Special matchings and permutations in Bruhat orders

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Abstract

For any permutation v, we show that the special matchings of v generate a Coxeter system. This gives new necessary conditions on an abstract poset to be isomorphic to a lower Bruhat interval of the symmetric group, and also answers in the affirmative, in the symmetric group case, a problem posed in [3].

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1 Introduction

In their celebrated paper [8], Kazhdan and Lusztig introduced a class of polynomials with integer coefficients, indexed by pairs of elements of any Coxeter group W, which have become known as the Kazhdan-Lusztig polynomials of W. These polynomials are of importance in several areas of mathematics, including representation theory and the geometry and topology of Schubert varieties. Recently ([2], [3]) the poset-theoretic concept of a special matching has been shown to play a fundamental role in the computation of the Kazhdan-Lusztig polynomials. Our purpose in this paper is to study in more detail the special matchings of permutations. Our main result (Theorem 5.1) is that, for any permutation v, the special matchings of v (see §2 for definitions), generate a Coxeter system. This gives new necessary conditions on an abstract poset to be isomorphic to a lower Bruhat interval of the symmetric group, and also answers in the affirmative, in the symmetric group case, a problem posed in [3].

2 Notation and Background

In this section we collect some definitions, notation and results that will be used in the rest of this work. We let $\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, \ldots\}$, and for $a, b \in \mathbb{N}$ we let $[a, b] \stackrel{\text{def}}{=} \{a, a+1, a+2, \ldots, b\}$ and $[a] \stackrel{\text{def}}{=} [1, a]$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). The cardinality of a set A will be denoted by $|A|$.

We will follow [10, Chap. 3] for notation and terminology concerning partially ordered sets. In particular, given a partially ordered set P and $x, y \in P$ we say that x *covers* y if $x \geq y$ and there is no $z \in P \setminus \{x, y\}$ such that $y \leq z \leq x$. We then write $y \lhd x$ (or $x \rhd y$). The *Hasse diagram* of P is the graph $H(P) = (P, E)$ where $E \stackrel{\text{def}}{=} \{\{x, y\} : x, y \in P \text{ and either } x \lhd y \text{ or } y \lhd x\}$. Given any (simple, loopless) graph $G = (V, E)$ a *matching* of G is an involution $M : V \to V$ such that $\{M(v), v\} \in E$ for all $v \in V$.

We will follow [7] for general Coxeter groups notation and terminology. In particular, a *Coxeter system* is a pair (W, S) where W is a group and $S \subset W$ is a set of generators for W subject only to relations of the form

$$(ss')^{m(s,s')} = 1,$$

where $m(s,s) = 1$ (i.e. any element of S is an involution), and $m(s,s') = m(s',s) \geq 2$, for $s \neq s' \in S$. We attribute the value ∞ to $m(s,s')$ if ss' has infinite order. It turns out that $m(s,s')$ is equal to the order of $ss' \in W$. The cardinality of S is usually called the *rank* of W. Given a Coxeter system (W,S) and $w \in W$ we denote by $\ell(w)$ the length of w, and we set $\ell(u,v) \stackrel{\text{def}}{=} \ell(v) - \ell(u)$, for all $u, v \in W$. We let

$$D_R(w) \stackrel{\text{def}}{=} \{s \in S : \ \ell(ws) < \ell(w)\},$$

and

$$D_L(w) \stackrel{\text{def}}{=} \{s \in S : \ \ell(sw) < \ell(w)\} = D_R(w^{-1}).$$

We call $D_R(w)$ and $D_L(w)$ respectively the right and the left *descent set* of w. We denote by e the identity of W, and we let $T \stackrel{\text{def}}{=} \{wsw^{-1} : w \in W, \ s \in S\}$ be the set of reflections of W. A subgroup W' of W is called a *reflection subgroup* if it is generated by a subset of T. The following result is due to Dyer [5].

Theorem 2.1 *Let W' be a reflection subgroup of W. Then there exists a set of reflections $S_{W'} \subset T$ such that $(W', S_{W'})$ is a Coxeter system.*

We denote by $\mathfrak{S}(n)$ the group of all bijections $\pi : [n] \to [n]$ (the symmetric group). As usual, we denote the transpositions in $\mathfrak{S}(n)$ by (i,j), where $1 \leq i < j \leq n$. It is well known that $(\mathfrak{S}(n), S)$, where $S \stackrel{\text{def}}{=} \{s_1 = (1,2), s_2 = (2,3), \ldots, s_{n-1} = (n-1,n)\}$, is a Coxeter system of rank $n-1$. We abuse notation by referring to this Coxeter system simply as $\mathfrak{S}(n)$. When no confusion arises, we identify S with the interval $[n-1]$, and hence we write i instead of s_i.

