V) - 1$ by swapping s_i and s_{i+1}, and this contradicts the minimality of V. So we can assume that s_i and s_{i+1} do not commute. This can happen only if either $s_i = t' \in K'$ and $s_{i+1} = t$ or $s_i = s' \in L'$ and $s_{i+1} = s$. For symmetry reasons we may assume we are in the first case. In particular we have $t't \leq v$. But this contradicts Theorem 3.5 since $M_{s,L}(t't) = t'ts$ and $(t'ts)^K = t'ts \notin W_{\bar{K}}$.

The case $s \in K$ and $t \in L$ follows by what we have already proved considering v^{-1} and the special matchings $M_{s,\bar{L}}, M_{s,\bar{K}} \in SM_{v^{-1}}$. $\square$

Given a special matching $M \in SM_v$, we say that M is *centered* in s if $M(e) = s$, i.e. if $M = M_{s,L}$ for some $L \subseteq S$.

Proposition 3.9. *Let $M_{s,L}, M_{s,H} \in SM_v$ be two special matchings centered in s both not commuting with the special matching $M_{t,K} \in SM_v$. Then $L = H$.*

Proof. By Lemma 3.6 we have that either $L \subseteq H$ or $H \subseteq L$ and we can clearly assume that $L \subseteq H$.
Suppose that $s \notin K$. Then, by Lemma 3.7, we have $t \notin H$. By contradiction, suppose $L \neq H$ and let $r \in H' \setminus L'$. By construction we have $r \in \bar{K} \cap \bar{L}$. Note that $r^L = r$ and $(rs)^K = rs$. Hence $M_{t,K}M_{s,L}(r) = M_{t,K}(rs) = rst$, so $rst \leq v$. But this contradicts Theorem 3.5 since $r \notin \bar{H}$ and so $(rst)^H = rst \notin W_{\bar{H}}$.
If $s \in K$ then, considering v^{-1} and the corresponding special matchings $M_{s,\bar{L}}, M_{s,\bar{H}}, M_{t,\bar{K}}$, we deduce from the first part that $\bar{L} = \bar{H}$, and hence $L = H$. $\square$

Let, for brevity, $\hat{W}_v \stackrel{\text{def}}{=} \hat{W}_{[e,v]}$.

Lemma 3.10. *Let $X, Y \in \hat{W}_v$ be such that $X = M_{s,L_1} \cdots M_{s,L_k}$ has a reduced expression consisting only of special matchings centered in s, and $Y = M_{t_1,K_1} \ldots M_{t_h,K_h}$, $t_i \neq s$ for all i, has a reduced expression consisting of special matchings which are not centered in s, and suppose that $\Phi(X) = \Phi(Y)$. Then $X = e$.*

Proof. Since the special matchings centered in s commute among each other by Lemma 3.7, we may assume that $L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_k$, by Lemma 3.6. Note that $s \not\leq \Phi(Y)(e)$ since Y has an expression consisting of special matchings which are not centered in s. If k is odd then $\Phi(Y)(e) = \Phi(X)(e) = s$ and this is not possible by the previous remark. If k is even and $k \geq 2$ let $t \in L_2' \setminus L_1'$. Then

$$\Phi(X)(t) = M_{s,L_1} \circ \cdots \circ M_{s,L_k}(t) = M_{s,L_1}M_{s,L_2}(t) = sts,$$

which is again a contradiction since $s \not\leq \Phi(Y)(t)$. So $k = 0$ and we are done. $\square$

Let $\{J_1, \ldots, J_n\}$ be a partition of S and $v = v_1 \cdots v_n$, with $v_i \in W_{J_i}$ and v_i irreducible in W_{J_i}. We have $[e, v] = [e, v_1] \times \cdots \times [e, v_n]$ and, by Proposition 3.1, the map $\Phi : \hat{W}_v \to \mathfrak{S}([e, v])$ is the direct product of the maps $\Phi_i : \hat{W}_{v_i} \to \mathfrak{S}([e, v_i])$ and hence we have

$$\mathrm{Ker}(\Phi) = \mathrm{Ker}(\Phi_1) \times \cdots \times \mathrm{Ker}(\Phi_n).$$

For these reasons, we can restrict our attention to the case where v is irreducible. We say that a matching $M \in SM_v$ is a *multiplication matching* if there exists $s \in S$ such that either $M(u) = su$ for all $u \leq v$ or $M(u) = us$ for all $u \leq v$. The next result shows that if two special matchings of an irreducible element do not commute then they are both multiplication matchings.

