Diamonds and Hecke algebra representations [1]

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1 Introduction

Kazhdan-Lusztig polynomials are polynomials in one variable indexed by pairs of elements of a Coxeter group ([9]). These polynomials have been the subject of considerable attention because they play a role in several areas of mathematics, including the algebraic geometry and topology of Schubert varieties and representation theory (see, e.g., [8] and [1])

Our purpose in this work is to show that Kazhdan and Lusztig's construction can be

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carried out in a much more general (and entirely combinatorial) context. More precisely, we introduce a new class of partially ordered sets, that we call diamonds, which have a very rich combinatorial and algebraic structure and which include all Coxeter groups partially ordered by Bruhat order. We prove that one can define a family of polynomials indexed by pairs of elements in the diamond which reduce to the Kazhdan-Lusztig polynomials in the case that the diamond is a Coxeter group. We then show that every diamond contains in a natural way a Coxeter group and hence a Hecke algebra. Finally we show that this Coxeter group and the corresponding Hecke algebra act naturally on the diamond, and that the resulting representations include those constructed by Kazhdan and Lusztig, but contain several new ones.

The organization of the paper is as follows. In the next section we recall some basic definitions and results that are used in the sequel. In Section 3 we introduce the main concept of this work, the class of diamonds. We show that any diamond has associated two families of polynomials, and that these coincide with the Kazhdan-Lusztig and R-polynomials defined in [9] if the diamond is a Coxeter group. In Section 4 we briefly show that many of the combinatorial properties known for the Kazhdan-Lusztig and R-polynomials of Coxeter groups continue to hold for diamonds. In Section 5 we associate to any diamond D a Coxeter group W_D, and we prove that the natural action of W_D on D extends to a representation of the Hecke algebra of W_D on the free module $\mathcal{M}_\mathcal{D}$ generated by D. In Section 6, using the polynomials defined in Section 3, we construct a filtration of the module $\mathcal{M}_\mathcal{D}$ which leads to representations of the Hecke algebra. We then show that this class of representations strictly includes all those derived by Kazhdan and Lusztig in [9].

2 Notation and Background

In this section we collect some definitions, notation and results that will be used in the rest of this work. We let $\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, \ldots\}$, and for $a, b \in \mathbb{N}$, $a \leq b$, we let $[a, b] \stackrel{\text{def}}{=} \{a, a+1, \ldots, b\}$ and $[a] \stackrel{\text{def}}{=} [1, a]$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). The cardinality of a set A will be denoted by $|A|$.
We follow [13, Chapter 3] for notation and terminology concerning partially ordered sets. In particular, given a partially ordered set (or poset, for short) P, we say that P has a *bottom element*, denoted by $\widehat{0}$, if there exists an element $\widehat{0} \in P$ satisfying $\widehat{0} \leq x$ for all $x \in P$. Similarly, P has a *top element* $\widehat{1}$ if there exists an element $\widehat{1} \in P$ satisfying $x \leq \widehat{1}$ for all $x \in P$. If $x \leq y$ we define the *(closed) interval* $[x, y] \stackrel{\text{def}}{=} \{z \in P : x \leq z \leq y\}$. If every interval of P is finite, then P is called a *locally finite poset*. We say that x is *covered* by y if $x < y$ and $[x, y] = \{x, y\}$, and we write $x \lhd y$ or $y \rhd x$. A standard way of depicting a poset P is by its *Hasse diagram*. This is the digraph with P as vertex set and having an (upward) directed edge from x to y if and only if $x \lhd y$. Two posets P and Q are *isomorphic* if there exists an order-preserving bijection $\phi : P \to Q$ whose inverse is also order-preserving. A poset P is

ranked if there exists a (rank) function $\rho : P \to \mathbb{N}$ such that $\rho(y) = \rho(x) + 1$ whenever $x \lhd y$.

We follow [8, Chapter 5] for notation and terminology concerning Coxeter systems. In particular, given a Coxeter system (W, S), we denote by $l(w)$ the length of $w \in W$ and by e the identity of W. For all $s, s' \in S$, we let $m(s, s')$ be the order of the element ss' (possibly ∞). For all $w \in W$, we let

$$\begin{aligned} D_R(w) &\stackrel{\text{def}}{=} \{s \in S : \ l(ws) < l(w)\}, \\ D_L(w) &\stackrel{\text{def}}{=} \{s \in S : \ l(sw) < l(w)\}. \end{aligned}$$

We call $D_R(w)$ and $D_L(w)$ respectively the *right* and the *left descent set* of w. We denote by $\leq$ the Bruhat order on W (see, e.g., [8], §5.9). For all $u, v \in W$, $u \leq v$, we let $l(u, v) \stackrel{\text{def}}{=} l(v) - l(u)$ and we consider the interval $[u, v]$ as a poset with the partial ordering induced by W. A Coxeter system (W, S) such that $|S| = 2$ is called *dihedral*. We say that an abstract poset is *dihedral* if it is poset-isomorphic to a dihedral Coxeter group under Bruhat order.

The *Hecke algebra* $\mathcal{H}(W)$ associated to W is the free $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module having the set $\{T_w : \ w \in W\}$ as a basis and multiplication uniquely determined by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } sw > w, \\ (q-1)T_w + qT_{sw} & \text{if } sw < w, \end{cases} \tag{1}$$

for all $w \in W$ and $s \in S$. This is an associative algebra having T_e as unity and each basis element is invertible in $\mathcal{H}(W)$. More precisely, we have the following result (see, [8], Proposition 7.4).

Proposition 2.1. *There exists a family of polynomials $\{R_{u,v}(q)\}_{u,v \in W} \subseteq \mathbb{Z}[q]$ satisfying*

$$(T_{w^{-1}})^{-1} = q^{-l(w)} \sum_{u \leq w} (-1)^{l(u,w)} R_{u,w}(q)\, T_u\,,$$

for all $w \in W$.

It is customary to let $R_{u,v}(q) \stackrel{\text{def}}{=} 0$ if $u \not\leq v$. The polynomials $R_{u,v}$ are called the *R-polynomials* of W and can be inductively computed by means of the following recursion. For notational convenience, for all $s \in S$ and $u \in W$, we let $c(s,u) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } su > u, \\ 0, & \text{if } su < u. \end{cases}$

Theorem 2.2. *Let $u, v \in W$ and $s \in D_L(v)$. Then*

$$R_{u,v}(q) = q^{c(s,u)} R_{su,sv}(q) + (q^{c(s,u)} - 1) R_{u,sv}(q). \tag{2}$$

Define an involution $\iota : \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \to \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ by $\iota(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}$. It can be shown that there exists a unique $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-semilinear ring automorphism $\iota : \mathcal{H}(W) \to \mathcal{H}(W)$ satisfying $\iota(T_w) = (T_{w^{-1}})^{-1}$ (see [8, Section 7.7]), and that the following holds.

Theorem 2.3. *There exists a unique basis $\mathcal{C}' = \{C'_w : \ w \in W\}$ of $\mathcal{H}(W)$ such that:*

1. $\iota(C'_w) = C'_w$;

2. $C'_w = q^{-\frac{l(w)}{2}} \sum_{u \leq w} P_{u,w}(q) T_u$;

3. $P_{u,w} \in \mathbb{Z}[q]$ *has degree at most* $\frac{1}{2}(l(u,w)-1)$ *if* $u < w$, *and* $P_{w,w} = 1$.

The polynomials $\{P_{u,v}(q)\}_{u,v \in W} \subseteq \mathbb{Z}[q]$ (where, for notational convenience, it is usual to set $P_{u,v}(q) \stackrel{\text{def}}{=} 0$ if $u \not\leq v$) are the *Kazhdan-Lusztig polynomials* of W.

Recall that a *matching* of a graph $G = (V, E)$ is an involution $M : V \to V$ such that $\{M(v), v\} \in E$ for all $v \in V$. Let P be a partially ordered set. Following [3], we say that a matching M of the Hasse diagram of P is a *special matching of* P if

$$u \lhd v \Longrightarrow M(u) \leq M(v),$$

for all $u, v \in P$ such that $M(u) \neq v$.

We conclude this section with some results which are proved in [4] and will be needed in the sequel. Given two special matchings M and N of a poset P, we denote by $\langle M, N \rangle$ the subgroup of the symmetric group on P generated by M and N.

Lemma 2.4. *Let M and N be two special matchings of a ranked poset P. Then the orbits of the action of $\langle M, N \rangle$ on P are dihedral posets.*

The following results show that the special matchings play a fundamental role in the computation of R-polynomials and Kazhdan-Lusztig polynomials. Theorem 2.5 is actually the main reason why Theorem 2.6 holds. A (right or left) *multiplication matching* is a matching given by (right or left) multiplication by a Coxeter generator.

Theorem 2.5. *Let (W, S) be a Coxeter system, $w \in W$ such that $[e, w]$ is not dihedral, and M be a special matching of $[e, w]$. Then there exists a multiplication matching N of $[e, w]$ such that $NM(u) = MN(u)$ for all $u \leq w$, and $N(w) \neq M(w)$.*

For all special matchings M of a poset P and all $u \in P$, we let $c(M, u) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } M(u) > u, \\ 0, & \text{if } M(u) < u. \end{cases}$

Theorem 2.6. *Let (W, S) be a Coxeter system, $w \in W$ and M be a special matching of $[e, w]$. Then*

$$\begin{aligned} R_{u,w}(q) &= q^c R_{M(u),M(w)}(q) + (q^c - 1)\, R_{u,M(w)}(q), \\ P_{u,w}(q) &= q^c P_{M(u),M(w)}(q) + q^{1-c} P_{u,M(w)}(q) - \sum_{\{z \,:\, M(z) \lhd z\}} \overline{\mu}(z, M(w)) q^{\frac{l(z,w)}{2}} P_{u,z}(q), \end{aligned}$$

for all $u \leq w$, where $c = c(M, u)$ and $\overline{\mu}(z, M(w))$ is the coefficient of $q^{\frac{l(M(w))-l(z)-1}{2}}$ in $P_{z,M(w)}(q)$.

3 Diamonds

In this section we introduce the main concept of this work, namely a class of ranked posets, that we call diamonds, which includes all Coxeter groups partially ordered by Bruhat order. We show that any diamond has associated two families of polynomials, and that these families coincide with the Kazhdan-Lusztig and R-polynomials defined in [9] if the diamond is a Coxeter group.

Diamonds are a special class of zircons (see [10]). Given a poset P, we say that M is a special matching of an element $w \in P$ if M is a special matching of $\{x \in P : x \leq w\}$. We denote by SM_w the set of all special matchings of w. A locally finite ranked poset Z is a *zircon* if SM_w is non-empty for all $w \in Z$, w not minimal. Note that any order ideal of a Coxeter group partially ordered by Bruhat order is a connected zircon. In fact, any Coxeter group W is ranked by the length function and, for all $w \in W$, any (left or right) multiplication by a (left or right) descent of w gives a special matching of w ([3, Proposition 4.1].