We will always assume that a Coxeter group W is partially ordered by (strong) *Bruhat order*. Recall (see, e.g., [7, §5.9]) that this means that $u \leq v$ if and only if there exist $r \in \mathbb{N}$ and $t_1, \ldots, t_r \in T$ such that $t_r \cdots t_1 u = v$ and $\ell(t_i \cdots t_1 u) > \ell(t_{i-1} \cdots t_1 u)$ for $i = 1, \ldots, r$. There is a well known characterization of Bruhat order on a Coxeter group (usually referred to as the *subword property*) that we will use repeatedly in this work, often without explicit mention. We recall it here for the reader's convenience. By a *subword* of a word $s_1 s_2 \cdots s_q$ we mean a word of the form $s_{i_1} s_{i_2} \cdots s_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq q$.

Theorem 2.2 *Let $u, v \in W$. The following are equivalent:*

1. *$u \leq v$;*

2. *there is a reduced expression for v that has a subword which is a reduced expression for u;*

3. *every reduced expression for v has a subword that is a reduced expression for u.*

A proof of the preceding result can be found, e.g., in [7, §5.10].

We now recall some results due to Tits [11]. Given $s, t \in S$ such that $m(s,t) < \infty$, let $\alpha_{s,t} = \underbrace{\ldots stst}_{m(s,t)}$, with exactly $m(s,t)$ letters.

Lemma 2.3 *Let $w \in W$ and $s, t \in D_L(w)$. Then $m(s,t) < \infty$ and there exists a reduced expression of w which starts with $\alpha_{s,t}$, that is*

$$v = \alpha_{s,t}\, v',$$

with $\ell(v) = m(s,t) + \ell(v')$.
Dually, if $s, t \in D_R(w)$, then there exists a reduced expression of w which ends with $\alpha_{s,t}$.

Two expressions are said to be linked by a braid move (respectively a nil move) if it is possible to obtain the first from the second by changing a factor $\alpha_{s,t}$ to a factor $\alpha_{t,s}$ (respectively by deleting a factor ss).

Theorem 2.4 (Tits' Word Theorem) *Let $u \in W$. Then:*

1. *any two reduced expressions of u are linked by a finite sequence of braid moves;*

2. *any expression of u (not necessarily reduced) is linked to any reduced expression of u by a finite sequence of braid and nil moves.*

Given $u, v \in W$ we let $[u, v] \stackrel{\text{def}}{=} \{x \in W : u \leq x \leq v\}$. We consider $[u, v]$ as a poset with the partial order induced by the Bruhat order on W. A *coatom* of the interval $[u, v]$ is an element $x \in W$ satisfying $u \leq x \lhd v$. We call $[u, v]$ a dihedral interval of length r if it is isomorphic as a poset to the Coxeter group of rank 2 generated by s_1 and s_2, with $m(s_1, s_2) = r$. We say that a graded poset P *avoids* $K_{3,2}$ if there are no elements $a_1, a_2, a_3, b_1, b_2 \in P$, all distinct, such that either $a_i \lhd b_j$ for all $i \in [3]$, $j \in [2]$, or $a_i \rhd b_j$ for all $i \in [3]$, $j \in [2]$. The following lemma is proved in [3].

Lemma 2.5 *Let $u, v \in W$, $u \leq v$. Then the interval $[u, v]$ avoids $K_{3,2}$. In particular, if the interval $[u, v]$ has two coatoms it is necessarily dihedral of length ≥ 2.*

The following lemma can be proved in exactly the same way as [6, Lemma 3.1], and we therefore omit its proof.

Lemma 2.6 *Let (W, S) be a Coxeter system and $t_1, \ldots, t_n \in T$.*

- *If $t_1 t_2 = t_3 t_4 \neq e$, then the corresponding positive roots α_{t_1}, α_{t_2}, α_{t_3}, α_{t_4} are coplanar.*

- *If $t_1, t_2, \ldots, t_n$ are such that the corresponding positive roots $\alpha_{t_1}, \alpha_{t_2}, \ldots \alpha_{t_n}$ are coplanar, then the reflection subgroup $\langle t_1, t_2, \ldots, t_n \rangle$ of W is a Coxeter group of rank 2.*

Proposition 2.7 *Let $u, v \in \mathfrak{S}(n)$, $u \leq v$, be such that the interval $[u, v]$ is dihedral. Then $\ell(u, v) \leq 3$.*

Proof. By Lemma 2.6 it follows easily that the reflection subgroup $W' \stackrel{\text{def}}{=} \langle \{ab^{-1} : u \leq a \lhd b \leq v\} \rangle$ is a Coxeter group of rank 2. By Theorem 1.4 of [6], we have that the interval $[u, v]$ is isomorphic, as a partially ordered set, to a subset of W'. Let, by Theorem 2.1, $t, t' \in T$ be such that $(W', \{t, t'\})$ is a Coxeter system. Since T is exactly the set of transpositions of $\mathfrak{S}(n)$, it follows immediately that $m(t, t') \leq 3$, and the proof is complete. $\square$

For $J \subseteq S$, let W_J be the parabolic subgroup of W generated by the set J, and

$$^{J}W \stackrel{\text{def}}{=} \{w \in W \; : \; D_L(w) \subseteq S \setminus J\}. \tag{1}$$

The following result is well known and a proof of it can be found, e.g., in [7].