Lemma 3.11. *Let $v \in W$ be irreducible, and $M_{s,L}, M_{t,K} \in SM_v$ be non-commuting. Then either $L = \{s\}$ and $K = \{t\}$ or $L = K = S$.*

Proof. By Lemma 3.7 $m(s,t)=3$ and either $s\notin K$ and $t\notin L$ or $s\in K$ and $t\in L$.
In the first case, by Proposition 3.8, $v=v_1v_2$ with $v_1\in W_{\bar{L}\cap\bar{K}}$ and $v_2\in W_{L\cup K\setminus\{s,t\}}$. Since v is irreducible we have that either $\bar{L}\cap\bar{K}=\emptyset$ or $\bar{L}\cap\bar{K}=S$. But $\{s,t\}\subseteq\bar{L}\cap\bar{K}$, so $\bar{L}=\bar{K}=S$ and therefore $L=\{s\}$ and $K=\{t\}$.
The second case follows from the first one by considering the special matchings $M_{s,\bar{L}}, M_{t,\bar{K}}\in SM_{v^{-1}}$. □

Note that this result implies that if v is irreducible and $M\in SM_v$ is not a multiplication matching then M commutes with all the special matchings of v.

Suppose that $M_{s_1,S},\dots,M_{s_r,S}\in SM_v$ and let $w=s_1\cdots s_r$. Then it is clear that $M_{s_1,S}\circ\ldots\circ M_{s_r,S}(u)=wu$ for all $u\le v$. Now suppose that W is finite and $v=w_0=s_1\cdots s_r$ is the top element of W. Then $M_{s,S}, M_{s,\{s\}}\in SM_{w_0}$ for all $s\in S$. It follows that the two elements of $\hat{W}_{w_0}$ $X=M_{s_1,S}\cdots M_{s_r,S}$ and $Y=M_{s_1,\{s_1\}}\cdots M_{s_r,\{s_r\}}$ are two distinct elements in $\hat{W}_{w_0}$ (unless in the trivial case $|S|=1$). Note that X and Y only depend on w_0 (and not on $s_1,\dots,s_r$) and that $X=X^{-1}$ and $Y=Y^{-1}$ since w_0 is an involution. Now consider $\Phi(XY^{-1})(u)=\Phi(X)(\Phi(Y^{-1})(u))=\Phi(X)(uw_0)=w_0uw_0$. If $W=D_{2n},E_7,E_8$ then, by Proposition 3.3, w_0 is in the center of W and hence $\Phi(XY^{-1})(u)=u$ for all $u\in W$. So XY^{-1} is a nontrivial element in $\mathrm{Ker}(\Phi)$. The next result shows that this is essentially the only possible nontrivial element in $\mathrm{Ker}(\Phi)$.

Theorem 3.12. *Let W be an SLT-Coxeter group and $v\in W$ be irreducible. If $W=D_{2n},E_7,E_8$ and $v=w_0$, then*
$$\mathrm{Ker}(\Phi)=\{e,Z_0\}$$
where $Z_0=M_{s_1,S}\cdots M_{s_r,S}M_{s_r,\{s_r\}}\cdots M_{s_1,\{s_1\}}$ and $w_0=s_1\cdots s_r$. Otherwise
$$\mathrm{Ker}(\Phi)=\{e\}.$$

Proof. Let
$$\begin{aligned}SM_1&=\{M_{s,L}\in SM_v:\ L=S\}\\ SM_2&=\{M_{s,L}\in SM_v:\ L=\{s\}\}\\ SM_3&=\{M_{s,L}\in SM_v:\ L\neq S,\{s\}\}.\end{aligned}$$