For M, N special matchings of P we denote by$\langle M, N\rangle(p)$ the orbit of $p \in P$ under the action of the group $\langle M, N\rangle$.

Definition 3.1. Let Z be a zircon, $w \in Z$, and let $M, N \in SM_w$. We say that M and N are *strictly coherent* if

$$|\langle M, N\rangle(x)| \text{ divides } |\langle M, N\rangle(w)| \tag{3}$$

for all $x \in Z$, $x \leq w$.
We say that M and N are *coherent* if there exists a sequence $(M_0, M_1, \ldots, M_k)$ of special matchings in SM_w such that $M_0 = M$, $M_k = N$, and M_i and M_{i+1} are strictly coherent for all $i = 0, 1, \ldots, k-1$.

For example, the two special matchings of the poset on the left of Figure 1 are strictly coherent, while the two special matchings of the poset on the right are not.

The following is the crucial definition of this work.

Definition 3.2. We say that a zircon D is a *diamond* if, for all $w \in D$, any two special matchings $M, N \in SM_w$ are coherent.

Let us make a few simple remarks on diamonds.

1. Not every zircon is a diamond. For example, consider the poset in Figure 1. Let M and N be respectively the dashed and the solid special matchings shown in the figure on the right. Then one can check that M and N are not coherent (there are 6 special matchings which split into two classes of coherence).

2. The class of diamonds is closed under the poset operations of direct product, disjoint union, and taking order ideals.

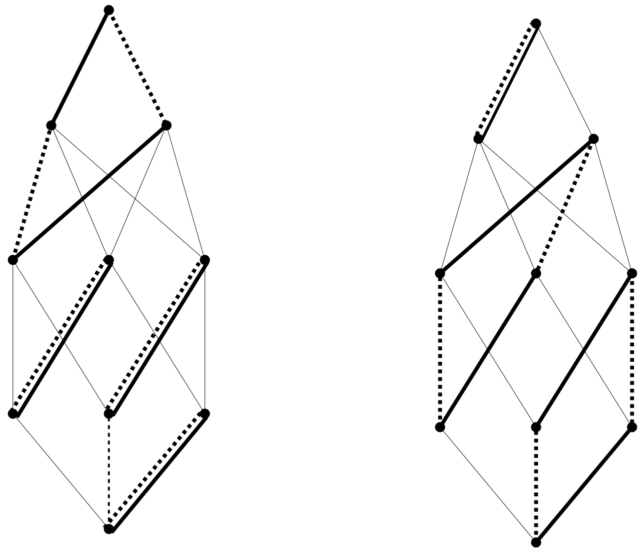

Figure 1: Strictly and non strictly coherent special matchings

3. The dual poset of a diamond is not necessarely a diamond. For example, the Bruhat interval $[e, s_2 s_1 s_3 s_2]$ in S_4 is a diamond but its dual is not even a zircon.

The following theorem, which is the main result of this section, shows that diamonds have "R-polynomials".

Theorem 3.3. *Let D be a diamond. Then there exists a unique family of polynomials $\{R_{u,w}\}_{u,w \in D} \subset \mathbb{Z}(q)$ such that, for any $u, w \in D$, we have*

$$R_{u,w}(q) = \begin{cases} q^c R_{M(u),M(w)}(q) + (q^c - 1)\, R_{u,M(w)}(q), & \text{if } u \leq w, \\ 1, & \text{if } u = w. \\ 0, & \text{if } u \not\leq w, \end{cases}$$

for all $M \in SM_w$, where $c = c(M, u)$.

Proof. The uniqueness part is clear. We prove the existence of the polynomials $R_{u,w}$ by induction on $\rho(w)$. This is clear if $\rho(w) \leq 1$. So assume $\rho(w) \geq 2$ and fix $u < w$. Let M and N be two special matchings of w. We need to prove that

$$q^c R_{M(u),M(w)}(q) + (q^c - 1)\, R_{u,M(w)}(q) = q^{\bar{c}} R_{N(u),N(w)}(q) + (q^{\bar{c}} - 1)\, R_{u,N(w)}(q) \qquad (4)$$

where $c = c(M, u)$ and $\bar{c} = c(N, u)$.
By definition of diamond, M and N are coherent, and we may clearly assume that they are strictly coherent.

Let $(W_{M,N}, \{M, N\})$ be the Coxeter system whose Coxeter generators are M and N subject to the relation given by $m(M, N) \stackrel{\text{def}}{=} \frac{|\langle M,N \rangle (w)|}{2}$. We denote by $\mathcal{H}(M, N)$ the Hecke algebra of $W_{M,N}$ and by $\mathcal{M}_{\langle M,N\rangle(u)}$ the free $\mathbb{Z}[q^{\frac{1}{2}},q^{-\frac{1}{2}}]$-module defined by

$$\mathcal{M}_{\langle M,N\rangle(u)} \stackrel{\text{def}}{=} \bigoplus_{v\in\langle M,N\rangle(u)} \mathbb{Z}[q^{\frac{1}{2}},q^{-\frac{1}{2}}]v.$$

We claim that there exists a unique action of $\mathcal{H}(M,N)$ on $\mathcal{M}_{\langle M,N\rangle(u)}$ such that, for all $v \in \langle M,N\rangle(u)$,

$$T_M(v) = \begin{cases} M(v), & \text{if } M(v) \rhd v, \\ qM(v) + (q-1)v, & \text{otherwise,} \end{cases} \tag{5}$$

and

$$T_N(v) = \begin{cases} N(v), & \text{if } N(v) \rhd v, \\ qN(v) + (q-1)v, & \text{otherwise.} \end{cases} \tag{6}$$

Since the Hecke algebra $\mathcal{H}(M,N)$ is generated by T_M and T_N subject only to the relations:

$$\begin{aligned} &T_M^2 = (q-1)T_M + qT_e, \\ &T_N^2 = (q-1)T_N + qT_e, \\ &\underbrace{T_M T_N T_M \cdots}_{m} = \underbrace{T_N T_M T_N \cdots}_{m}, \end{aligned}$$

where $m = m(M,N)$, the uniqueness part is clear and to prove the existence we only have to check that the action respects these relations. The quadratic relations are easy to verify. Let us prove that

$$\underbrace{T_M(T_N(T_M(\cdots}_{m}(v)))) = \underbrace{T_N(T_M(T_N(\cdots}_{m}(v)))) \tag{7}$$

for all $v \in \langle M,N\rangle(u)$. Let $(W,\{s,t\})$ be a dihedral Coxeter system of order $2d \stackrel{\text{def}}{=} |\langle M,N\rangle(u)|$. By Lemma 2.4, the orbit $\langle M,N\rangle(u)$ is isomorphic as a poset to W. Consider the isomorphism $\Phi : \langle M,N\rangle(u) \longrightarrow W$ sending $\underbrace{\cdots MNM}_{k}(u_0)$ to $\underbrace{\cdots sts}_{k}$, for all $k \in [2d]$, where u_0 is the smallest element in $\langle M,N\rangle(u)$. Extend Φ to a linear map $\Phi : \mathcal{M}_{\langle M,N\rangle(u)} \longrightarrow \mathcal{H}(W)$ by $\Phi(v) \stackrel{\text{def}}{=} T_{\Phi(v)}$ for all $v \in \langle M,N\rangle(u)$. Then it is clear that $\Phi(T_M(v)) = T_s\Phi(v)$ and $\Phi(T_N(v)) = T_t\Phi(v)$ for all $v \in \langle M,N\rangle(u)$. It follows that

$$\begin{aligned} \Phi(\underbrace{T_M(T_N(T_M(\cdots}_{d}(v)))) &= \underbrace{T_sT_tT_s\cdots}_{d}\Phi(v) \\ &= \underbrace{T_tT_sT_t\cdots}_{d}\Phi(v) \\ &= \Phi(\underbrace{T_N(T_M(T_N(\cdots}_{d}(v)))). \end{aligned}$$

Hence $\underbrace{T_M(T_N(T_M(\cdots}_{d}(v)))) = \underbrace{T_N(T_M(T_N(\cdots}_{d}(v))))$ and this proves (7) since d divides m because M and N are strictly coherent.

Now we introduce two module endomorphisms $\Psi_M, \Psi_N : \mathcal{M}_{\langle M,N\rangle(u)} \to \mathcal{M}_{\langle M,N\rangle(u)}$ by letting

$$\Psi_M(v) \stackrel{\text{def}}{=} q^{c(M,v)}M(v) + (q^{c(M,v)}-1)v$$

and

$$\Psi_N(v) \stackrel{\text{def}}{=} q^{c(N,v)} N(v) + (q^{c(N,v)} - 1)v$$

for all $v \in \langle M, N\rangle(u)$. This gives a representation of the Hecke algebra $\mathcal{H}(M,N)$. In fact, let $A : \mathcal{M}_{\langle M,N\rangle(u)} \to \mathcal{M}_{\langle M,N\rangle(u)}$ be the involution given by $A(\underbrace{\cdots MNM}_{k}(u_0)) = \underbrace{\cdots MNM}_{k}(u_1)$ for all $k \in [2d]$, where u_1 is the top element of $\langle M, N\rangle(u)$. Then $\Psi_M = A \circ T_M \circ A$ since we have that $AM = MA$ and $M(v) \lhd v$ if and only if $M(A(v)) \rhd A(v)$ for all $v \in \langle M, N\rangle(u)$. Similarly, $\Psi_N = A \circ T_N \circ A$. In particular

$$\underbrace{\cdots \Psi_M \Psi_N \Psi_M}_{m} = \underbrace{\cdots \Psi_N \Psi_M \Psi_N}_{m}. \tag{8}$$