Proposition 2.8 *Let $J \subseteq S$. Then*

1. *Every $w \in W$ has a unique factorization $w = w_J \, {}^{J}w$ such that $w_J \in W_J$ and ${}^{J}w \in {}^{J}W$.*

2. *For this factorization, $\ell(w) = \ell(w_J) + \ell({}^{J}w)$.*

There is of course a right version of the above definitions and results. Namely, if we let

$$W^J \stackrel{\text{def}}{=} \{w \in W \,:\, D_R(w) \subseteq S \setminus J\} = ({}^J W)^{-1}, \tag{2}$$

then every $w \in W$ can be uniquely factorized $w = w^J\, w_J$, where $w^J \in W^J$ and $w_J \in W_J$, and $\ell(w) = \ell(w^J) + \ell(w_J)$.

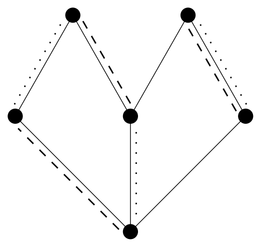
3 Generalities on special matchings

In this section we give the basic notions about special matchings of partially ordered sets. Recall that a matching of a graph $G = (V, E)$ is an involution $M : V \to V$ such that $\{M(v), v\} \in E$ for all $v \in V$. A matching of a graph may be visualized by coloring with the same color all edges of the form $\{M(v), v\}$.

Let P be a finite graded poset. We say that a matching M of the Hasse diagram of P is a special matching of P if

$$u \lhd v \Longrightarrow M(u) \leq M(v),$$

for all $u, v \in P$ such that $M(u) \neq v$. For example, the dotted matching of the following poset



is a special matching while the dashed one is not.

A different, but equivalent in the case of Eulerian posets, concept has also been introduced in [4]. The main application of special matchings is their link to Kazhdan-Lusztig polynomials, see [2, 3]. Note that a special matching has certain rigidity properties. For example, if $u \lhd v$ and $u \lhd M(u) \neq v$, then $M(v) \rhd v$ and $M(u) \lhd M(v)$. For the reader's convenience, we collect the following results which are proved in [3].

Lemma 3.1 *Let P be a graded poset, M be a special matching of P, and $x, y \in P$, $x \leq y$, be such that $M(x) \rhd x$ and $M(y) \lhd y$. Then M restricts to a special matching of $\{p \in P : x \leq p \leq y\}$.*

Lemma 3.2 *Let M and N be two special matchings of a finite graded poset P. Then the orbits of the action of the group generated by M and N on P are dihedral intervals.*

We denote by $\langle M, N\rangle(p)$ the orbit of $p \in P$ under the action of the group generated by M and N.

Proposition 3.3 *Let P be a graded poset that avoids $K_{3,2}$, $v \in P$, M and N be two special matchings of $\{u \in P : u \leq v\}$ such that $M(v) \neq N(v)$. Let $v' \lhd v$. Then*

$$|\langle M, N\rangle(v)| = |\langle M, N\rangle(v')|.$$

Similarly, if $u \leq v$ is such that $M(u) \neq N(u)$, $M(u) \rhd u$ and $N(u) \rhd u$, then

$$|\langle M, N\rangle(u)| = |\langle M, N\rangle(u')|$$

for all $u' \rhd u$.

We now restrict our attention to the case where P is a *lower Bruhat interval* of the symmetric group, i.e. an interval of the form $[e, v]$, with $v \in \mathfrak{S}(n)$. In this case we simply refer to a special matching of $[e, v]$ as a special matching of v.

The following is a further result on the rigidity of special matchings of a permutation and is proved in [2].

Lemma 3.4 *Let $v \in \mathfrak{S}(n)$ and M, N be two special matchings of v such that $M(u) = N(u)$ for all $u \leq v$ with $\ell(u) \leq 1$. Then*

$$M(u) = N(u)$$

for all $u \in [e, v]$.

