LATTICE PATHS, LEXICOGRAPHIC CORRESPONDENCE AND KAZHDAN-LUSZTIG POLYNOMIALS

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Abstract. In this paper we give a new closed formula for the Kazhdan-Lusztig polynomials of finite Coxeter groups and affine Weyl groups. This formula is computationally more efficient than any existing one, and it conjecturally holds for all Coxeter groups.

1. Introduction

In [7] Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials of W. These polynomials are related to the algebraic geometry and topology of Schubert varieties, and also play a crucial role in representation theory (see, e.g., [6, Chapter 7], [1, Chapter 5], and the references cited there).

In [2, 3] closed (i.e., nonrecursive) formulas have been given for these polynomials. Our purpose in this paper is to give a new closed formula for them, which is computationally more efficient than any existing one. This formula holds for finite Coxeter groups and affine Weyl groups, and it is conjectured to hold for any Coxeter group (see [4]). The new formula is based on the notion of lexicographic correspondence between Bruhat paths.

The organization of the paper is as follows. Basic definitions and results are collected in Section 2. In Section 3 the notion of lexicographic correspondence between Bruhat paths is introduced and the main result is stated and proved. Finally, in Section 4, we give some consequences of the main result, we present an algorithm implementing our formula and discuss its computational efficiency.

2. Preliminaries

Let $\mathbf{N} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}$. For $n \in \mathbf{N}$, we set $[n] \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$ and for $n, m \in \mathbf{Z}$, with $n \leq m$, we set $[n, m] \stackrel{\text{def}}{=} \{n, n+1, \dots, m\}$. A *lattice path* of length k is a map $\Gamma : [0, k] \to \mathbf{Z}$ such that $\Gamma(0) = 0$ and $|\Gamma(i) - \Gamma(i-1)| = 1$, for every $i \in [k]$. We

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also write $\ell(\Gamma) = k$. We denote by LP the set of all lattice paths. We set

$$\begin{aligned} Neg(\Gamma) &\stackrel{\text{def}}{=} \{i \in [0,k] : \Gamma(i) < 0\}, \\ Z(\Gamma) &\stackrel{\text{def}}{=} \{i \in [0,k] : \Gamma(i) = 0\}, \\ \Gamma_{\geq 0} &\stackrel{\text{def}}{=} |\{i \in [k] : \Gamma(i) \geq 0\}|, \\ d_+(\Gamma) &\stackrel{\text{def}}{=} |\{i \in [k] : |\Gamma(i) - \Gamma(i-1)| = 1\}|. \end{aligned}$$

Note that $\Gamma(\ell(\Gamma)) = 2d_+(\Gamma) - \ell(\Gamma)$. In particular, $\Gamma(\ell(\Gamma)) \equiv \ell(\Gamma) \pmod 2$.

We refer to [6] for basic notions about Coxeter groups. In particular, given a Coxeter group W, with set of generators S, the set of *reflections* of W is

$$T \stackrel{\text{def}}{=} \{wsw^{-1} : w \in W,\ s \in S\}.$$

Given $w \in W$, the *length* of w, denoted by $\ell(w)$, is the minimal k such that w can be written as a product of k generators. The *Bruhat graph* of W, denoted by $B(W)$, is the directed graph having W as vertex set and such that there is an edge $u \to v$ if and only if $v = ut$, with $t \in T$, and $\ell(u) < \ell(v)$. We label the edge by the reflection t and write

$$u \xrightarrow{t} v.$$

A *Bruhat path* is a (directed) path in the Bruhat graph of W. The *Bruhat order* of W is the partial order which is the transitive closure of $B(W)$: given $u, v \in W$, $u \leq v$ in the Bruhat order if and only if there is Bruhat path from u to v:

$$u = u_k \to u_{k-1} \to \cdots \to u_1 \to u_0 = v.$$

It is known that W, partially ordered by the Bruhat order, is a graded poset, with rank function given by the length. Let $u, v \in W$, with $u < v$. We set $\ell(u,v) \stackrel{\text{def}}{=} \ell(v) - \ell(u)$ and denote by $B(u,v)$ the set of all Bruhat paths from u to v. Given $\Delta \in B(u,v)$, we denote by $\ell(\Delta)$ its length (that is, the number of its edges). Note that $\ell(\Delta) \equiv \ell(u,v) \pmod 2$.

Let $\mathcal{H}$ be the *Hecke algebra* of W. Recall that $\mathcal{H}$ is the free module

$$\mathcal{H} \stackrel{\text{def}}{=} \bigoplus_{w \in W} \mathbf{Z}[q^{1/2}, q^{-1/2}]\, T_w$$

generated by a set of elements $\{T_w\}_{w \in W}$ (called the *natural basis* of $\mathcal{H}$) with the structure of an associative algebra uniquely defined by

$$T_w T_s = \begin{cases} qT_{ws} + (q-1)T_w, & \text{if } ws < w, \\ T_{ws}, & \text{if } ws > w, \end{cases}$$

for all $w \in W$ and $s \in S$. It is known that every element T_w is invertible. Kazhdan and Lusztig [7] defined a fundamental involution ι on $\mathcal{H}$ by $\iota(P)(q) \stackrel{\text{def}}{=} P(q^{-1})$ for every $P(q) \in \mathcal{L}$ and $\iota(T_w) \stackrel{\text{def}}{=} (T_{w^{-1}})^{-1}$ for all $w \in W$, and proved the following.

Theorem 2.1. *For every $v \in W$, there is a unique element $C_v \in \mathcal{H}$, such that $\iota(C_v) = C_v$ and*

$$C_v = q^{-\ell(v)/2} \sum_{u \leq v} P_{u,v}(q) T_u,$$

where $P_{u,v}(q) \in \mathbf{Z}[q]$, with $P_{v,v}(q) = 1$ and $\deg(P_{u,v}(q)) < \ell(u,v)/2$ if $u < v$.

The polynomials $P_{u,v}(q)$, uniquely defined by Theorem 2.1, have become known as the *Kazhdan-Lusztig polynomials* of W. They have applications to the algebraic geometry and topology of Schubert varieties, and play a crucial role in representation theory (see, e.g., [6, Chapter 7], [1, Chapter 5], and the reference cited there).

Reflection orderings play a crucial role in the computation of the Kazhdan-Lusztig polynomials. Following [6, Section 8.2], we call *reflection subgroup* of W any subgroup W' generated by a subset of T. It is known that W' is also a Coxeter group, with set of generators $S' \stackrel{\text{def}}{=} \{t' \in W' \cap T : \ell(t't) > \ell(t') \quad \forall t \in T \setminus \{t'\}\}$. We call W' a *dihedral reflection group* if $|S'| = 2$. Following [5], we say that a total ordering $\prec$ on T is a *reflection ordering* if, for any dihedral reflection subgroup W', with $S' = \{a, b\}$, we have either $a \prec aba \prec \cdots \prec bab \prec b$ or $b \prec bab \prec \cdots \prec aba \prec a$. The existence of reflection orderings has been proved in [5, Section 2]. Note that, by definition, if $\prec$ is a reflection ordering, then $\prec^r$, the ordering defined by $a \prec^r b$ if and only if $b \prec a$, is also a reflection ordering.