For $f = \sum_v f_v(q)v \in \mathcal{M}_{\langle M,N\rangle(u)}$ and $z < w$ we let $f^z \stackrel{\text{def}}{=} \sum_v f_v(q) R_{v,z}(q)$. Note that, in this notation, (4) can be reformulated as

$$(\Psi_M(u))^{M(w)} = (\Psi_N(u))^{N(w)}. \tag{9}$$

By our induction hypothesis, for all $f \in \mathcal{M}_{\langle M,N\rangle(u)}$ and all $z < w$ such that $M(z) \lhd z$ we have

$$\begin{aligned} f^z &= \sum_v f_v(q) R_{v,z} \\ &= \sum_{\{v: M(v) \lhd v\}} f_v(q) R_{M(v),M(z)} + \sum_{\{v: M(v) \rhd v\}} f_v(q) \Big(q R_{M(v),M(z)} + (q-1) R_{v,M(z)} \Big) \end{aligned}$$

and

$$\begin{aligned} \Psi_M(f)^{M(z)} &= \Big(\sum_v f_v(q) \Psi_M(v) \Big)^{M(z)} \\ &= \Big(\sum_{\{v: M(v) \lhd v\}} f_v(q) M(v) + \sum_{\{v: M(v) \rhd v\}} f_v(q)) \Big(q M(v) + (q-1) v \Big) \Big)^{M(z)} \\ &= \sum_{\{v: M(v) \lhd v\}} f_v(q) R_{M(v),M(z)} + \sum_{\{v: M(v) \rhd v\}} f_v(q) \Big(q R_{M(v),M(z)} + (q-1) R_{v,M(z)} \Big). \end{aligned}$$

So we can deduce that, for all $f \in \mathcal{M}_{\langle M,N\rangle(u)}$

$$f^z = (\Psi_M(f))^{M(z)}, \quad \text{for all } z < w \text{ such that } M(z) \lhd z,$$

and similarly that

$$f^z = (\Psi_N(f))^{N(z)}, \quad \text{for all } z < w \text{ such that } N(z) \lhd z.$$

By alternated use of these formulae we have

$$(\Psi_M(u))^{M(w)} = (\Psi_N \Psi_M(u))^{NM(w)} = \cdots = (\underbrace{\cdots \Psi_M \Psi_N \Psi_M}_{m}(u))^{\overbrace{\cdots MNM}^{m}(w)},$$

and similarly

$$(\Psi_N(u))^{N(w)} = (\underbrace{\cdots \Psi_N \Psi_M \Psi_N}_{m}(u))^{\underbrace{\cdots NMN}_{m}(w)}.$$

Clearly $\underbrace{\cdots MNM}_{m}(w) = \underbrace{\cdots NMN}_{m}(w)$ and so equation (9) follows from (8). $\square$

Note that the hypothesis that D is a diamond is essential in Theorem 3.3. For example, let D be the zircon in Figure 2, M be the dashed special matching and N be the dotted one.

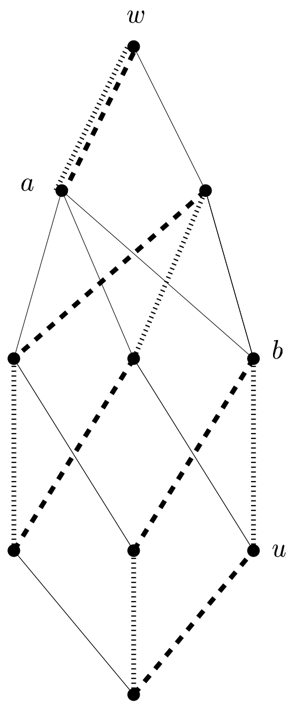

Figure 2: A zircon which is not a diamond

Then

$$q^{c(M,u)} R_{M(u),M(w)}(q) + (q^{c(M,u)} - 1)\, R_{u,M(w)}(q) = R_{\hat{0},a} = (q-1)^3$$

while

$$q^{c(N,u)} R_{N(u),N(w)}(q) + (q^{c(N,u)} - 1)\, R_{u,N(w)}(q) = q R_{b,a}(q) + (q-1)\, R_{u,a}(q) = q(q-1) + (q-1)^3.$$

The R-polynomials of a diamond have the following basic properties. For all $x \in D$, we let $\varepsilon_x \stackrel{\text{def}}{=} (-1)^{\rho(x)}$.

Proposition 3.4. *Let D be a diamond and $u, v \in D$, $u \leq v$. Then:*

1. *$R_{u,v}(q)$ is a monic polynomial of degree $\rho(v) - \rho(u)$;*

2. *$R_{u,v}(0) = \varepsilon_u \varepsilon_v$;*

3. *$R_{u,v}(q) = \varepsilon_u \varepsilon_v q^{\rho(v)-\rho(u)} R_{u,v}(\frac{1}{q})$;*

4. $\sum_{u\leq a\leq v}\varepsilon_u\varepsilon_a R_{u,a}(q)R_{a,v}(q)=\delta_{u,v}$ *(Kronecker delta).*

Proof. The first three statements are straightforward by induction on $\rho(v)$. The last one follows exactly in the same way as in the proof of Theorem 6.7 of [5]. $\Box$

The Kazhdan-Lusztig polynomials of a diamond are defined by the following theorem, whose proof (after 3. and 4. of Proposition 3.4) follows by Theorem 6.5 of [12].

Theorem 3.5. *Let D be a diamond. Then there is a unique family of polynomials $\{P_{u,v}(q)\}_{u,v\in D}\subseteq \mathbb{Z}[q]$ satisfying the following conditions:*

1. $P_{u,v}(q)=0$ *if* $u\not\leq v$*;*
2. $P_{u,u}(q)=1$*;*
3. $\deg(P_{u,v}(q))\leq \frac{1}{2}(\rho(v)-\rho(u)-1)$*, if* $u<v$*;*
4. *if* $u\leq v$*, then*
$$q^{\rho(v)-\rho(u)}\,P_{u,v}\left(\frac{1}{q}\right)=\sum_{u\leq z\leq v}R_{u,z}(q)\,P_{z,v}(q)\,.$$

It is usually not easy to check that a poset is a diamond using the definition. The following result gives some simple sufficient conditions for a zircon to be a diamond (and it can be applied, for example, to all simply laced Coxeter groups). We say that a poset P is *n-gon-avoiding* if it does not contain a dihedral interval of length $\frac{n}{2}$. We say that P is *lower n-gon-avoiding* if it does not contain a dihedral interval of length $\frac{n}{2}$ containing a minimal element. Furthermore, we say that P *avoids* $K_{3,2}$ if there are no elements $a_1,a_2,a_3,b_1,b_2\in P$, all distinct, such that either $a_i\lhd b_j$ for all $i\in[3]$, $j\in[2]$ or $a_i\rhd b_j$ for all $i\in[3]$, $j\in[2]$.

Lemma 3.6. *Let P be a ranked poset with $\hat{0}$ that avoids $K_{3,2}$, and $u,v\in P$, $u\leq v$. Let M and N be two special matchings of $[\hat{0},v]$. If $|\langle M,N\rangle(u)|=2m>2$, then there exist $u'\in P$ and a dihedral poset I such that $\hat{0},M(\hat{0}),N(\hat{0})\in I$, $\langle M,N\rangle(u')\subseteq I$ and $|\langle M,N\rangle(u')|=2m$.*

Proof. The assertion follows in a way similar to Lemma 4.3 of [4]. $\Box$

Theorem 3.7. *Let Z be a connected zircon which is both lower 8-gon-avoiding and $K_{3,2}$-avoiding. Suppose that for all $w\in Z$, $\rho(w)\geq 2$, and for all $M\in SM_w$ there exists a special matching $M'\in SM_w$ such that $M(w)\neq M'(w)$. Then Z is a diamond.*

Proof. We have to prove that for all $w\in Z$ and all $M,N\in SM_w$, M and N are coherent. We proceed by induction on $\rho(w)$, the result being clear if $\rho(w)=1$.

So assume $\rho(w)\geq 2$. First we prove that, if $M(w)\neq N(w)$, then M and N are strictly coherent, i.e. $|\langle M,N\rangle(u)|$ divides $|\langle M,N\rangle(w)|$ for all $u\leq w$. So set $2n\overset{\text{def}}{=}|\langle M,N\rangle(w)|$, where $n\geq 2$ since $M(w)\neq N(w)$, and let $u\leq w$ and $2m\overset{\text{def}}{=}|\langle M,N\rangle(u)|$. We have to prove that

m divides n so we may assume $m \geq 2$. Since any connected zircon has $\hat{0}$ (see [11]) we apply Lemma 3.6 to $\langle M,N\rangle(w)$ and $\langle M,N\rangle(u)$ in order to prove that there exists a lower dihedral interval containing an orbit of cardinality $2n$ and a lower dihedral interval containing an orbit of cardinality $2m$. Hence $\{m,n\} \subset \{2,3\}$ since Z is lower 8-gon-avoiding.
If $M(\hat{0}) \neq N(\hat{0})$ then the two dihedral intervals must coincide, so $m = n$.
If $M(\hat{0}) = N(\hat{0})$, then the two dihedral intervals share at least two elements ($\hat{0}$ and $M(\hat{0}) = N(\hat{0})$), and so $m = n = 2$.

Now suppose that $M(w) = N(w)$. By our hypotheses, there exists a special matching $M' \in SM_w$ such that $M(w) \neq M'(w)$. Then, by what we have already proved, M and M' are strictly coherent as well as M' and N. $\square$

As mentioned at the beginning of this section the class of diamonds includes all Coxeter groups.

Theorem 3.8. *All Coxeter groups partially ordered by Bruhat order are diamonds.*

Proof. Let (W,S) be a Coxeter system and let M and N be two special matchings of an element $w \in W$. We need to prove that M and N are coherent.
Suppose first that $[e,w]$ is not dihedral. If M and N are both given by left (or right) multiplication, then M and N are strictly coherent. Now, suppose that M is left multiplication by a certain $s \in S$, N is right multiplication by a certain $t \in S$ and $ss_2s_3\cdots s_r$ is a reduced expression of w. Call ρ_r the special matching given by right multiplication by s_r. Then M and ρ_r are strictly coherent, as well as ρ_r and N. So M and N are coherent. If M and N are not both multiplication matchings, then the assertion follows by Theorem 2.5.

Now suppose that $[e,w]$ is a dihedral interval of length n. Then every matching of $[e,w]$ is special and so SM_w is in bijection with the set of all n-sequences with entries in $\{+,-\}$, ending with $+$. In fact, for all $i = [n-1]$, let $\{v_i^+, v_i^-\} \stackrel{\text{def}}{=} \{v \in [e,w] : l(v) = i\}$, $v_n^+ = w$, and send the special matching M to the sequence $(x_1\, x_2 \ldots x_{n-1}\, +)$ where $x_i = +$ if $M(v_i^+) \lhd v_i^+$ and $x_i = -$ if $M(v_i^-) \lhd v_i^-$ (note that exactly one of this conditions applies). For example, the sequence associated to the dotted special matching in Figure 3 is $(--++-+)$.

Any two such sequences give rise to a composition of n by looking at the positions where they coincide. In fact, let $n_1 < n_2 < \ldots < n_k = n$ be the positions where the two sequences coincide. Then we can consider the composition $(n_1, n_2 - n_1, \ldots, n_k - n_{k-1})$. For example, the sequences $(--+++-+-+)$ and $(+---+---+)$ give rise to the composition $(2,3,1,2,1)$ of 9 since they have the same entries in positions 2, 5, 6, 8, and clearly 9. Two special matchings are strictly coherent if all the terms in the associated composition divide the last term.