The next result is a complete characterization of the special matchings of $v \in \mathfrak{S}(n)$ and is proved in [2]. For this we first need some notation. For all $i \in [n-1]$ denote by $\lambda_i, \rho_i : \mathfrak{S}(n) \to \mathfrak{S}(n)$ the multiplications on the left and on the right by s_i, i.e. $\lambda_i(v) = s_i v$ and $\rho_i(v) = v s_i$ for all $v \in \mathfrak{S}(n)$. If $i \in D_L(v)$, then the restriction of λ_i to $[e, v]$ satisfies the axioms of a special matching, by [7, Lemma 7.4]. Analogously for ρ_i if $i \in D_R(v)$.
Now fix $i \in [n-1]$ and let $J \stackrel{\mathrm{def}}{=} [i]$ and $K \stackrel{\mathrm{def}}{=} [i, n-1]$. Then we set

$$\begin{aligned} l_i(u) &= u_J s_i \, {}^J u \\ r_i(u) &= u_K s_i \, {}^K u, \end{aligned}$$

where $u = u_J\, {}^J u$ and $u = u_K\, {}^K u$ are the decompositions of u relative to the parabolic subgroups $\mathfrak{S}(n)_J$ and $\mathfrak{S}(n)_K$ (see Proposition 2.8). When no confusion arises, we also denote by λ_i, ρ_i, l_i and r_i any restriction of these applications to a proper subset of $\mathfrak{S}(n)$. Note that we have $l_1 = \lambda_1$, $r_1 = \rho_1$, $l_{n-1} = \rho_{n-1}$ and $r_{n-1} = \lambda_{n-1}$.

Theorem 3.5 *Let $v \in \mathfrak{S}(n)$ and M be a special matching of v. Then M is one of λ_i, ρ_i, l_i or r_i, where i is such that $s_i \stackrel{\text{def}}{=} M(e)$.*

We say that a special matching M of v is of type λ if $M = \lambda_i$ for some $i \in [n-1]$, and we similarly define special matchings of type ρ, l and r. Note that a special matching may have more than one type. For example, the unique matching of the trivial interval $[e, s_i]$ has all the types.

The following results follow directly from the proof of Theorem 3.5 and we state them for future reference.

Corollary 3.6 *Let $v \in \mathfrak{S}(n)$, $i \in [n-1]$, $J = [i]$, and $K = [i, n-1]$.*

1. *If l_i is a special matching of v, then $s_{i+1}s_is_{i-1} \not\leq v$. In particular ${}^J u \in \mathfrak{S}(n)_K$, for all $u \leq v$.*

2. *If r_i is a special matching of v, then $s_{i-1}s_is_{i+1} \not\leq v$. In particular ${}^K u \in \mathfrak{S}(n)_J$, for all $u \leq v$.*

Corollary 3.7 *Let $u, v \in \mathfrak{S}(n)$, $u \leq v$, $J = [i]$ and $K = [i, n-1]$.*

1. *Let l_i be a special matching of v and $u = u_1u_2$ with $u_1 \in \mathfrak{S}(n)_J$ and $u_2 \in \mathfrak{S}(n)_K$. Then we have either $u_1 = u_J$ or $u_1 = u_Js_i$. In particular $l_i(u) = u_1s_iu_2$.*

2. *Let r_i be a special matching of v and $u = u_1u_2$ with $u_1 \in \mathfrak{S}(n)_K$ and $u_2 \in \mathfrak{S}(n)_J$. Then we have either $u_1 = u_K$ or $u_1 = u_{JK}s_i$. In particular $r_i(u) = u_1s_iu_2$.*

4 The Commutation Graph

In this section we study the commutation relations among pairs of special matchings of a permutation v. We encode these in a simple graph that we call the commutation

graph of the special matchings of v.
We start with two technical lemmas that will be needed later. Actually Lemma 4.1 is a generalization of [9, Lemma 3.1].

Lemma 4.1 *Let (W, S) be any Coxeter system and let $J, K \subseteq S$, with $J \cap K = \emptyset$. Suppose that $w = w_j w_k$, with $w_j \in W_J$ and $w_k \in W_K$, and that $s_j \in J \cap D_R(w)$. Then $s_j \in D_R(w_j)$ and s_j commutes with every letter appearing in any reduced expression of w_k.*

Proof. Note that we have $w_k \in {}^{J}W$ and hence, by Proposition 2.8, $\ell(w) = \ell(w_j) + \ell(w_k)$. We proceed by induction on $\ell(w_k)$, the assertion being clear if $\ell(w_k) = 0$.
So suppose $\ell(w_k) \geq 1$ and let $s \in D_R(w_k)$. By the Lifting Lemma (see [7, Lemma 7.4]), $s_j \in D_R(ws)$, and we can consider the factorization $ws = w_j \cdot w_k s$, with $\ell(w_k s) < \ell(w_k)$. So, by induction, s_j commutes with every letter in $w_k s$, namely with every letter in w_k except possibly s. Suppose, by contradiction, that s_j does not commute with s. By Lemma 2.3, w admits a reduced expression of the form $w' \alpha_{s,s_j}$, where α_{s,s_j} has more than two letters. Hence $s \leq ws$ and this forces $s \leq w_k s$; but this is a contradiction because we have already proved that s_j commutes with every letter in $w_k s$. $\square$
Clearly, a left version of Lemma 4.1 holds.