From now on, let $\prec$ be a fixed reflection ordering on T. Let $u, v \in W$, with $u < v$, and Δ be a Bruhat path from u to v of length k:

$$\Delta = \left(u = u_k \xrightarrow{t_k} u_{k-1} \xrightarrow{t_{k-1}} \cdots \xrightarrow{t_2} u_1 \xrightarrow{t_1} u_0 = v\right).$$

The set of *descents* of Δ is

$$Des(\Delta) \stackrel{\text{def}}{=} \{i \in [k-1] : t_{i+1} \succ t_i\} \cup \{k\}.$$

The path Δ is said to be *increasing* if $Des(\Delta) = \{k\}$, *decreasing* if $Des(\Delta) = [k]$, and *monotone* if it is either increasing or decreasing.

The following result is due to Cellini [4].

Theorem 2.2. *Let W be a finite Coxeter group or affine Weyl group. If $t_1, t_2, \ldots, t_k \in T$ are such that $t_1 \prec t_2 \prec \cdots \prec t_k$, then*

$$t_1 t_2 \ldots t_k \neq e.$$

Theorem 2.2 has the following consequence.

Corollary 2.3. *Let W be a finite Coxeter group or affine Weyl group and $u, v \in W$, with $u < v$. Let*

$$\Delta = \left(u \xrightarrow{t_k} \bullet \xrightarrow{t_{k-1}} \cdots \xrightarrow{t_2} \bullet \xrightarrow{t_1} v\right) \quad \text{and} \quad \Delta' = \left(u \xrightarrow{t'_h} \bullet \xrightarrow{t'_{h-1}} \cdots \xrightarrow{t'_2} \bullet \xrightarrow{t'_1} v\right)$$

be, respectively, any increasing Bruhat path from u to v and any decreasing Bruhat path from u to v, with $(h, k) \neq (1, 1)$. Then

$$t_k \prec t'_h \quad \text{and} \quad t_1 \succ t'_1.$$

Proof. We prove that $t_1 \succ t'_1$, the proof of the other relation being totally symmetric. Suppose $t_1 \prec t'_1$. Then

$$t_k \prec t_{k-1} \prec \cdots \prec t_2 \prec t_1 \prec t'_1 \prec t'_2 \prec \ldots t'_{h-1} \prec t'_h.$$

But $u t_k t_{k-1} \ldots t_2 t_1 = v = u t'_h t'_{h-1} \ldots t'_2 t'_1$, thus $t_k t_{k-1} \ldots t_2 t_1 t'_1 t'_2 \ldots t'_{h-1} t'_h = e$, which is in contradiction with Theorem 2.2. □

In [4] it is conjectured that Theorem 2.2 holds for every Coxeter group. If this conjecture is true, then all the results of this paper also hold for every Coxeter group.

Closed (i.e., nonrecursive) formulas for the Kazhdan-Lusztig polynomials have been given in [2, 3]. Let

$$L(u,v) \stackrel{\text{def}}{=} \{(\Gamma,\Delta) \in LP \times B(u,v) : \ell(\Gamma) = \ell(\Delta),\ Neg(\Gamma) = Des(\Delta)\}\,. \tag{1}$$

Note that if $(\Gamma,\Delta) \in L(u,v)$, then necessarily $\Gamma(\ell(\Gamma)) < 0$. Then, we have the following (see [3, Theorem 7.3]).

Theorem 2.4. *Let W be a Coxeter group and let $u, v \in W$, with $u < v$. Then,*

$$P_{u,v}(q) = \sum_{(\Gamma,\Delta)\in L(u,v)} (-1)^{\Gamma_{\geq 0}+d_+(\Gamma)} q^{\frac{\ell(u,v)+\Gamma(\ell(\Gamma))}{2}}. \tag{2}$$

3. Main result

Let W be a Coxeter group and $u, v \in W$, with $u < v$. It is well-known (see, e.g., [5]) that the number of paths from u to v of length k which are increasing with respect to a given reflection ordering $\prec$ does not depend on the choice of $\prec$. In particular, this number is the same for the two orderings $\prec$ and $\prec^r$. Thus, for a fixed reflection ordering $\prec$, denoting by $B_k^{\prec}(u,v)$ and $B_k^{\succ}(u,v)$ the sets of Bruhat paths of length k which are, respectively, increasing and decreasing, with respect to $\prec$, we have $|B_k^{\prec}(u,v)| = |B_k^{\succ}(u,v)|$. This allows us to give the following.

Definition 3.1. *Identify every Bruhat path with its sequence of labels and totally order $B_k^{\prec}(u,v)$ and $B_k^{\succ}(u,v)$ lexicographically, according to $\prec$. Let Δ and Δ' be the r-th Bruhat paths, respectively, in $B_k^{\prec}(u,v)$ and in $B_k^{\succ}(u,v)$, with $r \in [|B_k^{\prec}(u,v)|]$. Then Δ' is said to be the* lexicographic correspondent *of Δ and conversely.*

When W is the symmetric group S_n, it is known that the reflections are the transpositions and that a reflection ordering on them is the lexicographic ordering. For example, for $u = 12345$ and $v = 42531 \in S_5$, all the increasing and decreasing Bruhat paths from u to v are listed in Table 1, where, for simplicity, a path is denoted by the sequence of its labels and the transposition (i, j) is denoted by ij. In the table, the matching between lexicographic correspondents is shown.

Length	Increasing paths		Decreasing paths
7	$(12, 13, 14, 23, 34, 35, 45)$	$\longleftrightarrow$	$(45, 35, 34, 25, 24, 15, 12)$
5	$(12, 13, 14, 23, 35)$	$\longleftrightarrow$	$(35, 25, 24, 15, 12)$
	$(12, 14, 24, 35, 45)$	$\longleftrightarrow$	$(45, 34, 25, 15, 12)$
	$(13, 14, 34, 35, 45)$	$\longleftrightarrow$	$(45, 35, 34, 15, 14)$
3	$(13, 14, 15)$	$\longleftrightarrow$	$(35, 15, 14)$
	$(14, 35, 45)$	$\longleftrightarrow$	$(45, 34, 15)$

Table 1. Lexicographic correspondence for $(u, v) = (12345, 42531)$.