Let $\overline{x} = (x_1 \ldots x_{n-1}\, +)$ and $\overline{y}$ be the sequence obtained from $\overline{x}$ by changing the entry x_i. It is enough to show that there exists a chain of sequences starting with $\overline{x}$ and ending with $\overline{y}$,

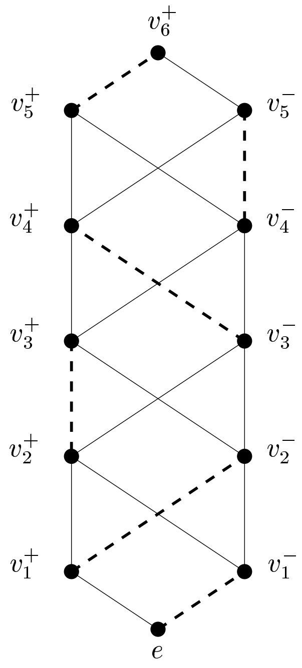

Figure 3: The special matching $(--++-+)$

and such that all the terms in the composition associated to any two consecutive sequences divide the last term. Then the assertion will follow by transitivity. If $i = n-1$, then the chain $(\overline{x}, \overline{y})$ satisfies the properties since the associated composition is $(1, \ldots, 1, 2)$. If $i \neq n-1$, then we can consider the sequence $\overline{z}$ obtained from $\overline{x}$ by changing the entry x_{n-1}. The chain $(\overline{x}, \overline{z}, \overline{y})$ has the required properties. $\square$

Remark. Our definitions of R-polynomials and Kazhdan-Lusztig polynomials are consistent with the ones for Coxeter groups given in [9]. Namely, if D is a diamond that is isomorphic to a lower Bruhat interval $[e, w]$ of a Coxeter group W, and $\phi : D \to [e, w]$ is a poset isomorphism, then $R_{x,y}(q) = R_{\phi(x),\phi(y)}(q)$ and $P_{x,y}(q) = P_{\phi(x),\phi(y)}(q)$ for all $x, y \in D$. The proof of this fact is clear by Theorems 2.6, 3.3 and 3.5. In particular, if D is a diamond isomorphic to a Boolean algebra then $R_{x,y}(q) = (q-1)^{\rho(x,y)}$ and $P_{x,y}(q) = 1$ for all $x, y \in D$, and if D is a dihedral poset then $R_{x,y}(q) = q^{\rho(x,y)} - 2q^{\rho(x,y)-1} + 2q^{\rho(x,y)-2} - \ldots \pm 2q \mp 1$ and $P_{x,y}(q) = 1$ for all $x, y \in D$.

One may ask if all diamonds are subposets of some Coxeter group. The poset in Figure 4 gives a negative answer to this question. This can be seen by noting that its two coatoms have 5 coatoms in common, and this is not allowed in Coxeter groups by Theorem 3.1 of [4]. On the other hand, one can easily check that it is a diamond (the only two special matchings of the entire poset are shown in the picture).

The two main conjectures on Kazhdan-Lusztig polynomials are the so-called non-negativity and combinatorial invariance conjectures (see, e.g., [1, p. 171 and p. 161]). Namely, it is con-

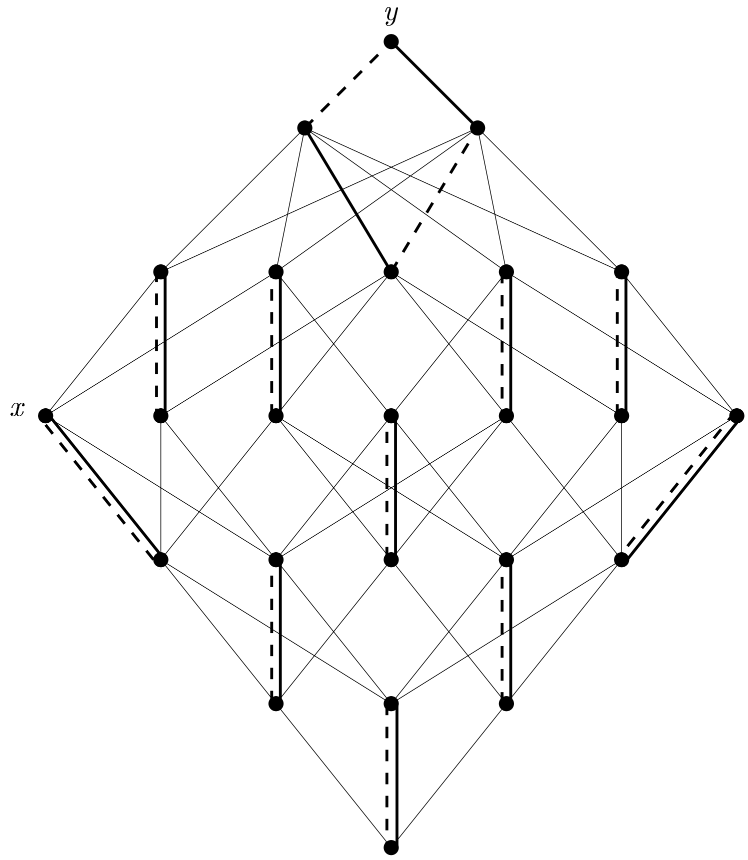

Figure 4: A diamond that is not a Bruhat interval

jectured that, for all u, v in a Coxeter group W, the Kazhdan-Lusztig polynomial $P_{u,v}$ has nonnegative coefficients and that it depends only on the isomorphism type of the interval $[u, v]$ as an abstract poset. It is interesting to note that both of these conjectures are false, in general, for diamonds. For example, consider the elements x and y in the diamond in Figure 4. Then one can check that $P_{x,y}(q) = 1 - q$. Furthermore, the interval $[x, y]$ is isomorphic to the interval $[e, s_1 s_2 s_1]$ of the symmetric group, but $P_{e,s_1 s_2 s_1}(q) = 1$.

4 Further properties

In this section we show that many combinatorial properties and formulas for the Kazhdan-Lusztig and R-polynomials of Coxeter groups continue to hold for diamonds. As the proofs of these results are essentially identical to (and sometimes simpler than) the corresponding ones for Coxeter groups, we omit them. The results of this section are not needed in the rest of the paper.

Definition 4.1. Let $v \in D$. A sequence $(M_1, \ldots, M_r)$ (where $r \stackrel{\text{def}}{=} \rho(v)$) is a *regular sequence (of special matchings)* for v if, for all $i \in [r]$, M_i is a special matching of $M_{i+1} \cdots M_r(v)$.

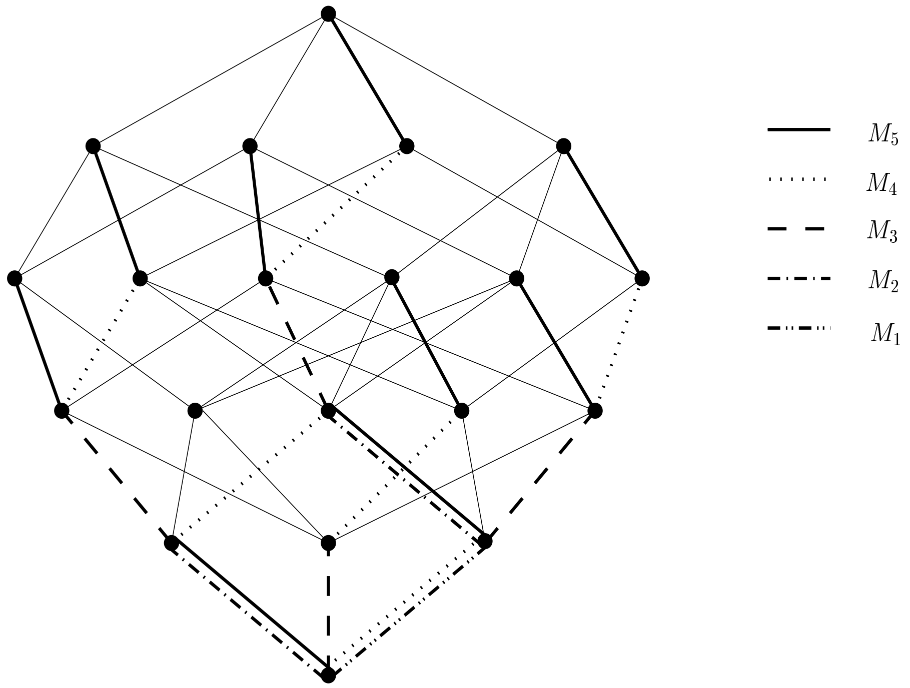

Figure 5: A regular sequence of special matchings

Note that, in particular, $M_1 \cdots M_r(v) = \hat{0}$. For example, the sequence $(M_1, M_2, M_3, M_4, M_5)$ in Figure 5 is a regular sequence.

The following result is the analogue, for any diamond, of a well known result for Coxeter groups.

Lemma 4.2. *Let $v \in D$, and $(M_1, \ldots, M_r)$ be a regular sequence for v. Then the composition $M_{i_k} \cdots M_{i_1}(\hat{0})$ is defined for any $1 \le i_1 < i_2 < \cdots < i_k \le r$, and for all $u \le v$ there exist $1 \le i_1 < \ldots < i_k \le r$ such that $(M_{i_1}, \ldots, M_{i_k})$ is a regular sequence for u.*

Let $v \in D$ and $\mathcal{M} = (M_1, \ldots, M_r)$ be a regular sequence for v (so $r = \rho(v)$). Given $S = \{i_1, \ldots, i_k\} \subseteq [r]$ with $i_1 < i_2 < \ldots < i_k$, we let

$$\pi(S) \stackrel{\text{def}}{=} M_{i_k} \cdots M_{i_1}(e)$$

and we define, following [4], for each $j \in [r]$,

$$\varepsilon_j(S) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } M_j(y) \lhd y, \\ 0, & \text{if } M_j(y) \rhd y, \end{cases}$$

where $y \stackrel{\text{def}}{=} \pi(S \cap [j-1])$. We also let $d_1(S,r) \stackrel{\text{def}}{=} \sum_{j \in [r] \setminus S} \varepsilon_j(S)$ and $d_2(S) \stackrel{\text{def}}{=} \sum_{j \in S} \varepsilon_j(S)$. Note that $(M_{i_1}, \ldots, M_{i_k})$ is a regular sequence for $M_{i_k} \cdots M_{i_1}(e)$ if and only if $d_2(S) = 0$. Let, for brevity,

$$d(S,r) \stackrel{\text{def}}{=} d_1(S,r) + d_2(S).$$

We say that S is *distinguished*, with respect to $\mathcal{M}$, if $d_1(S,r)=0$. We denote by $\mathcal{D}(\mathcal{M})$ the set of all subsets of $[r]$ which are distinguished with respect to $\mathcal{M}$, and we let, for $u \in W$,

$$\mathcal{D}(\mathcal{M})_u \stackrel{\text{def}}{=} \{S \in \mathcal{D}(\mathcal{M}) \,:\, \pi(S)=u\}.$$

Our first result of this section is a closed formula for the R-polynomials which generalizes Theorem 1.3 of [6] and Theorem 9.4 of [4]. For all $u,v \in D$, let $\widetilde{R}_{u,v}(q)$ be the polynomial defined by $R_{u,v}(q) = q^{\frac{1}{2}(\rho(v)-\rho(u))}\widetilde{R}_{u,v}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})$.