Lemma 4.2 *Let (W, S) be any Coxeter system, $w \in W$ and $K \subseteq S$. Suppose that $s \in D_L(w)$ but $s \notin K$. Then $s \in D_L({}^{K}w)$.*

Proof. Recall the factorization of Proposition 2.8: $w = w_K \, {}^{K}w$. We proceed by induction on $\ell(w_K)$, the assertion being clear if $\ell(w_K) = 0$.
So suppose $\ell(w_K) \geq 1$ and let $s' \in D_L(w_K)$. By the Lifting Lemma (see [7, Lemma 7.4]), since $s \neq s'$, $s \in D_L(s'w)$, and we have the factorization $s'w = s'w_K \cdot {}^{K}w$, with $\ell(s'w_K) < \ell(w_K)$. So, by induction, $s \in D_L({}^{K}w)$. $\square$
Clearly, a right version of Lemma 4.2 holds.

From now on we call, for brevity, a dihedral interval of length 1, 2 and 3 respectively a segment, a square and a hexagon.

Proposition 4.3 *Let $v \in \mathfrak{S}(n)$, M and N be two special matchings of v, and $u_0 \leq v$.*

1. *If $\langle M, N\rangle(u_0)$ is a hexagon, then $\langle M, N\rangle(u)$ is a hexagon for all $u \leq v$.*

2. *If $\langle M, N\rangle(u_0)$ is either a segment or a square, then $\langle M, N\rangle(u)$ is either a segment or a square for all $u \leq v$.*

Proof. Clearly 1. and 2. are equivalent by Proposition 2.7 and Lemma 3.2. Let us prove 1.
With no loss of generality, we may assume that u_0 is the top element of an orbit. We first prove the statement for $u \leq u_0$ by induction on $\ell(u_0)$. Suppose that there exists $u_1 \lhd u_0$, $u_1 \notin \{M(u_0), N(u_0)\}$, such that $u \leq u_1$. Then, by the definition of a special matching, $M(u_1) \lhd u_1$ and $N(u_1) \lhd u_1$ and, by Proposition 3.3, $\langle M, N\rangle(u_1)$ is a hexagon, and we can conclude by induction. If such u_1 does not exist then, by Lemma 2.5, $[u, u_0]$ is a dihedral interval containing $M(u_0)$ and $N(u_0)$. Then, by Lemma 2.5 and Proposition 2.7, we can easily see that $u \in \langle M, N\rangle(u_0)$ and we are done. In particular, the orbit $\langle M, N\rangle(e)$ is a hexagon. An upside-down argument with $u_0 = e$ now shows that the orbit $\langle M, N\rangle(u)$ is a hexagon for all $u \geq e$ and the proof is complete. $\square$

Now we can conclude that the commutation rules of special matchings really look like the Coxeter relations for the symmetric group.

Corollary 4.4 *Let $v \in \mathfrak{S}(n)$, and M and N be two special matchings of v. Then either $MN = NM$ or $MNM = NMN$.*

Now we focus our attention on non-commuting pairs of special matchings. Note that two special matchings M and N do not commute if and only if $MN(e) \neq NM(e)$, by Proposition 4.3. It follows that any special matching does not commute with at most four other special matchings. More precisely, we have the situation in the following table.

special matchings	does not commute with
λ_i	λ_{i-1}, λ_{i+1}, l_{i-1}, r_{i+1}
ρ_i	ρ_{i-1}, ρ_{i+1}, r_{i-1}, l_{i+1}
l_i	r_{i-1}, r_{i+1}, λ_{i+1}, ρ_{i-1}
r_i	l_{i-1}, l_{i+1}, λ_{i-1}, ρ_{i+1}

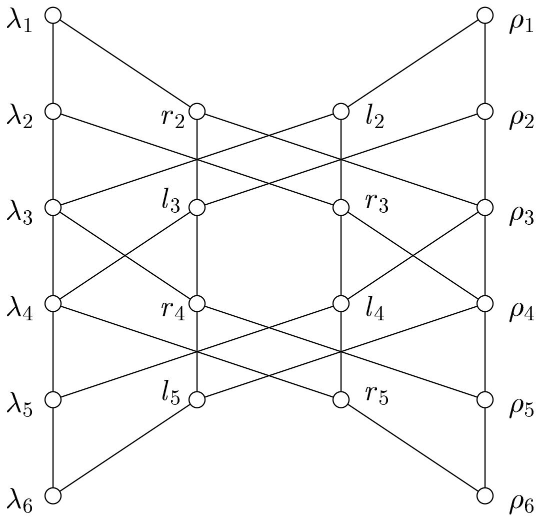


Figure 1: Special matchings in $\mathfrak{S}(7)$

We shall see that the situation is actually much simpler. We define the commutation graph of the special matchings of v to be the graph $G_v = (S_v, E_v)$, where S_v is the set of special matchings of v, and E_v is the set of non-commuting pairs of special matchings. By what we have proved so far, if $v \in \mathfrak{S}(7)$, its commutation graph can be obtained from the graph in Figure 1 by deleting some vertices and the corresponding adjacent edges. Note that l_1, l_6, r_1, r_6 do not appear in this graph since they are necessarily also either of type λ or of type ρ.