Going back to the general set up, let $(\Gamma, \Delta) \in L(u, v)$, with

$$\Delta = \left(u = u_k \xrightarrow{t_k} u_{k-1} \xrightarrow{t_{k-1}} \cdots \xrightarrow{t_2} u_1 \xrightarrow{t_1} u_0 = v \right).$$

Let $\{z_0, z_1, \ldots, z_r\} \stackrel{\text{def}}{=} Z(\Gamma)$, with $0 = z_0 < z_1 < \cdots < z_r$. For every $j \in [r]$, let Δ_j be the subpath of Δ going from u_{z_j} to $u_{z_{j-1}}$. Note that, by the definitions of $L(u, v)$ and $Des(\Delta)$, every subpath Δ_j is either increasing or decreasing. For every $j \in [r]$, let Δ'_j be the lexicographic correspondent of Δ_j and let Δ' be the Bruhat path from u to v obtained from Δ by substituting every Δ_j with Δ'_j. Suppose

$$\Delta' = \left(u = u'_k \xrightarrow{t'_k} u'_{k-1} \xrightarrow{t'_{k-1}} \cdots \xrightarrow{t'_2} u'_1 \xrightarrow{t'_1} u'_0 = v \right).$$

Note that $u'_i = u_i$ if and only if $i \in [z_r, k]$ or $i = z_j$, for $j \in [0, r]$. For every $j \in [r]$, let $\varphi_j(\Gamma) : [0, k] \to \mathbf{Z}$ be the lattice path defined by

$$\varphi_j(\Gamma)(i) \stackrel{\text{def}}{=} \begin{cases} -\Gamma(i), & \text{if } i \in [z_{j-1}, z_j], \\ \Gamma(i), & \text{otherwise} \end{cases}$$

and let $\varphi_j(\Delta)$ be the Bruhat path from u to v obtained from Δ by substituting the subpath Δ_j with Δ'_j. Note that $\varphi_j(\varphi_j(\Gamma)) = \Gamma$ and $\varphi_j(\varphi_j(\Delta)) = \Delta$ for all $j \in [r]$. An example of pair $(\Gamma, \Delta) \in L(u, v)$, with the action of φ_j, is shown in Figure 1. Note that the pair $(\varphi_j(\Gamma), \varphi_j(\Delta))$ is not necessarily in $L(u, v)$. One of our first concerns will be to give precise conditions for this to happen.

As the picture suggests, it is useful to identify the element u_i with the point $(i, \Gamma(i))$ and the edge $u_i \to u_{i-1}$ with the segment from $(i-1, \Gamma(i-1))$ to $(i, \Gamma(i))$, for $i \in [\ell(\Delta)]$, so that the same pictorial object represents simultaneously Γ and Δ. We now associate with every z_j a set of four reflections. Let $j \in [r]$. Set

$$(a, b) \stackrel{\text{def}}{=} \begin{cases} (t'_{z_j+1}, t_{z_j+1}), & \text{if } \Gamma(z_j + 1) > 0, \\ (t_{z_j+1}, t'_{z_j+1}), & \text{if } \Gamma(z_j + 1) < 0 \end{cases} \tag{3}$$

and

$$(c, d) \stackrel{\text{def}}{=} \begin{cases} (t_{z_j}, t'_{z_j}), & \text{if } \Gamma(z_j - 1) > 0, \\ (t'_{z_j}, t_{z_j}), & \text{if } \Gamma(z_j - 1) < 0. \end{cases} \tag{4}$$

With the identification $u_i \leftrightarrow (i, \Gamma(i))$ just described, the four reflections a, b, c and d label in every case the same edges, as illustrated in Figure 2.

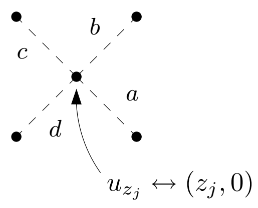

FIGURE 2. The reflections a, b, c and d associated with a zero.

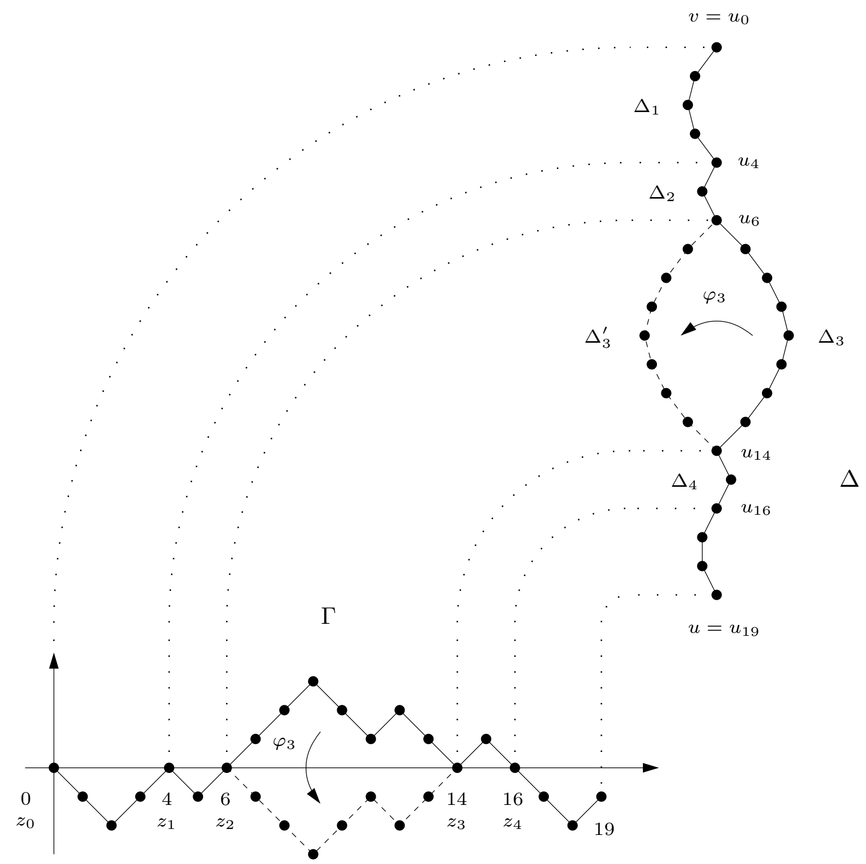

FIGURE 1. A pair $(\Gamma, \Delta) \in L(u, v)$ and the action of φ_j.

Let $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4$ be the only permutation of $\{a, b, c, d\}$ (written in one-line notation) such that $\sigma_1 \prec \sigma_2 \prec \sigma_3 \prec \sigma_4$. Note that, by Corollary 2.3, in every case $a \prec b$ and $c \prec d$. We define the *type* of z_j by

$$\Lambda(z_j) \overset{\text{def}}{=} \begin{cases} V_\sigma, & \text{if } \Gamma(z_j - 1) > 0 < \Gamma(z_j + 1), \\ D_\sigma, & \text{if } \Gamma(z_j - 1) > 0 > \Gamma(z_j + 1), \\ U_\sigma, & \text{if } \Gamma(z_j - 1) < 0 < \Gamma(z_j + 1), \\ P_\sigma, & \text{if } \Gamma(z_j - 1) < 0 > \Gamma(z_j + 1). \end{cases} \tag{5}$$

The four different types are illustrated in Figure 3.

For each type, since $z_j \notin Des(\Delta)$, only certain permutations σ can occur:

- If $\Lambda(z_j) = V_\sigma$, then $b \prec c \quad \Rightarrow \quad \sigma \in \mathcal{V} \overset{\text{def}}{=} \{abcd\}$.
- If $\Lambda(z_j) = D_\sigma$, then $a \prec c \quad \Rightarrow \quad \sigma \in \mathcal{D} \overset{\text{def}}{=} \{abcd, acbd, acdb\}$.
- If $\Lambda(z_j) = U_\sigma$, then $b \prec d \quad \Rightarrow \quad \sigma \in \mathcal{U} \overset{\text{def}}{=} \{abcd, acbd, cabd\}$.
- If $\Lambda(z_j) = P_\sigma$, then $a \prec d \quad \Rightarrow \quad \sigma \in \mathcal{P} \overset{\text{def}}{=} \{abcd, acbd, acdb, cabd, cadb\}$.