Theorem 4.3. *Let $v \in D$ and $\mathcal{M}=(M_1,\ldots,M_r)$ be a regular sequence for v. Then*

$$\widetilde{R}_{u,v}(q) = \sum_{S \in \mathcal{D}(\mathcal{M})_u} q^{\rho(v)-|S|},$$

for all $u \in D$.

The preceding result has the following consequence, which may also be proved directly, and which associates a canonical labeling to any element of a diamond, once a regular sequence has been fixed.

Corollary 4.4. *Let $v \in D$ and $(M_1,\ldots,M_r)$ be a regular sequence for v. Then π is a bijection between $\{S \subseteq [r] : d_1(S,r)=d_2(S)=0\}$ and $\{u \in D : u \leq v\}$.*

We now derive a nonrecursive formula for the Kazhdan-Lusztig polynomials, which is the analogue, for diamonds, of Theorem 4.2 of [2]. Given a chain $\mathcal{C} = a_0 < a_1 < \cdots < a_i$ (of length $l(\mathcal{C})=i$) in D, we define

$$R_{a_0,\ldots,a_i}(q) \stackrel{\text{def}}{=} R_{a_0,a_1}(q)L_{\frac{\rho(a_i)-\rho(a_1)-1}{2}}(R_{a_1,\ldots,a_i}(q))$$

(where $L_k(\sum_{i\geq 0} a_i q^i) \stackrel{\text{def}}{=} \sum_{i\leq k} a_i q^i$) if $i \geq 2$, and $R_{a_0,\ldots,a_i}(q) \stackrel{\text{def}}{=} R_{a_0,a_1}(q)$ if $i=1$. We call the polynomial $R_{a_0,\ldots,a_i}(q)$ just defined the *R-polynomial* of the chain $a_0 < a_1 < \cdots < a_i$. Recall that a chain from u to v is a chain whose first element is u and whose last element is v.

Theorem 4.5. *Let $u,v \in D$, $u<v$. Then,*

$$P_{u,v}(q) - q^{\rho(u,v)}P_{u,v}\left(\frac{1}{q}\right) = \sum_{\mathcal{C} \in C(u,v)} (-1)^{l(\mathcal{C})} R_{\mathcal{C}}(q),$$

where $C(u,v)$ is the set of all the chains from u to v.

Given $u,v \in D$, $u \leq v$, and a composition $\alpha=(\alpha_1,\ldots,\alpha_r)$ of length $l(\alpha)=r$, we define

$$\begin{aligned} c_\alpha(u,v) &\stackrel{\text{def}}{=} \textstyle\sum_{(a_0,\ldots,a_r)\in C_r(u,v)} \prod_{j=1}^{r}[q^{\alpha_j}](\widetilde{R}_{a_{j-1},a_j}), \\ b_\alpha(u,v) &\stackrel{\text{def}}{=} \textstyle\sum_{\beta \succeq \alpha} (-1)^{l(\beta)-l(\alpha)} c_\beta(u,v) \end{aligned}$$

where $C_r(u,v)$ denotes the set of all chains of length r from u to v, $[q^k](R(q))$ is the coefficient of q^k in the polynomial $R(q)$, and $(\alpha_1,\ldots,\alpha_r) \preceq (\beta_1,\ldots,\beta_t)$ if there exist $0 = i_0 < i_1 < \cdots < i_{t-1} < i_t = r$ such that $\sum_{j=i_{k-1}+1}^{i_k} \alpha_j = \beta_k$ for $k = 1,\ldots,t$. Note that, if $r \geq 2$, then

$$c_{\alpha_1,\ldots,\alpha_r}(u,v) = \sum_{u \leq a \leq v} c_{\alpha_1,\ldots,\alpha_{r-1}}(u,a) c_{\alpha_r}(a,v) .$$

Proceeding as in [4, Section 11], one can prove the following nonrecursive combinatorial formula for the Kazhdan-Lusztig polynomials of a diamond, which generalizes (52) of [2].

Theorem 4.6. *Let $u, v \in D$, $u < v$. Then*

$$P_{u,v}(q) - q^{\rho(v)-\rho(u)} P_{u,v}\left(\frac{1}{q}\right) = \sum_{\beta \in C} q^{\frac{\rho(v)-\rho(u)-|\beta|}{2}} \Upsilon_\beta(q)\, b_\beta(u,v),$$

where the polynomial $\Upsilon_\beta(q)$ is as in [4, Section 2].

5 The Hecke algebra of a diamond

In this section we associate to any diamond D a Coxeter group, and hence a Hecke algebra, and we show that there is a natural action of this Hecke algebra on the free module $\mathcal{M}_D$ generated by the diamond. We then construct, using the polynomials defined in Section 3, another basis of $\mathcal{M}_D$ and we express the action with respect to this basis.

Let SM_D be the collection of all special matchings of D. We denote by (W_D, SM_D) the Coxeter system whose Coxeter generators are the elements of SM_D and whose Coxeter matrix is given by $m(M,N) \stackrel{\text{def}}{=} o(MN)$, the period of MN as a permutation of D (eventually ∞). We denote by $\mathcal{H}(D)$ the Hecke algebra of W_D and by $\mathcal{M}_D$ the free $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module defined by

$$\mathcal{M}_D \stackrel{\text{def}}{=} \bigoplus_{u \in D} \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]u.$$

Our first result shows that the natural action of W_D on D extends to an action of $\mathcal{H}(D)$ on $\mathcal{M}_D$.

Theorem 5.1. *Let D be a diamond. Then there exists a unique structure of $\mathcal{H}(D)$-module on $\mathcal{M}_D$ such that*

$$T_M(u) = \begin{cases} M(u), & \text{if } M(u) \rhd u, \\ qM(u) + (q-1)u, & \text{otherwise}, \end{cases} \tag{10}$$

for all $u \in D$ and $M \in SM_D$.

Proof. The Hecke algebra $\mathcal{H}(D)$ is generated by the set $\{T_M : M \in SM_D\}$ subject only to the relations:

$$T_M^2 = (q-1)T_M + qT_e, \qquad \text{for all } M \in SM_D,$$
$$\underbrace{T_M T_N T_M \cdots}_{m(M,N)} = \underbrace{T_N T_M T_N \cdots}_{m(M,N)}, \qquad \text{for all } M, N \in SM_D \text{ with } m(M,N) < \infty.$$

Hence the uniqueness part is clear. To prove the existence we only have to check that the action respects these relations. This can be done using the same argument as in the proof of (7). $\square$

Now we introduce an operator ι on $\mathcal{M}_D$. Define a ring automorphism $\iota : \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \to \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ by $\iota(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}$ and let

$$\iota(v) \stackrel{\text{def}}{=} \varepsilon_v q^{-\rho(v)} \sum_{u \leq v} \varepsilon_u \, R_{u,v}(q) \, u,$$

where $\varepsilon_x = (-1)^{\rho(x)}$ for all $x \in D$. Combine these assignments to obtain a $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-semilinear map with respect to ι.

Proposition 5.2. *The map ι is an involution on $\mathcal{M}_D$.*

Proof. By 3. of Proposition 3.4, we have

$$\iota^2(v) = \sum_{x \leq v} \Big(\sum_{x \leq u \leq v} \varepsilon_x \varepsilon_u R_{x,u} R_{u,v} \Big) x$$

and the assertion follows by 4. of Proposition 3.4. $\square$

The following result is a compact reformulation of Theorem 3.3 in terms of the action of $\mathcal{H}(D)$ on $\mathcal{M}_D$. As no confusion arises, we use the same symbol ι also for the involution of the Hecke algebra $\mathcal{H}(D)$.

Theorem 5.3. *The map $\iota : \mathcal{M}_D \to \mathcal{M}_D$ is a $\mathcal{H}(D)$-semilinear map with respect to $\iota : \mathcal{H}(D) \to \mathcal{H}(D)$, i.e.*

$$\iota \circ h = \iota(h) \circ \iota$$

for all $h \in \mathcal{H}(D)$.

Proof. We may clearly assume that $h = T_M$ for some $M \in SM_D$. Let $u \in D$. It is enough to show that $\iota(T_M(u)) = \iota(T_M)(\iota(u))$.

Suppose first that $u \lhd M(u)$. Then, by (10), we have that

$$\iota(T_M(u)) = \iota(M(u)) = -\varepsilon_u q^{-\rho(u)-1} \sum_x \varepsilon_x R_{x,M(u)} \, x.$$

On the other hand

$$
\begin{aligned}
\iota(T_M)(\iota(u)) &= (q^{-1}T_M - (1-q^{-1}))(\varepsilon_u q^{-\rho(u)} \sum_x \varepsilon_x R_{x,u}\, x) \\
&= \varepsilon_u q^{-\rho(u)} \Big(\sum_{x \lhd M(x)} (q^{-1}\varepsilon_x R_{x,u}\, M(x) - (1-q^{-1})\varepsilon_x R_{x,u}\, x) \\
&\quad + \sum_{x \rhd M(x)} (q^{-1}\varepsilon_x R_{x,u}(qM(x) + (q-1)x) - (1-q^{-1})\varepsilon_x R_{x,u}\, x) \Big) \\
&= \varepsilon_u q^{-\rho(u)} \Big(- \sum_{x \rhd M(x)} q^{-1}\varepsilon_x R_{M(x),u}\, x - \sum_{x \lhd M(x)} (1-q^{-1})\varepsilon_x R_{x,u} x \\
&\quad - \sum_{x \lhd M(x)} \varepsilon_x R_{M(x),u} x \Big) \\
&= \varepsilon_u q^{-\rho(u)} \Big(- \sum_{x \rhd M(x)} q^{-1}\varepsilon_x R_{x,M(u)}\, x - \sum_{x \lhd M(x)} q^{-1}\varepsilon_x R_{x,M(u)}\, x \Big)
\end{aligned}
$$

by Theorem 3.3 and the assertion follows in this case.