Lemma 4.5 *Let $v \in \mathfrak{S}(n)$.*

1. *If l_i, $r_{i+1} \in S_v$, then $v = v' s_i s_{i+1} s_i$, with $v' \in \mathfrak{S}(n)_{[n-1]\setminus\{i,i+1\}}$ and $\ell(v) = \ell(v') + 3$.*

2. *If r_i, $l_{i+1} \in S_v$, then $v = s_i s_{i+1} s_i v''$, with $v'' \in \mathfrak{S}(n)_{[n-1]\setminus\{i,i+1\}}$ and $\ell(v) = \ell(v') + 3$.*

Proof. We prove only 1. because 2. is its dual statement.
Let $u \in [e, v]$ be such that $l_i(u) > u$ and $r_{i+1}(u) > u$. We claim that $s_{i+1} \not\leq u$.
In fact, let $J \stackrel{\text{def}}{=} [i]$ and $K \stackrel{\text{def}}{=} [i, n-1]$, and decompose $u = u_J\, {}^J u$, where, by Corollary 3.6, ${}^J u \in \mathfrak{S}(n)_K$. We have $s_{i+1}s_{i+2} \not\leq {}^J u$, otherwise $s_i s_{i+1} s_{i+2} \leq l_i(u) \leq v$ which contradicts 2. of Corollary 3.6. Hence, if $s_{i+1} \leq {}^J u$, we have that ${}^J u = u' s_{i+1} u''$ with $u' \in \mathfrak{S}(n)_{[i+2,n-1]}$ and $u'' \in \mathfrak{S}(n)_{\{i\}}$, and $\ell(u) = \ell(u') + 1 + \ell(u'')$. So $u = u_J\, {}^J u = u_J u' s_{i+1} u'' = u' u_J s_{i+1} u''$ and hence, by Corollary 3.7, $r_{i+1}(u) =$

$u's_{i+1}u_Js_{i+1}u''$. By 1. of Corollary 3.6, such decomposition forces $s_is_{i-1} \not\leq u_J$. Since $l_i(u) \rhd u$, we have $u_Js_i \rhd u_J$, so $s_i \not\leq u_J$. Then, since $r_{i+1}(u) = u's_{i+1}u_Js_{i+1}u''$, we would have $r_{i+1}(u) < u$, which is a contradiction.
A symmetric argument shows that $s_i \not\leq u$.
Since $\langle l_i, r_{i+1}\rangle(e)$ is a hexagon, all the orbits of the action of the group $\langle l_i, r_{i+1}\rangle$ on $[e, v]$ are hexagons by Proposition 4.3. Suppose u is the bottom element of the hexagon containing v, so that $v = l_ir_{i+1}l_i(u)$. By what we proved above $u = u_1u_2$ with $u_1 \in \mathfrak{S}(n)_{[i-1]}$ and $u_2 \in \mathfrak{S}(n)_{[i+2,n-1]}$. So

$$\begin{aligned} v &= l_ir_{i+1}l_i(u) = l_ir_{i+1}(u_1s_iu_2) \\ &= l_ir_{i+1}(u_2u_1s_i) = l_i(u_2s_{i+1}u_1s_i) \\ &= u_1s_iu_2s_{i+1}s_i = u_1u_2s_is_{i+1}s_i \end{aligned}$$

and we are done. $\square$

Theorem 4.6 *Let $v \in \mathfrak{S}(n)$.*

1. *If $l_i, r_{i+1} \in S_v$, then*

$$l_i \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\, r_{i+1}$$

is a connected component of G_v.

2. *If $r_i, l_{i+1} \in S_v$, then*

$$r_i \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\, l_{i+1}$$

is a connected component of G_v.

Proof. We prove only 1. because 2. is its dual statement.
Figure 2 shows all the possible neighbors of l_i and r_{i+1} in the commutation graph (see Figure 1).