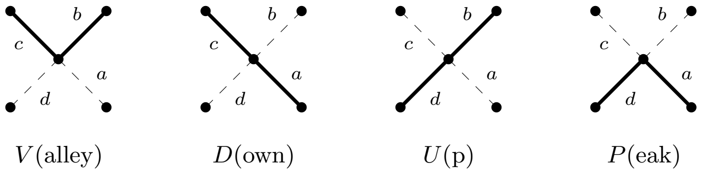

Figure 3. The four types of zeroes.

Note that $\mathcal{V} \subset \mathcal{D} \subset \mathcal{P}$ and $\mathcal{V} \subset \mathcal{U} \subset \mathcal{P}$. For $z_0 = 0$, we set

$$\Lambda(z_0) \stackrel{\text{def}}{=} \begin{cases} V_{abcd}, & \text{if } \Gamma(1) > 0, \\ D_{abcd}, & \text{if } \Gamma(1) < 0, \end{cases} \tag{6}$$

(as if $\Gamma(-1) = 1$). For simplicity, we set $T_{\mathcal{S}} \stackrel{\text{def}}{=} \{T_\sigma : \sigma \in \mathcal{S}\}$, for $T \in \{V, D, U, P\}$ and $\mathcal{S} \in \{\mathcal{V}, \mathcal{D}, \mathcal{U}, \mathcal{P}\}$. So that

$$\Lambda(z_j) \in V_{\mathcal{V}} \cup D_{\mathcal{D}} \cup U_{\mathcal{U}} \cup P_{\mathcal{P}}.$$

We can now describe precisely for which $j \in [r]$ we have $(\varphi_j(\Gamma), \varphi_j(\Delta)) \in L(u, v)$.

Proposition 3.2. *Let $j \in [r]$ be such that Δ_j is increasing. Then*

$$(\varphi_j(\Gamma), \varphi_j(\Delta)) \in L(u, v).$$

Proof. We have to show that $Neg(\varphi_j(\Gamma)) = Des(\varphi_j(\Delta))$. Since $(\Gamma, \Delta) \in L(u, v)$, for $i \in [\ell(\Gamma)] \setminus [z_{j-1}, z_j]$, we have $i \in Neg(\varphi_j(\Gamma))$ if and only if $i \in Des(\varphi_j(\Delta))$. For $i \in [z_{j-1} + 1, z_j - 1]$, we have $\varphi_j(\Gamma)(i) < 0$ and, since Δ'_j is decreasing, $i \in Des(\varphi_j(\Delta))$. It remains to check that $z_{j-1}, z_j \notin Des(\varphi_j(\Delta))$. Since Δ_j is increasing, we have

$$\Lambda(z_{j-1}) \in V_{\mathcal{V}} \cup U_{\mathcal{U}} \quad \text{and} \quad \Lambda(z_j) \in V_{\mathcal{V}} \cup D_{\mathcal{D}}.$$

Let a, b, c and d be the reflections associated with z_j, as defined in (3) and (4). If $\Lambda(z_j) = V_{abcd}$, then $b \prec c \prec d$. And if $\Lambda(z_j) \in D_{\mathcal{D}}$, then $a \prec c \prec d$. In both cases, it follows that $z_j \notin Des(\varphi_j(\Delta))$. The reasoning for z_{j-1} is similar. □

Proposition 3.3. *Let $j \in [r]$ be such that Δ_j is decreasing. Then*

$$(\varphi_j(\Gamma), \varphi_j(\Delta)) \in L(u, v) \quad \Leftrightarrow \quad \Lambda(z_{j-1}) \in D_{\mathcal{V}} \cup P_{\mathcal{U}} \quad \textit{and} \quad \Lambda(z_j) \in U_{\mathcal{V}} \cup P_{\mathcal{D}}.$$

Proof. Reasoning as in the proof of Proposition 3.2, we only have to check that $z_{j-1}, z_j \notin Des(\varphi_j(\Delta))$. Since Δ_j is decreasing, we have

$$\Lambda(z_{j-1}) \in D_{\mathcal{D}} \cup P_{\mathcal{P}} \quad \text{and} \quad \Lambda(z_j) \in U_{\mathcal{U}} \cup P_{\mathcal{P}}.$$

Let a, b, c and d be the reflections associated with z_j. If $\Lambda(z_j) \in U_\sigma$, then

$$z_j \notin Des(\varphi_j(\Delta)) \quad \Leftrightarrow \quad b \prec c \quad \Leftrightarrow \quad \sigma \in \mathcal{V}.$$

And if $\Lambda(z_j) = P_\sigma$, then

$$z_j \notin Des(\varphi_j(\Delta)) \quad \Leftrightarrow \quad a \prec c \quad \Leftrightarrow \quad \sigma \in \mathcal{D}.$$

The reasoning for z_{j-1} is similar. □

We now define the *type* of (Γ, Δ) by

$$\Lambda(\Gamma, \Delta) \stackrel{\text{def}}{=} \Lambda(z_0)\Lambda(z_1)\dots\Lambda(z_r). \tag{7}$$

Note that the set

$$\mathcal{L}(u, v) \stackrel{\text{def}}{=} \{\Lambda(\Gamma, \Delta) : (\Gamma, \Delta) \in L(u, v)\}$$

is a regular language on the alphabet

$$\{V_{\mathcal{V}}, D_{\mathcal{V}}, D_{acbd}, D_{acdb}, U_{\mathcal{V}}, U_{acbd}, U_{cabd}, P_{\mathcal{V}}, P_{acbd}, P_{acdb}, P_{cabd}, P_{cadb}\} \tag{8}$$

(where the singleton $T_{\mathcal{V}} = \{T_{abcd}\}$ denotes the element T_{abcd}). In fact, the language $\mathcal{L}(u, v)$ is generated by the finite automaton $\mathcal{A}$ schematically shown in Figure 4, where, for simplicity, each directed edge represents the complete directed bipartite graph from all the vertices in its tail to all those in its head. The words in $\mathcal{L}(u, v)$ are then in bijection with the paths from *start* to *end* in the automaton.

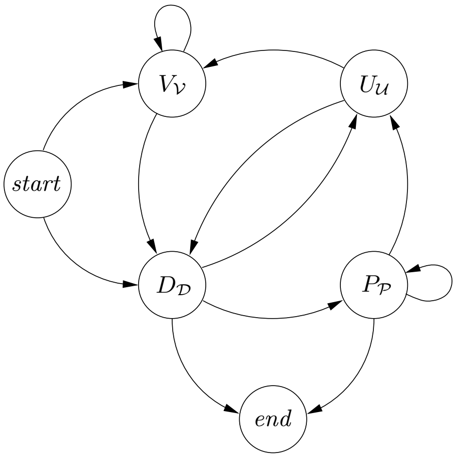

FIGURE 4. The automaton $\mathcal{A}$ generating the language $\mathcal{L}(u, v)$.

Labelling the vertices *start* and *end* respectively by 1 and 14 and the other vertices by the numbers from 2 to 13 in the same order as they are written in (8), the adjacency matrix of the automaton $\mathcal{A}$ is then

$$A = \begin{pmatrix}
0 & \boxed{1} & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & \boxed{1} & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \boxed{1} & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{9}$$

(where the meaning of the boxes will be made clear in the next section).