Suppose now that $u \rhd M(u)$. Then applying what we have just proved to $M(u)$ we get

$$
\iota(u) = \iota(T_M(M(u))) = \iota(T_M)(\iota(M(u))) = T_M^{-1}(\iota(M(u)).
$$

Therefore, applying T_M to both sides of the previous identity we get, by Proposition 5.1, $T_M(\iota(u)) = \iota(M(u))$. Hence

$$
\begin{aligned}
\iota(T_M(u)) &= \iota(qM(u) + (q-1)u) \\
&= q^{-1}\iota(M(u)) + (q^{-1} - 1)\iota(u) \\
&= q^{-1}T_M(\iota(u)) + (q^{-1} - 1)\iota(u) \\
&= (q^{-1}T_M + q^{-1} - 1)(\iota(u)) \\
&= \iota(T_M)(\iota(u)),
\end{aligned}
$$

and the result again follows. □

For all $w \in D$, we let

$$
D_w \stackrel{\text{def}}{=} q^{-\frac{\rho(w)}{2}} \sum_{u \leq w} P_{u,w} u, \tag{11}
$$

As in the case of Coxeter groups, the elements D_w are characterized by the following property.

Theorem 5.4. *Let D be a diamond. For each $w \in D$ there exists a unique element $\tilde{D}_w \in \mathcal{M}_D$ having the following two properties*

1. $\iota(\tilde{D}_w) = \tilde{D}_w$*;*

2.
$$\tilde{D}_w = q^{-\frac{\rho(w)}{2}} \sum_{v \le w} \tilde{P}_{v,w} v, \tag{12}$$

where $\tilde{P}_{v,w} \in \mathbb{Z}[q]$ has degree $\le \frac{1}{2}(\rho(w)-\rho(v)-1)$ if $v < w$, and $\tilde{P}_{w,w} = 1$.

Proof. Let us prove that every family $\{\tilde{D}_w\}_{w \in D}$ satisfying 2. satisfies 1. if and only if
$$q^{\rho(w)-\rho(u)} \iota(\tilde{P}_{u,w}) = \sum_{u \le v \le w} R_{u,v} \tilde{P}_{v,w}. \tag{13}$$
In fact, applying ι to both sides of (12), we get
$$\iota(\tilde{D}_w) = q^{\frac{\rho(w)}{2}} \sum_{v \le w} \varepsilon_v q^{-\rho(v)} \iota(\tilde{P}_{v,w}) \sum_{u \le v} \varepsilon_u R_{u,v} u.$$
Property 1. holds if and only if the coefficient of u in this expression equals the coefficient of u in (12), i.e.
$$q^{-\frac{\rho(w)}{2}} \tilde{P}_{u,w} = q^{\frac{\rho(w)}{2}} \sum_{u \le v \le w} \varepsilon_u \varepsilon_v q^{-\rho(v)} \iota(\tilde{P}_{v,w}) R_{u,v}$$
for all $u \in D$. Applying ι to both sides of this expression and then applying 4. of Proposition 3.4 we get that this holds if and only if (13) is satisfied. But, by Theorem 3.5 and the constraint on the degree, (13) implies that $\tilde{P}_{u,w} = P_{u,w}$ for all u, w. $\square$

The action of $\mathcal{H}(D)$ with respect to the basis $\{D_v\}_{v \in D}$ of $\mathcal{M}_D$ is given by the following result. Its proof is similar to that of Theorem 8.3 of [4] and is therefore omitted. For $u, v \in D$, $u < v$, we let $\overline{\mu}(u,v)$ be the coefficient of $q^{\frac{1}{2}(\rho(v)-\rho(u)-1)}$ in $P_{u,v}(q)$ if $\rho(v)-\rho(u)$ is odd, and $\overline{\mu}(u,v) \stackrel{\mathrm{def}}{=} 0$ otherwise.

Theorem 5.5. *Let $M \in SM_D$. Then, for all $v \in D$*
$$T_M(D_v) = \begin{cases} -D_v, & \text{if } M(v) < v, \\ qD_v + q^{\frac{1}{2}} D_{M(v)} + q^{\frac{1}{2}} \sum \overline{\mu}(u,v) D_u, & \text{if } M(v) > v, \end{cases} \tag{14}$$
where the sum is taken over all $u < v$ such that $M(u) < u$. $\square$

As a consequence of Theorem 5.5 we get the following result which also gives an algorithm to compute the elements D_v and the Kazhdan-Lusztig polynomials of D, and generalizes formula (2.2c) of [9].

Corollary 5.6. *Let $u, v \in D$, $u < v$, and $M \in SM_v$. Then*
$$\begin{aligned} D_v &= C'_M(D_{M(v)}) - \sum_{\{z \,:\, M(z) \lhd z\}} \bar{\mu}(z, M(v)) D_z, \\ P_{u,v}(q) &= q^c P_{M(u),M(v)}(q) + q^{1-c} P_{u,M(v)}(q) - \sum_{\{z \,:\, M(z) \lhd z\}} \overline{\mu}(z, M(v)) q^{\frac{\rho(z,v)}{2}} P_{u,z}(q), \end{aligned}$$
where $c = c(M,u)$. $\square$

The previous corollary implies the following useful fact. For any $u, v \in D$, $u < v$, we have

$$P_{u,v} = P_{M(u),v} \tag{15}$$

for all $M \in SM_v$.

Another consequence of the previous corollary is the following result whose proof is entirely similar to the one of Corollary 9.9 of [4].

Corollary 5.7. *Let D be a diamond which has Kazhdan-Lusztig polynomials with nonnegative coefficients. Let $v \in D$ and $(M_1, \ldots, M_r)$ be a regular sequence for v. Then there exists a family $\mathcal{E}$ of subsets of $[r]$ such that*

$$P_{u,v}(q) = \sum_{\{S \in \mathcal{E} : \pi(S) = u\}} q^{d(S,r)},$$

for all $u \leq v$. $\square$

6 Hecke algebra representations

In this section we show that $\mathcal{M}_D$ has a filtration as $\mathcal{H}(D)$-module associated to the basis $\{D_v\}_{v \in D}$ defined in (11). Let D be a diamond and $S \subseteq SM_D$ be a subset of the set of special matchings of D. We denote by W_D^S the parabolic subgroup of W_D generated by S, and by $\mathcal{H}(D, S)$ the corresponding Hecke algebra (so that $\mathcal{H}(D, SM_D) = \mathcal{H}(D)$). For any $x \in D$, we define

$$D_S(x) = \{M \in S : M(x) \lhd x\}.$$

Note that it may happen that $D_S(x) = \emptyset$ even if $x \neq \hat{0}$ and $S = SM_D$ (see, for example, Figure 4). Recall from Section 5 the definition of $\overline{\mu}(x, y)$ for $x, y \in D$, $x < y$, and set $\overline{\mu}(x, y) \stackrel{\text{def}}{=} \overline{\mu}(y, x)$ when $x > y$. Moreover, let

$$E \stackrel{\text{def}}{=} \{\{x, y\} : \ x < y \text{ or } x > y \text{ and } \overline{\mu}(x, y) \neq 0\}.$$

Here are a few immediate observations.

Lemma 6.1. *Let $x, y \in D$, $x < y$. Then the following hold:*

(i) if $\{x, y\} \in E$, then $\rho(y) - \rho(x)$ is odd;

(ii) if $\rho(y) - \rho(x) = 1$, then $\overline{\mu}(x, y) = 1$ and $\{x, y\} \in E$;

(iii) if $\rho(y) - \rho(x) > 1$ and there exists $M \in S$ such that $M(y) \lhd y$ and $M(x) \rhd x$, then $\{x, y\} \notin E$.

Proof. The first two statements follow directly from the definitions. For the third, note that $P_{x,y} = P_{M(x),y}$ by (15), so the degree of $P_{x,y}$ is $< \frac{1}{2}(\rho(y) - \rho(x) - 1)$ and $\overline{\mu}(x, y) = 0$. $\square$

We define the following graph which has doubly-labeled edges.

Definition 6.2. The *diamond graph of* D (with respect to S) is the directed graph $\Gamma_D^S = (D, A)$ whose set A of labeled edges $x \xrightarrow[M]{\overline{\mu}} y$ are of the following two types:

(i) $x \neq y$, $\{x, y\} \in E$, $M \in D_S(x) \setminus D_S(y)$ and $\overline{\mu} = \overline{\mu}(x, y)$;

(ii) $x = y$, $M \in S$ and $\overline{\mu} = \begin{cases} q^{\frac{1}{2}}, & \text{if } M \notin D_S(x), \\ -q^{-\frac{1}{2}}, & \text{if } M \in D_S(x). \end{cases}$

Hence, there are exactly $|S|$ loops attached to every node. Note that Γ_D^S can have multiple (parallel or antiparallel) edges between a given pair of nodes. In such cases, these edges always have distinct labels. The down-directed edges in Γ_D^S are easy to characterize: they are the edges of all the matchings in S.

Lemma 6.3. *Suppose that $\rho(x) > \rho(y)$. Then there is an edge in Γ_D^S from x to y if and only if $M(x) = y$ for some $M \in S$.*

Proof. If $x = M(y)$, then clearly $x \to y$ is an M-labeled edge in Γ_D^S. The converse follows from *(iii)* of Lemma 6.1. $\square$

We can now introduce the main concepts of this section.

Definition 6.4. Let $x, y \in D$. We say the following:

(i) $x \preceq y$ if there exists a directed path in Γ_D^S from x to y;

(ii) $x \sim y$ if $x \preceq y$ and $y \preceq x$;

(iii) an equivalence class of D under $\sim$ is a *diamond cell*;

(iv) given two diamond cells $\mathcal{C}$ and $\mathcal{C}'$, $\mathcal{C} \preceq \mathcal{C}'$ if $x \preceq y$ for some (equivalently, all) $x \in \mathcal{C}$, $y \in \mathcal{C}'$.

We can consider the induced subgraphs of Γ_D^S on each cell $\mathcal{C}$, and in fact from now on we will usually think of cells as the induced graphs $\Gamma_{\mathcal{C}}$. Hence the cells are the strongly connected components of Γ_D^S. For example, let D be the dihedral poset of length 4. There are 8 special matchings of D. We can choose a subset $S = \{M_1, M_2, M_3, M_4\} \subset SM_D$ of cardinality 4 such that the Coxeter system (W_D^S, S) is of type $\tilde{A}_3$. The diamond graph Γ_D^S (without loops, without labels $\overline{\mu}$ equal to 1 and where i stands for M_i for $i \in [4]$) is in Figure 6. Therefore there are 3 diamond cells $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, with $|\mathcal{C}_1| = |\mathcal{C}_3| = 1$, $|\mathcal{C}_2| = 6$, and $\mathcal{C}_1 \preceq \mathcal{C}_2 \preceq \mathcal{C}_3$.