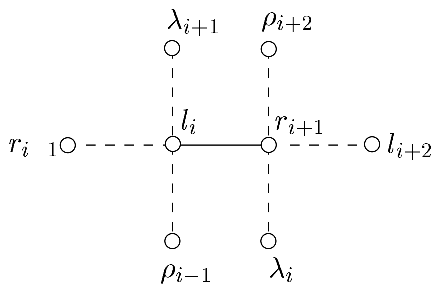


Figure 2: Neighbors of l_i and r_{i+1}

Recall that, by Lemma 4.5,

$$v = v' s_i s_{i+1} s_i, \tag{3}$$

with $v' \in \mathfrak{S}(n)_{[n-1]\setminus\{i,i+1\}}$.
If λ_{i+1} is a special matching of v then, by Lemma 4.1, $s_{i+2} \not\leq v$, which forces $\lambda_{i+1} = r_{i+1}$, by Lemma 3.4. If λ_i is a special matching of v, then, by Lemma 4.1, $s_{i-1} \not\leq v$, which implies $\lambda_i = l_i$, by Lemma 3.4.
Again by Lemma 4.1, we have that ρ_{i-1} and ρ_{i+2} are not special matchings of v.

Let us check that l_{i+2} and r_{i-1} are not special matchings of v. We show it for l_{i+2}, the same argument being valid also for r_{i-1}. Suppose, by contradiction, that l_{i+2} is a special matching of v. By 2. of Lemma 4.5, we have

$$v = s_{i+1} s_{i+2} s_{i+1} v'', \tag{4}$$

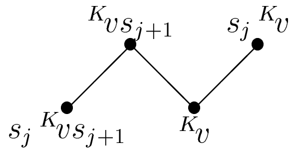
with $v'' \in \mathfrak{S}(n)_{[n-1]\setminus\{i+1,i+2\}}$. But the two expressions in (3) and (4) are incompatible. In fact, from (4) we have $i+1 \in D_L(v)$ which forces, by (3), $s_{i+2} \not\leq v$. But this is a contradiction with (4). $\square$

We go on in our analysis of the commutation graph by showing two other forbidden local configurations.

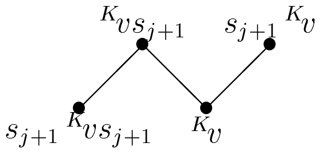
Theorem 4.7 *Let $v \in \mathfrak{S}(n)$. Then $\{\lambda_{j-i}, r_j, \rho_{j+1}\} \not\subseteq S_v$ and $\{\rho_{j-i}, l_j, \lambda_{j+1}\} \not\subseteq S_v$ for all $j \in [2, n-2]$.*

Proof. We prove the statement only for the first case, the proof for the second one being entirely similar.
By contradiction, suppose that λ_{j-1}, r_j and ρ_{j+1} are all special matchings of v. Let $J = [j]$ and $K = [j, n-1]$, and decompose $v = v_K \, {}^K v$. We claim that $s_j \not\leq {}^K v$. In fact ${}^K v s_{j+1} \notin {}^K W$ since ${}^K W \cap [e, v] \subseteq W_J \cap [e, v]$ by 2. of Corollary 3.6. Then, by

the definition of ${}^K W$, there exists $k \in K$ such that $s_k \in D_L({}^K v s_{j+1})$. Necessarily $k = j$ or $k = j+1$, as ${}^K v \in W_J$.
If $k = j$ we have



By [7, Lemma 7.4]) we have ${}^K v s_{j+1} = s_j\, {}^K v$. But this is not possible since $s_{j+1} \not\leq s_j\, {}^K v$.
So $k = j+1$ and we have



and hence, by [7, Lemma 7.4], ${}^K v s_{j+1} = s_{j+1}\, {}^K v$ so $s_{j+1} \in D_R(s_{j+1}\, {}^K v$ which implies that $s_j \not\leq {}^K v$ by Lemma 4.1. So the claim is proved.

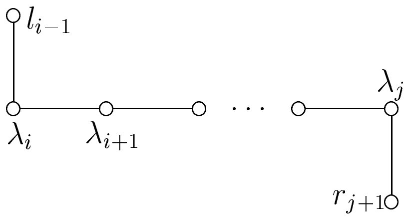
A similar argument applied to v^{-1} and to J, together with Corollary 3.7, shows that either $s_j \not\leq v_K$ or $v_K = v' s_j$ with $s_j \not\leq v'$. Since r_j is a special matching of v, we must be in the last situation. So we have

$$v = v' s_j\, {}^K v$$

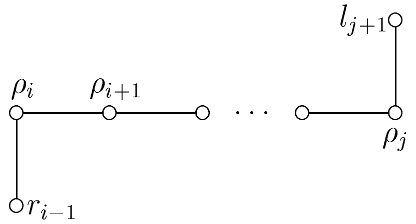
with $v' \in W_{[j+1,n-1]}$ and ${}^K v \in W_{[j-1]}$. All these conditions are in contradiction with Lemma 4.1 (e.g. apply Lemma 4.1 to $j+1 \in D_R(v)$), and the proof is complete. $\square$

The next result shows us what a connected component of the commutation graph looks like.