We now define

$$\mathcal{F} \stackrel{\text{def}}{=} (V_{\mathcal{V}} \cup U_{cabd})(V_{\mathcal{V}} \cup D_{\mathcal{D}}) \cup (D_{\mathcal{V}} \cup P_{cabd})(U_{\mathcal{V}} \cup P_{\mathcal{D}}), \tag{10}$$

(where T_{cabd} denotes the singleton $\{T_{cabd}\}$ and $AB \stackrel{\text{def}}{=} \{ab : a \in A, b \in B\}$) and

$$M(u,v) \stackrel{\text{def}}{=} \{(\Gamma,\Delta) \in L(u,v) : \Lambda(\Gamma,\Delta) \text{ does not contain any 2-factor in } \mathcal{F}\}. \tag{11}$$

The following remark will be used often in the rest of the paper and it follows from the definitions of $M(u,v)$ and $\mathcal{F}$, and from the fact that in $\Lambda(\Gamma,\Delta)$ a V_{abcd} or a U_{cabd} are necesssarily followed by an element in $V_{\mathcal{V}} \cup D_{\mathcal{D}}$.

Remark 3.4. *Let $(\Gamma,\Delta) \in M(u,v)$. Then $\Lambda(\Gamma,\Delta)$ avoids V_{abcd} and U_{cabd}.*

We can now state and prove the main result of this work.

Theorem 3.5. *Let W be a finite Coxeter group or affine Weyl group and $u,v \in W$, with $u < v$. Then*

$$P_{u,v}(q) = \sum_{(\Gamma,\Delta)\in M(u,v)} (-1)^{\Gamma_{\geq 0} + d_+(\Gamma)}\, q^{\frac{\ell(u,v)+\Gamma(\ell(\Gamma))}{2}}. \tag{12}$$

Proof. Fix $\ell \in [\ell(u,v)]$, with $\ell \equiv \ell(u,v) \pmod 2$, $\gamma \in [\ell]$, with $\gamma \equiv \ell \pmod 2$ and $Z \subseteq [0,\ell] \cap 2\mathbf{Z}$, with $0 \in Z$. We set

$$\widetilde{L}_{\ell,\gamma,Z}(u,v) \stackrel{\text{def}}{=} \{(\Gamma,\Delta) \in L(u,v) \setminus M(u,v) : \ell(\Gamma) = \ell,\ \Gamma(\ell) = -\gamma,\ Z(\Gamma) = Z\}.$$

We will prove that

$$\sum_{(\Gamma,\Delta)\in \widetilde{L}_{\ell,\gamma,Z}(u,v)} (-1)^{\Gamma_{\geq 0}} = 0. \tag{13}$$

This implies our result, since, by Theorem 2.4, we have

$$\begin{aligned} P_{u,v}(q) = & \sum_{(\Gamma,\Delta)\in M(u,v)} (-1)^{\Gamma_{\geq 0} + d_+(\Gamma)}\, q^{\frac{\ell(u,v)+\Gamma(\ell(\Gamma))}{2}} \\ & + \sum_{(\ell,\gamma,Z)} (-1)^{\frac{\ell-\gamma}{2}} q^{\frac{\ell(u,v)-\gamma}{2}} \sum_{(\Gamma,\Delta)\in \widetilde{L}_{\ell,\gamma,Z}(u,v)} (-1)^{\Gamma_{\geq 0}}. \end{aligned}$$

Let $(\Gamma,\Delta) \in \widetilde{L}_{\ell,\gamma,Z}(u,v)$ and let $\Lambda_0\Lambda_1 \ldots \Lambda_r \stackrel{\text{def}}{=} \Lambda(\Gamma,\Delta)$. We define

$$p \stackrel{\text{def}}{=} \min\{j \in [r] :\ \Lambda_{j-1}\Lambda_j \in \mathcal{F}\}.$$

and

$$\Phi(\Gamma,\Delta) \stackrel{\text{def}}{=} (\varphi_p(\Gamma), \varphi_p(\Delta)).$$

We prove equation (13) by showing that Φ is an involution of $\widetilde{L}_{\ell,\gamma,Z}(u,v)$ and that it is sign-reversing. By the definition of $\varphi_j(\Gamma)$, we have

$$(-1)^{(\varphi_p(\Gamma))_{\geq 0}} = -(-1)^{\Gamma_{\geq 0}}.$$

So we only need to show that

$$\Phi(\Gamma,\Delta) \in \widetilde{L}_{\ell,\gamma,Z}(u,v) \quad \text{and} \quad \Phi(\Phi(\Gamma,\Delta)) = (\Gamma,\Delta).$$

First of all, we have $\Phi(\Gamma,\Delta) \in L(u,v)$. If Δ_p is increasing, this follows directly from Proposition 3.2. If Δ_p is decreasing, this is a consequence of Proposition 3.3, since $\Gamma_{p-1}\Gamma_p \in \mathcal{F}$ implies $\Lambda(z_{p-1}) \in D_{\mathcal{V}} \cup P_{cabd}$ and $\Lambda(z_p) \in U_{\mathcal{V}} \cup P_{\mathcal{D}}$. And, of course, $\ell(\varphi_p(\Gamma)) = \ell$, $\varphi_p(\Gamma)(\ell) = -\gamma$ and $Z(\varphi_p(\Gamma)) = Z$.

Now, let $\Lambda'_0\Lambda'_1 \ldots \Lambda'_r \stackrel{\text{def}}{=} \Lambda(\Phi(\Gamma, \Delta))$. Note that $\Lambda'_j = \Lambda_j$, for every $j \in [p-2]$. So, by the definition of p, $\Lambda'_{j-1}\Lambda'_j \notin \mathcal{F}$, for $j \in [p-2]$. We will prove that $\Lambda'_{p-1}\Lambda'_p \in \mathcal{F}$ and that, if $p \geq 2$, then $\Lambda'_{p-2}\Lambda'_{p-1} \notin \mathcal{F}$. This implies that $\Lambda'_{p-1}\Lambda'_p$ is the leftmost 2-factor of $\Lambda(\Phi(\Gamma, \Delta))$ in $\mathcal{F}$ and that

$$\Phi(\Phi(\Gamma, \Delta)) = (\varphi_p(\varphi_p(\Gamma)), \varphi_p(\varphi_p(\Delta))) = (\Gamma, \Delta).$$

There are four cases to consider.

1. $\Lambda_{p-1} = V_{abcd}$ (then, necessarily, $\Lambda_p \in V_{\mathcal{V}} \cup D_{\mathcal{D}}$).

Then $\Lambda'_{p-1}\Lambda'_p \in D_{\mathcal{V}}(U_{\mathcal{V}} \cup P_{\mathcal{D}}) \subset \mathcal{F}$. Assume that $\Lambda'_{p-2}\Lambda'_{p-1} \in \mathcal{F}$. Then $\Lambda_{p-2} = \Lambda'_{p-2} \in \{V_{abcd}, U_{cabd}\}$, so $\Lambda_{p-2}\Lambda_{p-1} \in \{V_{abcd}V_{abcd}, U_{cabd}V_{abcd}\} \subset \mathcal{F}$, which contradicts the definition of p.