Given the previous definitions, (14) can now be more succinctly written as

$$T_M(D_w) = q^{\frac{1}{2}} \sum_x \overline{\mu}(x, w) D_x, \tag{16}$$

with summation over all $x \in D$ having an M-labeled arrow to w in Γ_D^S.

The *natural representation* of $\mathcal{H}(D, S)$ is the mapping $\mathrm{Nat}_{D,S} : \mathcal{H}(D, S) \to \mathrm{End}_{\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]}(\mathcal{M}_D)$ defined by

$$\mathrm{Nat}_{D,S}(T_Q) : x \mapsto T_Q(x)$$

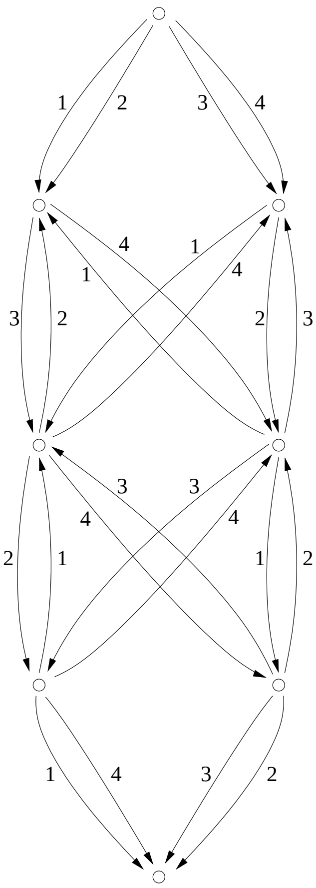

Figure 6: The diamond graph Γ_D^S

for all $x \in \mathcal{M}_D$ and all $Q \in W_D^S$ (the restriction of the action studied in Section 5). We define the following filtration of $\mathcal{M}_D$. Let $\{\mathcal{C}_i\}_{i\in I}$ be the collection of all cells indexed so that $\mathcal{C}_i \preceq \mathcal{C}_j$ implies $i < j$ (where I can be either $\mathbb{N}$ or $[0,n]$ for some $n \in \mathbb{N}$). Let $\mathcal{M}_{\mathcal{C}}$ be the $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-submodule spanned by $\{D_x : x \in \mathcal{C}\}$ and let

$$\mathcal{I}_i = \bigoplus_{j \leq i} \mathcal{M}_{\mathcal{C}_j}.$$

It follows directly from (16) and the previous definitions that $\{\mathcal{I}_i\}_{i\in I}$ is a filtration of $\mathcal{M}_D$ as a $\mathcal{H}(D,S)$-module:

- $\mathcal{I}_i$ is a $\mathcal{H}(D,S)$-submodule of $\mathcal{M}_D$ for all $i \in I$,

- $\mathcal{I}_i \subseteq \mathcal{I}_j$ for $i \leq j$,

- $\mathcal{M}_D = \bigcup_i \mathcal{I}_i$.

Definition 6.5. We let $\mathcal{D}_{\mathcal{C}_0} = \mathcal{I}_0$ and $\mathcal{D}_{\mathcal{C}_i} = \mathcal{I}_i/\mathcal{I}_{i-1}$ for all $i > 0$. We call $\mathcal{D}_{\mathcal{C}}$ the *diamond representation* determined by the cell $\mathcal{C}$.

Note that the diamond representation $\mathcal{D}_{\mathcal{C}}$ depends only on $\mathcal{C}$ and not on the indexing chosen for the cells. More precisely,

$$\mathcal{D}_{\mathcal{C}} \cong \frac{\bigoplus_{\mathcal{C}' \preceq \mathcal{C}} \mathcal{M}_{\mathcal{C}'}}{\bigoplus_{\mathcal{C}' \prec \mathcal{C}} \mathcal{M}_{\mathcal{C}'}}.$$

Theorem 6.6. *Let $Q \in W_D^S$, $M_1 \dots M_k$ (with $M_i \in S$) be a reduced expression for Q, $A(T_Q) = (a_{x,y}(T_Q))_{x,y \in D}$ be the matrix of $\mathrm{Nat}_{D,S}(T_Q)$ with respect to the basis $\{D_v : v \in D\}$ (so that $T_Q(D_y) = \sum_{x \in D} a_{x,y}(Q) D_x$). Arrange the columns and rows of $A(T_Q)$ cellwise in any total order refining $\preceq$. Then*

1) the block form of $A(T_Q)$ is of the following type

	$\mathcal{C}_0$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\cdots$
$\mathcal{C}_0$	$A_{\mathcal{C}_0}(Q)$	$*$	$*$	
$\mathcal{C}_1$	0	$A_{\mathcal{C}_1}(Q)$	$*$	
$\mathcal{C}_2$	0	0	$A_{\mathcal{C}_2}(Q)$	
$\vdots$				$\ddots$

where for each cell $\mathcal{C}$ we let $A_{\mathcal{C}}(T_Q) = (a_{x,y}(T_Q))_{x,y \in \mathcal{C}}$ be the diagonal-block submatrix (note that not only the number of the blocks can be infinite but also the size of a single block),

2) $A_{\mathcal{C}}(T_Q)$ is the matrix of the diamond representation $\mathcal{D}_{\mathcal{C}}$ with respect to the basis given by the classes of the elements $\{D_x : x \in \mathcal{C}\}$,

3) for all $x, y \in D$,

$$a_{x,y}(T_Q) = q^{\frac{k}{2}} \sum \overline{\mu}(x_0, x_1)\, \overline{\mu}(x_1, x_2) \dots \overline{\mu}(x_{k-1}, x_k),$$

where the sum is over all paths $x = x_0 \xrightarrow[M_1]{} x_1 \xrightarrow[M_2]{} \cdots \xrightarrow[M_k]{} x_k = y$ in Γ_D^S.

Proof. The first two assertions are straightforward from what we have already done. To prove the third, note that equation (16) can be restated as

$$a_{x,y}(T_M) = \begin{cases} q^{\frac{1}{2}} \overline{\mu}(x, y), & \text{if } (x \xrightarrow[M]{} y) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Now use the fact that $T_Q = T_{M_1} \cdots T_{M_k}$ and so $A(T_Q) = A(T_{M_1}) \cdots A(T_{M_k})$. $\square$

Corollary 6.7. *Let D and S be such that $\mathcal{H}(D,S)$ is semisimple. Then,*

$$\mathrm{Nat}_{D,S} \cong \bigoplus_{\mathcal{C}} \mathcal{D}_{\mathcal{C}},$$

where $\mathcal{C}$ runs over all the cells of D.

Proof. Using the homomorphism theorem, by induction on i one can show that

$$\mathcal{M}_D \cong \bigoplus_{j \leq i} \mathcal{D}_{\mathcal{C}_j} \ \oplus \ \mathcal{M}_D/\mathcal{I}_i$$

for all $i \in I$. Then the result follows since $\mathcal{M}_D = \bigcup \mathcal{I}_i$. □

Note that every (left, right or two-sided) Kazhdan-Lusztig representation of a Coxeter group W (see [1, Chapter 6] or [9]) is isomorphic to a diamond representation. In fact, set $D = W$ partially ordered by Bruhat order and S be the set of all special matchings given by left multiplication by some Coxeter generator of W. Then $W_D^S \cong W$ (and hence $\mathcal{H}(D,S) \cong \mathcal{H}(W)$) in a canonical way, and the action of $\mathcal{H}(W)$ on $\mathcal{M}_D$ is isomorphic to the left regular representation. Since, as observed in Section 3, the Kazhdan-Lusztig polynomials of D as a diamond coincide with the Kazhdan-Lusztig polynomials of W as a Coxeter group, the diamond cells of D coincide with the left Kazhdan-Lusztig cells of W, and the representations are isomorphic. Similarly, we obtain the right Kazhdan-Lusztig cells of W by considering the set of all special matchings given by right multiplication by some Coxeter generator of W. Furthermore, if we consider the set of all special matchings arising by multiplication on the left and on the right by some Coxeter generator of W, then W_D^S is isomorphic to $W \times W$ and $\mathcal{M}_D$ has a structure of W-bimodule. In this case, the diamond cells coincide with the two-sided cells and the corresponding representations are isomorphic. However, other choices of S (and other choices of D) lead to representations which are either not obtainable by Kazhdan-Lusztig's method, or obtainable in a much more difficult way. We close this work by giving several such examples.

Let $(U_n, \{s_1, s_2, \ldots, s_n\})$ be the universal Coxeter system of rank n (so $s_1^2 = \cdots = s_n^2 = e$ are the only relations in U_n, see, e.g., [1, Example 1.2.2]). Using Kazhdan and Lusztig's approach we obtain from U_n only representations of U_n itself. Seeing U_n as a diamond, however, we obtain a richer picture. For example, if D is the interval $[e, s_2s_1s_2s_1s_2s_3s_2s_3]$ in U_3 then one can check (e.g. using Theorem 7.6 of [4]) that $|SM_D| = 12$, and that the Coxeter diagram of (W_D, SM_D) is the one shown in Figure 7.
This is a more interesting Coxeter group than the U_3 we started from, and contains as parabolic subgroups many hyperbolic Coxeter groups (see, e.g., [8, §6.9]) as well as the affine Weyl group $\tilde{B}_3$.
Note that in this case we obtain representations of infinite Coxeter groups by a (constructive and) finite procedure while Kazhdan and Lusztig's method to construct the cell representations of an infinite Coxeter group W is (constructive but) potentially infinite (since to

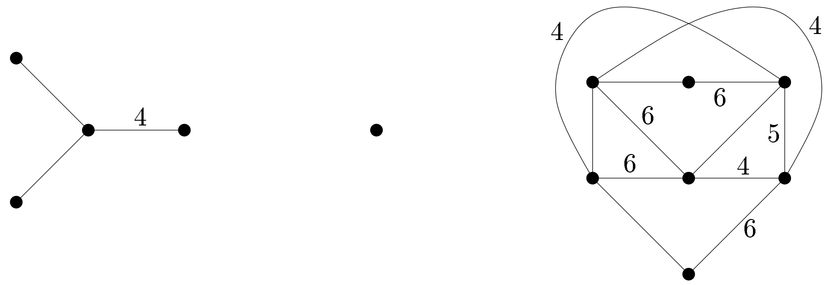

Figure 7: The Coxeter diagram of W_D

construct the cell that contains a given element $u \in W$ one has to find in particular all the $v \in W$ such that $\deg(P_{u,v}) = \frac{1}{2}(l(u,v) - 1))$. Also note that, since the Kazhdan-Lusztig polynomials of a diamond that is isomorphic to a lower interval in a Coxeter group coincide with those of the Coxeter group (see the remark at the end of §3), and since the Kazhdan-Lusztig polynomials of the universal Coxeter groups have been explicitly computed (see [7]) our construction here is also entirely explicit.