Let $X\in\hat{W}_v$ be such that $\Phi(X)=Id$. By Lemma 3.11 and the observation following it, X can be factorized as $X=X_1X_2X_3$ where X_i belongs to the parabolic subgroup $(\hat{W}_v)_{SM_i}$. If $X_1=M_{s_1,S}\cdots M_{s_i,S}$ and $X_2=M_{r_1,\{r_1\}}\cdots M_{r_j,\{r_j\}}$ we let $u_1=s_1\cdots s_i$ and $u_2=r_1\cdots r_j$.
We begin by treating the particular case $X_3=e$. Then for all $u\le v$ we have

$$u=\Phi(X)(u)=\Phi(X_1)(\Phi(X_2)(u))=\Phi(X_1)(uu_2^{-1})=u_1uu_2^{-1}. \tag{1}$$

For $u=e$ we get $u_1=u_2$ and, since equation (1) must be satisfied in particular for all $u\in S$, it implies that u_1 belongs to the center $Z(W)$ of W. If $u_1=e$ then $X_1=X_2=e$ so $X=e$. If $u_1\neq e$, then, by Proposition 3.3, $u_1=w_0$ and W is finite of type D_k (k

even), E_7, E_8. But u_1 belongs to the parabolic subgroup $W_{D_L(v)}$ by construction and so we also have $v = w_0$.

Now suppose that $X_3 \neq e$ and let $M_{s,L} \leq X_3$. If $M_{t,S}, M_{t,\{t\}} \not\leq X$ for all $t \in \mathrm{Ne}(s)$, then the special matchings centered in s commute with all the others and, by Lemma 3.10, all special matchings centered in s can be simplified. So we may assume that there exists $t \in \mathrm{Ne}(s)$ such that either $M_{t,S} \leq X$ or $M_{t,\{t\}} \leq X$. We assume that $M_{t,S} \leq X$. Therefore we have $t \notin L$ otherwise $M_{s,L}$ and $M_{t,S}$ should not commute by Lemma 3.7, contradicting Lemma 3.11. Hence we can deduce similarly that $M_{t,\{t\}} \not\leq X_2$. Since $(\hat{W}_v)_{SM_3}$ is abelian we can write $X_3 = Y_3 Z_3$ where Y_3 is given by the product of elements which are centered in t and Z_3 is given by the product of elements which are not centered in t. So, recalling that the elements in SM_3 are central, we have

$$\Phi(X_1 Y_3) = \Phi(X_2^{-1} Z_3^{-1}).$$

Note that $X_2^{-1} Z_3^{-1}$ is given by special matchings which are not centered in t. If $\ell(Y_3)$ is even then

$$\Phi(X_1 Y_3)(e) = \Phi(X_1)(e) = u_1,$$

with $t \leq u_1$, and this is a contradiction since $t \not\leq \Phi(X_2^{-1} Z_3^{-1})(e)$. So $\ell(Y_3)$ is odd and

$$\Phi(X_1 Y_3)(e) = \Phi(X_1)(t) = u_1 t.$$

As before we have $t \not\leq u_1 t$, since $u_1 t = \Phi(X_2^{-1} Z_3^{-1})(e)$. This implies that $u_1 = u_1' t$ with $t \not\leq u_1'$. So we may write $X_1 = Y_1 M_{t,S}$, where Y_1 does not contain any special matching centered in t. So we have

$$\Phi(M_{t,S} Y_3) = \Phi(Y_1^{-1} X_2^{-1} Z_3^{-1}).$$

By Lemma 3.10 we can conclude that $M_{t,S} Y_3 = e$. So

$$X_1 X_2 X_3 = Y_1 M_{t,S} X_2 Y_3 Z_3 = Y_1 M_{t,S} Y_3 X_2 Z_3 = Y_1 X_2 Z_3.$$

But in $Y_1 X_2 Z_3$ there are no special matchings centered in t, so the result follows by induction.

Similarly if $M_{t,\{t\}} \leq X$. □

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