2. $\Lambda_{p-1}\Lambda_p \in D_{\mathcal{V}}(U_{\mathcal{V}} \cup P_{\mathcal{D}})$.

Then $\Lambda'_{p-1}\Lambda'_p \in V_{\mathcal{V}}(V_{\mathcal{V}} \cup D_{\mathcal{D}}) \subset \mathcal{F}$. Assume that $\Lambda'_{p-2}\Lambda'_{p-1} \in \mathcal{F}$. Then $\Lambda_{p-2} = \Lambda'_{p-2} \in \{V_{abcd}, U_{cabd}\}$, so $\Lambda_{p-2}\Lambda_{p-1} \in \{V_{abcd}D_{abcd}, U_{cabd}D_{abcd}\} \subset \mathcal{F}$, which contradicts the definition of p.

3. $\Lambda_{p-1} = U_{cabd}$ (then, necessarily, $\Lambda_p \in V_{\mathcal{V}} \cup D_{\mathcal{D}}$).

Then $\Lambda'_{p-1}\Lambda'_p \in \{P_{cabd}\}(U_{\mathcal{V}} \cup P_{\mathcal{D}}) \subset \mathcal{F}$. Assume that $\Lambda'_{p-2}\Lambda'_{p-1} \in \mathcal{F}$. Then $\Lambda_{p-2}P_{cabd} \in \mathcal{F}$, which is a contradiction, since $cabd \notin \mathcal{D}$.

4. $\Lambda_{p-1}\Lambda_p \in P_{cabd}(U_{\mathcal{V}} \cup P_{\mathcal{D}})$.

Then $\Lambda'_{p-1}\Lambda'_p \in \{U_{cabd}\}(V_{\mathcal{V}} \cup D_{\mathcal{D}}) \subset \mathcal{F}$. Assume that $\Lambda'_{p-2}\Lambda'_{p-1} \in \mathcal{F}$. Then $\Lambda_{p-2}U_{cabd} \in \mathcal{F}$, which is a contradiction, since $cabd \notin \mathcal{V}$.

This completes the proof of equation (13) and hence of the theorem. □

4. Consequences

In this section we discuss some features and consequences of the formula given in Theorem 3.5. First of all, we show that the conditions in the set $M(u, v)$, defined in (11), can essentially be reduced to restrictions on the Bruhat paths, with a simple condition on the lattice paths paired with them.

Let W be a Coxeter group and $u, v \in W$, with $u < v$. Let $\Delta \in B(u, v)$:

$$\Delta = \left(u = u_k \xrightarrow{t_k} u_{k-1} \xrightarrow{t_{k-1}} \cdots \xrightarrow{t_2} u_1 \xrightarrow{t_1} u_0 = v\right).$$

The set of *ascents* of Δ is

$$Asc(\Delta) = \{i \in [k-1] : t_{i+1} < t_i\} \cup \{0\}.$$

We also distinguish between *internal* ascents:

$$\overset{0}{Asc}(\Delta) = \{i \in [k-1] : [i-1, i+1] \subseteq Asc(\Delta)\},$$

and *boundary* ascents:

$$\partial Asc(\Delta) = Asc(\Delta) \setminus \overset{0}{Asc}(\Delta).$$

Note that, if there exists a lattice path Γ such that $(\Gamma, \Delta) \in L(u,v)$, then necessarily $\partial Asc(\Delta) \subseteq Z(\Gamma) \subseteq Asc(\Delta)$, and it is possible that $Z(\Gamma) \neq \partial Asc(\Delta)$. So, since $Z(\Gamma) \subset 2\mathbf{Z}$, there exists a lattice path Γ such that $(\Gamma, \Delta) \in L(u,v)$ if and only if $\partial Asc(\Delta) \subset 2\mathbf{Z}$. If this happens we say that the path Δ is *even*. We denote by $B_e(u,v)$ the set of all even paths from u to v.

One of the crucial advantages of $M(u,v)$ over $L(u,v)$ (besides the fact that it is usually much smaller, as we will show later), is that if $(\Gamma, \Delta) \in M(u,v)$ then $Z(\Gamma)$ only depends on Δ, which is not the case if $(\Gamma, \Delta) \in L(u,v)$. In fact we have the following.

Proposition 4.1. *Let* $(\Gamma, \Delta) \in M(u,v)$*. Then* $Z(\Gamma) = \partial Asc(\Delta)$*.*

Proof. For all $z_j \in Z(\Gamma)$, with $j \in [r]$, we have $z_j \in \overset{0}{Asc}(\Delta)$ if and only if $\Lambda(z_j) = V_{abcd}$. However, by Remark 3.4, $\Lambda(\Gamma, \Delta)$ does not contain any V_{abcd}. Also note that, by definition, $z_0 = 0 \in \partial Asc(\Delta)$. Thus, $z_j \in \partial Asc(\Delta)$ for every $j \in [0,r]$. $\square$

Note that the definitions of $\Lambda(z_j)$ and $\Lambda(\Gamma, \Delta)$, given in (5), (6) and (7), only depend on the set $Z = Z(\Gamma)$ and on Δ. Moreover, by Proposition 4.1, given $(\Gamma, \Delta) \in M(u,v)$, the set $Z(\Gamma)$ is only determined by Δ. These facts suggest to define the *type* of Δ by

$$\Lambda(\Delta) \overset{\text{def}}{=} \Lambda(\Gamma, \Delta),$$

where Γ is any lattice path such that $(\Gamma, \Delta) \in L(u,v)$ and $Z(\Gamma) = \partial Asc(\Delta)$. Then, we say that Δ is *suitable* if $\Lambda(\Delta)$ does not contain any 2-factor in the set $\mathcal{F}$, defined in (10). We denote by $B_s(u,v)$ the set of all suitable paths from u to v. Then $M(u,v)$ can be decomposed as follows:

$$M(u,v) = \bigcup_{\Delta \in B_s(u,v)} \left\{ \Gamma \in LP : \begin{array}{rcl} \ell(\Gamma) &=& \ell(\Delta) \\ Neg(\Gamma) &=& Des(\Delta) \\ Z(\Gamma) &=& \partial Asc(\Delta) \end{array} \right\} \times \{\Delta\}. \tag{14}$$

Now, let $\Delta \in B_e(u,v)$:

$$\Delta = \left(u = u_k \xrightarrow{t_k} u_{k-1} \xrightarrow{t_{k-1}} \cdots \xrightarrow{t_2} u_1 \xrightarrow{t_1} u_0 = v \right).$$