Let D be a dihedral poset of rank n. Recall from the proof of Theorem 3.8 that a special matching of D can be encoded by a sequence of $+$ and $-$ signs of length n ending with $+$ in the following way. For all $i \in [n-1]$, let v_i^+ and v_i^- be the two elements of rank i in D and let v_n^+ be the top element of D. Then the sequence $A = a_1\, a_2 \ldots a_n$ encodes the unique special matching M such that $M(v_i^{a_i}) \lhd v_i^{a_i}$ for all $i \in [n]$. Recall also that two special matchings M and N of D give rise to a composition of n by looking at the positions where the corresponding sequences coincide. The order $o(MN)$ is given by the least common multiple of the entries of the associated composition.

The following example shows that already in the case of Weyl groups there are irreducible diamond representations which can not be obtained as Kazhdan-Lusztig representations. Let D be the dihedral poset of length 7 and consider the set $S = \{M, N, O\}$ of special matchings of D encoded by the following binary sequences:

$$\begin{aligned} M &= +++++++ \\ N &= +-+---+ \\ O &= -++++-+. \end{aligned}$$

Then the corresponding diamond graph Γ_D^S is depicted in Figure 8, where we have omitted loops and $\bar{\mu}$ labels, and where edges labeled by M, N and O are represented by dotted, dashed and solid lines respectively.

The Coxeter group W_D^S is isomorphic to the hyperoctahedral group B_3 (since $m(M,N) = 4$, $m(N,O) = 3$, $m(M,O) = 2$). The two vertices in the shaded area form a diamond cell $\mathcal{C}$

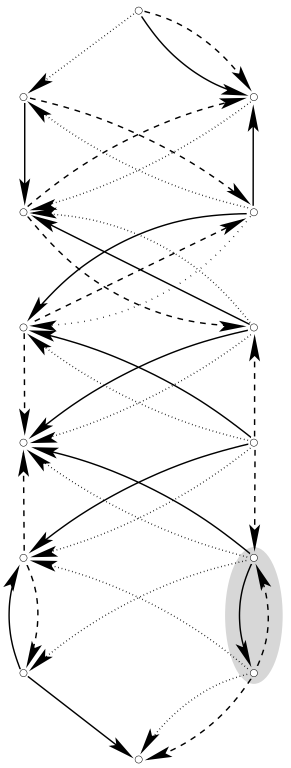

Figure 8: The diamond graph Γ_D^S

and following Theorem 6.6 we have

$$A_{\mathcal{C}}(T_M) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_{\mathcal{C}}(T_N) = \begin{pmatrix} q & 0 \\ q^{\frac{1}{2}} & -1 \end{pmatrix}, \quad A_{\mathcal{C}}(T_O) = \begin{pmatrix} -1 & q^{\frac{1}{2}} \\ 0 & q \end{pmatrix}.$$

Hence this is a 2-dimensional irreducible representation of (the Hecke algebra of) B_3. However, this representation is not a Kazhdan-Lusztig cell representation of B_3. In fact, as it easy to check, there are no Kazhdan-Lusztig (left, right or two-sided) cells of cardinality 2 in B_3. (Note that, in general, taking a parabolic subgroup W_J of a Coxeter group W and the subgraph of the Kazhdan-Lusztig graph of W induced by the directed edges labeled by elements of J yields representations of W_J which are isomorphic to the cell representations of W_J as an abstract Coxeter system). A similar analysis shows that among the ten irreducible representations of B_3, there are eight diamond representations and only four of these can be obtained as Kazhdan-Lusztig representations of B_3.

We now show that there are infinitely many infinite Coxeter groups of rank 3 whose only finite dimensional Kazhdan-Lusztig representation is the trivial one (Proposition 6.9), while they afford non-trivial finite dimensional diamond representations (Proposition 6.10).
We need first the following result about a forbidden local configuration in some Coxeter groups.

Lemma 6.8. *Let (W,S) be a Coxeter system and $s_1,s_2,s_3 \in S$ be such that $m(s_i,s_j) \geq 3$, for $i \neq j$. Then there is no $w \in W$ such that*

$$l(w) = l(s_iw) + 1 = l(s_jw) + 1 = l(s_ks_jw) + 2,$$

where $\{i,j,k\} = \{1,2,3\}$.

Proof. By contradiction suppose that w is a minimal element such that $l(w) = l(s_iw) + 1 = l(s_jw) + 1 = l(s_ks_jw) + 2$ (see Figure 9).

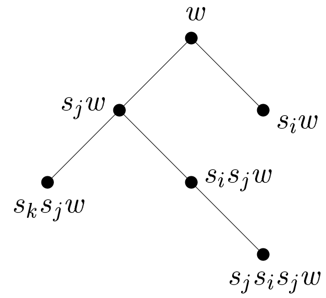

Figure 9: Proof of Lemma 6.8

By construction we have $s_is_jw < s_jw$ (because $s_iw < w$) and, since $m(s_i,s_j) \geq 3$ we also have $s_js_is_jw < s_is_jw$. So, if we let $w' = s_jw$ we have

$$l(w') = l(s_kw') + 1 = l(s_iw') + 1 = l(s_js_iw') + 2.$$

This is a contradiction since $w' < w$. □

We can now prove that the only finite dimensional Kazhdan-Lusztig representation of most Coxeter systems of rank 3 is the trivial one.

Proposition 6.9. *Let $(W,\{s_1,s_2,s_3\})$ be a Coxeter system of rank 3 such that $m(s_1,s_2) \geq 4$, $m(s_1,s_3) \geq 4$ and $m(s_2,s_3) \geq 3$. Let $\mathcal{C}$ be a finite Kazhdan-Lusztig (left, right or two-sided) cell of W. Then $\mathcal{C} = \{e\}$.*

Proof. It is enough to prove the statement for a left cell $\mathcal{C}$. In fact, if $\mathcal{C}$ is a right cell the statement follows by duality; if $\mathcal{C}$ is a two-sided cell, then it is union of left cells. Let w be a maximal element of a finite left cell. Note that w has at most two left descents since W, being infinite, does not have elements admitting 3 left descents.

Suppose that w has exactly two left descents, say s_i and s_j. We have an arrow pointing downward from s_kw to w in the graph Γ_W^S, where S is the set of left multiplications by s_1, s_2 and s_3. Since w is a maximal element of a cell there are no arrows pointing upward from w to s_kw. This implies that s_i and s_j are left descents of s_kw. But this is a contradiction since the element s_kw would have 3 left descents.

Now suppose that w has exactly one left descent, say s_i, and let j be such that $m(s_i,s_j) \geq 4$. As above, since w is maximal in its cell, we have $s_is_jw < s_jw$ and $s_is_kw < s_kw$ (see

Figure 10). Then, the conditions $m(s_i, s_k) \geq 3$ and $m(s_i, s_j) \geq 4$ imply that $s_k s_i w < s_i w$, $s_j s_i w < s_i w$, and $s_i s_j s_i w < s_j s_i w$ (see Figure 10). Therefore the element $w' = s_i w$ satisfies

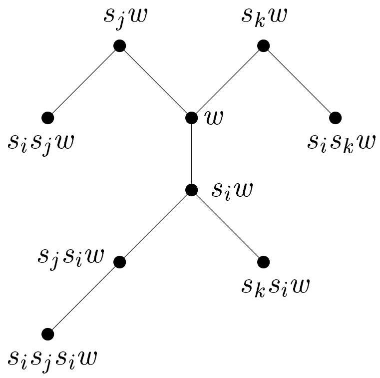

Figure 10: Proof of Proposition 6.9

$$l(w') = l(s_j w') + 1 = l(s_k w') + 1 = l(s_i s_j w') + 2,$$

which contradics Lemma 6.8.

If w has no left descents then $w = e$ and $\mathcal{C} = \{e\}$. $\square$

Our last result shows that, on the other hand, every Coxeter system of rank 3 has non-trivial finite dimensional diamond representations.

Proposition 6.10. *Let W be a Coxeter group of rank 3. Then there exists a finite dihedral poset D and a set of three special matchings $S \subset SM_D$ such that $W_D^S \cong W$.*

Proof. Let s_1, s_2 and s_3 be the Coxeter generators of W, and $a = m(s_1, s_2)$, $b = m(s_1, s_3)$ and $c = m(s_2, s_3)$. If $a = b = c$ then we can choose

$$M = \underbrace{+ \cdots +}_{2a}$$

$$N = \underbrace{- \cdots -}_{a-1}\underbrace{+ \cdots +}_{a+1}$$

$$O = \underbrace{- \cdots -}_{a-1} + \underbrace{- \cdots -}_{a-1} +.$$

If $a \neq b$ then we construct the sequences $(m_1\, m_2 \ldots)$, $(n_1\, n_2 \ldots)$, $(o_1\, o_2 \ldots)$ of the special matchings M, N and O in the following way. The matching M is given by the sequence whose entries are all $+$ signs. The sequences of N and O start both with $+-$ and then they continue using the following algorithm: if $n_{i-1} = -$ then

$$n_i = \begin{cases} - & \text{if there exists } j \in [i - a + 1, i - 1] \text{ such that } n_j = + \\ + & \text{otherwise.} \end{cases}$$

Similarly, if $o_{i-1} = -$ then

$$o_i = \begin{cases} - & \text{if there exists } j \in [i-b+1, i-1] \text{ such that } o_j = + \\ + & \text{otherwise.} \end{cases}$$

If $n_{i-1} \cdot o_{i-1} = -$ then

$$n_i \cdot o_i = \begin{cases} - & \text{if there exists } j \in [i-c+1, i-1] \text{ such that } n_j \cdot o_j = + \\ + & \text{otherwise.} \end{cases}$$

If $n_{i-1} = o_{i-1} = +$ all the sequences stop. We claim that the algorithm eventually stops. By contradiction, suppose that the algorithm does not stop for some a, b, c. Since to find n_i and o_i we keep track only of the entries with index in $[i - \max(a, b, c), i - 1]$, it is clear that the three sequences are periodic. Moreover, since the entries in position i can also be recovered by the entries with index in $[i + 1, i + \max(a, b, c)]$, the sequences should be periodic from the beginning. This implies that the algorithm stops since the three sequences start with a $+$. By construction we have $o(MN) = a$, $o(MO) = b$ and $o(NO) = c$ and the proof is complete. $\square$

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