Let $\partial Asc(\Delta) = \{z_0, z_1, \ldots, z_r\}$, with $0 = z_0 < z_1 < \cdots < z_r$. For every $j \in [r]$, let Δ_j be the subpath of Δ going from u_{z_j} to $u_{z_{j-1}}$. Then, every subpath Δ_j is either increasing or decreasing. Moreover, there are no two consecutive subpaths Δ_j and Δ_{j+1} which are both increasing. Also, let Δ_{r+1} be the subpath of Δ going from u to u_{z_r}. Since, by definition, $\ell(\Delta) \in Des(\Delta)$, the subpath Δ_{r+1}, that might consist of one single edge, is decreasing. We set

$$Dec(\Delta) \overset{\text{def}}{=} \{j \in [r] : \Delta_j \text{ is decreasing}\}$$

and $dec(\Delta) \overset{\text{def}}{=} |Dec(\Delta)|$. Also, set $h_\Delta \overset{\text{def}}{=} \ell(\Delta_{r+1})$ and $\alpha_j \overset{\text{def}}{=} \ell(\Delta_j)/2$, for $j \in [r]$. Let $C_n \overset{\text{def}}{=} \frac{1}{n+1}\binom{2n}{n}$ be the n-th Catalan number, with $C_0 \overset{\text{def}}{=} 1$. We define the *Catalan constant* of Δ by

$$c_\Delta \overset{\text{def}}{=} \prod_{j=1}^{r} C_{\alpha_j - 1}.$$

Finally, for $h \in \mathbf{P}$, consider the family of polynomials $\{F_h(q)\}_{h\in\mathbf{P}}$, defined by

$$F_h(q) \stackrel{\text{def}}{=} \sum_{d=0}^{\lfloor \frac{h-1}{2} \rfloor} \frac{h-2d}{h}\binom{h}{d}(-q)^d.$$

Then, we have the following.

Corollary 4.2. *Let W be a finite Coxeter group or affine Weyl group and $u, v \in W$, with $u < v$. Then*

$$P_{u,v}(q) = \sum_{\Delta \in B_s(u,v)} (-1)^{\frac{\ell(\Delta)-h_\Delta}{2}+dec(\Delta)} c_\Delta \, q^{\frac{\ell(u,v)-h_\Delta}{2}} F_{h_\Delta}(q). \tag{15}$$

Proof. It follows from equation (12), by considering the decomposition of $M(u,v)$ given in (14), and counting all possible Γ that can be paired with a given Δ. In fact, recall that the number of lattice paths Γ from $(0,0)$ to $(2n,0)$ with $\Gamma(i) \geq 0$ for all $i \in [0,2n]$ (*Dyck paths*) is C_n (see, e.g., [9, Corollary 6.2.3]) and the number of those such that $\Gamma(i) > 0$ for all $i \in [1,2n-1]$ (*lifted Dyck paths*) is C_{n-1}. Also, the number of lattice paths Γ from $(0,0)$ to $(h,-h+2d)$ with $\Gamma(i) < 0$ for all $i \in [h]$ (*skew Dyck paths*) is $\frac{h-2d}{h}\binom{h}{d}$ (see, e.g. [9, Exercise 6.20]). Now, if $(\Gamma,\Delta) \in M(u,v)$, then, by Remark 3.4, $\Lambda(\Delta)$ does not contain any V_{abcd}. Then, to every Δ_j in Δ corresponds in Γ a lifted Dyck path, and to the path Δ_{r+1} corresponds a skew Dyck path. Finally, since

$$\Gamma_{\geq 0} = \ell(\Gamma) - |Neg(\Gamma)| = \ell(\Delta) - |Des(\Delta)| = \ell(\Delta) - \sum_{j \in Dec(\Delta)} (2\alpha_j - 1),$$

we have $(-1)^{\Gamma_{\geq 0}} = (-1)^{\ell(\Delta)+dec(\Delta)}$ and the result follows. □

From a computational point of view, formula (12) provides a more efficient way of computing the Kazhdan-Lusztig polynomials, compared to the other known closed formulas for them (namely, Theorem 2.4 and [2, Theorem 4.1]). In fact, [2, Theorem 4.1] involves a summation over all multichains from u to v, whereas Theorem 2.4 involves a summation over Bruhat paths (hence, a special kind of chains). To get an idea of how many of the summands in (2) "survive" in (12), we start by considering the set of all possible types of pairs (Γ,Δ) in $M(u,v)$:

$$\mathcal{M}(u,v) \stackrel{\text{def}}{=} \{\Lambda(\Gamma,\Delta) : (\Gamma,\Delta) \in M(u,v)\}.$$

The set $\mathcal{M}(u,v)$, subset of $\mathcal{L}(u,v)$, is still a regular language, generated by the finite automaton $\mathcal{B}$ schematically shown in Figure 5. The adjacency matrix B of $\mathcal{B}$ is obtained from A, given in (9), by changing the entries in boxes to zeroes.

We denote by $\mathcal{L}_n(u,v)$ and $\mathcal{M}_n(u,v)$, for $n \in \mathbf{N}$, the sets of words of length n respectively in $\mathcal{L}(u,v)$ and $\mathcal{M}(u,v)$, assuming the two languages in the common alphabet (8). The set $\mathcal{L}_n(u,v)$ is in bijection with the set of paths of length $n+1$ from *start* to *end* in $\mathcal{A}$. Therefore (see, e.g., [8, Section 4.7]), the number of such paths is the $(1,14)$-entry in A^{n+1}. A similar reasoning holds for $\mathcal{M}_n(u,v)$, so

$$|\mathcal{L}_n(u,v)| = (A^{n+1})_{1,14} \quad \text{and} \quad |\mathcal{M}_n(u,v)| = (B^{n+1})_{1,14}\,.$$

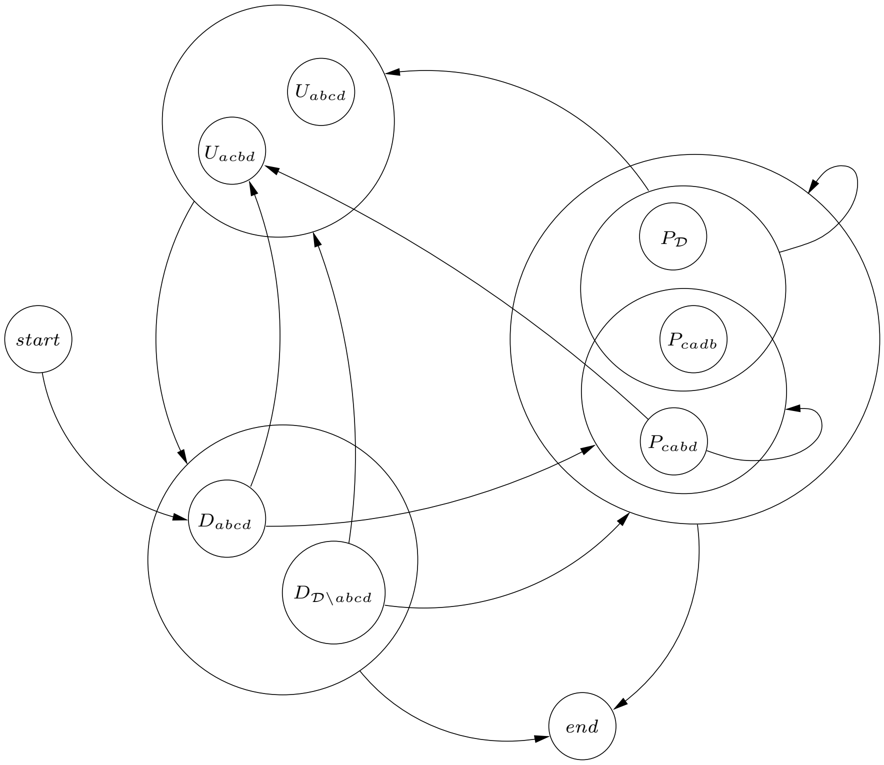

FIGURE 5. The automaton $\mathcal{B}$ generating the language $\mathcal{M}(u,v)$.

Passing through the diagonal forms of A and A^*, we obtain

$$\begin{aligned} |\mathcal{L}_n(u,v)| &= \frac{3}{104}\left(13-3\sqrt{13}\right)\left(3+\sqrt{13}\right)^{n+1}+\frac{3}{104}\left(13+3\sqrt{13}\right)\left(3-\sqrt{13}\right)^{n+1}, \\ |\mathcal{M}_n(u,v)| &= \frac{3}{440}\left(15-\sqrt{5}\right)\left(3+\sqrt{5}\right)^{n+1}+\frac{3}{440}\left(15+\sqrt{5}\right)\left(3-\sqrt{5}\right)^{n+1}+\frac{6}{11}(-1)^{n+1}. \end{aligned}$$

In particular,

$$\lim_{n\to\infty} |\mathcal{M}_n(u,v)|/|\mathcal{L}_n(u,v)| = 0,$$

and the rate of convergence is exponential. This is strong indication that the set $B_s(u,v)$ itself is exponentially smaller than $B(u,v)$, as $\ell(u,v) \to \infty$, and we conjecture that this is the case. This is also supported by computational experiments. For example, if $u = 12354$ and $v = 45321 \in S_5$, with $\ell(u,v) = 8$, then $|B_s(u,v)| = 162$, whereas $|B(u,v)| = 7146$, and

$$\max\{|B_s(u,v)|/|B(u,v)| : u,v \in S_5,\ \ell(u,v) = 7\} = 59/963 \simeq 0.061.$$

Finally, the computational advantage of formula (15), is guaranteed by the fact that there is an efficient algorithm for generating all the suitable paths between two elements u and v, without first generating all the Bruhat paths from u to v. The last part of this work is devoted to the presentation of this algorithm.

The main function is described (with a metaprogramming language) in Table 2. It calls a recursive function, described in Tables 3 and 4. The algorithm requires a preliminary call to the function *generateMonotonePaths* (u, v), which returns the array *numMonotonePaths*, whose $[z, w, \ell]$-entry is the number of monotone Bruhat paths from z to w of length ℓ, for every $z, w \in [u, v]$, with $z < w$, either $z = u$ or $\ell(z, v)$ even, and $\ell(w, v)$ even, and for every $\ell \in [\ell(z, w)]$, with $\ell \equiv \ell(z, w) \pmod 2$, and the arrays *increasingPaths* and *decreasingPaths*, whose $[z, w, \ell]$-entries are the arrays of the Bruhat paths from z to w of length ℓ, which are, respectively, increasing and decreasing, both ordered by the lexicographic order. Thus, for k from 1 to *numMonotonePaths* $[z, w, \ell]$, the paths *increasingPaths* $[z, w, \ell]$ $[k]$ and *decreasingPaths* $[z, w, \ell]$ $[k]$ are lexicographic correspondents of each other.

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generateMonotonePaths $(u, v) \to$ (*numMonotonePaths* $[z, w, \ell]$,
increasingPaths $[z, w, \ell]$ $[k]$,
decreasingPaths $[z, w, \ell]$ $[k]$)

function *generateSuitablePaths* (u, v)
  answer := $\emptyset$
  for i **from** 2 **to** $\ell(u, v) - 1$ **step** 2 **do**
    for $z \in [u, v]$ **with** $\ell(v, z) = i$ **do**
      for ℓ **from** i **to** 1 **step** -2 **do**
        for k **from** 1 **to** *numMonotonePaths* $[z, v, \ell]$ **do**
          incPath := *increasingPaths* $[z, v, \ell]$ $[k]$
          decPath := *decreasingPaths* $[z, v, \ell]$ $[k]$
          firstType := D_{abcd}
          Δ := *decPath*
          c := label on the edge from z in *incPath*
          d := label on the edge from z in *decPath*
          recursiveSuitablePaths (u, z, $firstType$, c, d) $\to$ *recPaths*
          for *recPath* $\in$ *recPaths* **do**
            answer := *answer* **union** (*recPath*, Δ)
          end for (*recPath*)
        end for (k)
      end for (ℓ)
    end for (z)
  end for (i)
  for ℓ **from** $\ell(u, v)$ **to** 1 **step** -2 **do**
    for k **from** 1 **to** *numMonotonePaths* $[u, v, \ell]$ **do**
      Δ := *decreasingPaths* $[u, v, \ell]$ $[k]$
      answer := *answer* **union** Δ
    end for (k)
  end for (ℓ)
  return *answer*
end function

TABLE 2. Generating suitable paths: main function.

```
function recursiveSuitablePaths (u, v, precType, c, d)
  answer := ∅
  for i from 2 to ℓ(u,v) step 2 do
    for z ∈ [u,v] with ℓ(v,z) = i do
      for ℓ from i to 1 step −2 do
        for k from 1 to numMonotonePaths [z,v,ℓ] do
          incPath := increasingPaths [z,v,ℓ] [k]
          decPath := decreasingPaths [z,v,ℓ] [k]
          a := label on the edge towards v in decPath
          b := label on the edge towards v in incPath
          σ := reordering permutation of {a,b,c,d}
          if precType ∈ {U_𝒰} then
            newTypes := {D_σ}
          else
            newTypes := {U_σ, P_σ}
          end if
          for newType ∈ newTypes do
            if (newType = 𝒟_σ and not a ≺ c) or
              (newType = 𝒰_σ and not b ≺ d) or
              (newType = 𝒫_σ and not a ≺ d) or
              (precType, newType) ∈ ℱ
            then break
            if newType = 𝒰_σ then
              Δ := incPath
            else
              Δ := decPath
            end if
            newc := label on the edge from z in incPath
            newd := label on the edge from z in decPath
            recursiveSuitablePaths (u, z, newType, newc, newd) → recPaths
            for recPath ∈ recPaths do
              answer := answer union (recPath, Δ)
            end for (recPath)
          end for (newType)
        end for (k)
      end for (ℓ)
    end for (z)
  end for (i)
  ...
```

Table 3. Generating suitable paths: recursive function - Part 1.

```
...
for ℓ from ℓ(u,v) to 1 step −2 do
  for k from 1 to numMonotonePaths [u,v,ℓ] do
    incPath := increasingPaths [z,v,ℓ] [k]
    decPath := decreasingPaths [z,v,ℓ] [k]
    a := label on the edge towards v in decPath
    b := label on the edge towards v in incPath
    σ := reordering permutation of {a,b,c,d}
    if precType ∈ {U_𝒰} then
      newType := D_σ
    else
      newType := P_σ
    end if
    if (newType = 𝒟_σ and not a ≺ c) or
      (newType = 𝒫_σ and not a ≺ d) or
      (precType, newType) ∈ ℱ
    then break
    Δ := decPath
    answer := answer union Δ
  end for (k)
end for (ℓ)
return answer
end function
```

TABLE 4. Generating suitable paths: recursive function - Part 2.

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