Theorem 4.8 *Let $v \in \mathfrak{S}(n)$. Let $\lambda_i, \lambda_{i+1}, \ldots, \lambda_j$ with $j \geq i$ be special matchings of v, and suppose that λ_{i-1} and λ_{j+1} are not special matchings of v. Then their connected component in G_v is a subgraph of*

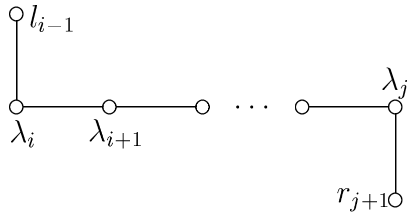


Dually, if $\rho_i, \ldots, \rho_j$ are special matchings of v and ρ_{i-1} and ρ_{j+1} are not, then their connected component in the commutation graph is a subgraph of



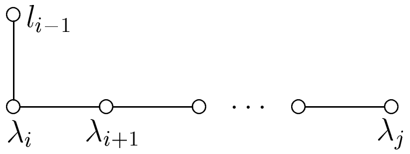
Proof. We prove the first statement only.
Observe that if λ_k, λ_{k+1} and r_{k+1} are special matchings of v for some k, then $r_{k+1} = \lambda_{k+1}$. In fact, if $k, k+1 \in D_L(v)$, then $v = s_k s_{k+1} s_k v'$ with $\ell(v) = \ell(v') + 3$, by Lemma 2.3. So, if r_{k+1} is a special matching of v, we have $s_{k+2} \not\leq v'$, otherwise $s_k s_{k+1} s_{k+2} \leq v$, which is not possible by Corollary 3.6. Hence $r_{k+1} = \lambda_{k+1}$, by Lemma 3.4 . Similarly, if λ_k, λ_{k+1} and l_k are special matchings of v for some k then $l_k = \lambda_k$.
Now we look at the possible neighbors of λ_i in G_v. The matching $\lambda_{i-1} \notin S_v$ by hypothesis. If $r_{i+1} \in S_v$ then, by what we have just proved, $r_{i+1} = \lambda_{i+1}$. Similarly for the neighbors of $\lambda_{i+1}, \ldots, \lambda_j$. $\square$

The next example shows that the graphs appearing in the statement of Theorem 4.8 can actually be connected components of the commutation graph.
Example. Let, for $3 \leq i < j \leq n-3$, $v = s_{i-2} s_{j+2} w$ where w is the longest element in the parabolic subgroup $W_{[i,j+1]}$. Then

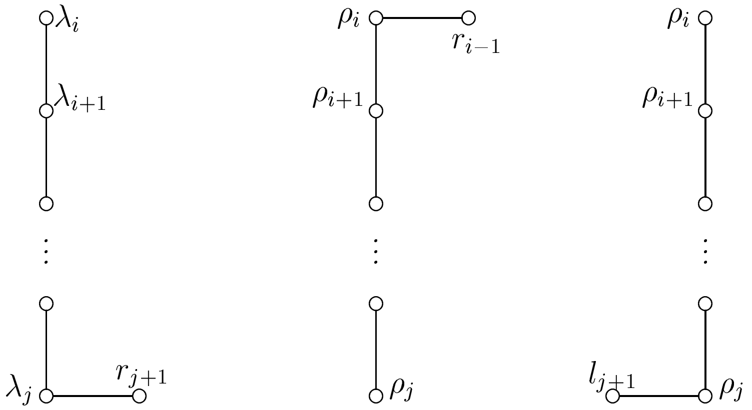


is actually a connected component in the commutation graph of the special matchings of v.
Our next goal is to understand if there are any further relations among the special matchings of a permutation v, other than commutation relations. In other words, we want to understand if the group generated by the special matchings is the Coxeter group whose Coxeter diagram is the commutation graph, or a proper quotient of it.

Lemma 4.9 *Let $i < j$ and $v \in \mathfrak{S}(n)$. Suppose that $\{l_{i-1}, \lambda_i, \ldots, \lambda_j\} \subseteq S_v$ so*

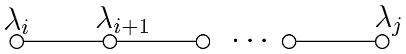


is the subgraph induced by $\{l_{i-1}, \lambda_i, \ldots, \lambda_j\}$ in G_v. Then every connected component having at least a special matching indexed in $[i-1, j]$ does not have vertices with index smaller than $i-1$.
Analogous statements hold if we change the subgraph with any of the following



Proof. Let us check all other possible special matchings indexed by $i-1$.
By the proof of Theorem 4.8, if λ_{i-1} is a special matching, then $\lambda_{i-1} = l_{i-1}$.
Let $J = [i-1]$ and decompose $v = v_J\, {}^J v$. Since l_{i-1} is a special matching, we have $s_{i-1} \in D_R(v_J)$. We may assume that $s_{i-2} \leq v_J$, else, by Corollary 3.6, $s_{i-2} \not\leq v$, and v would not have any special matching indexed by $i-2$, and the result would be trivial. If r_{i-1} is a special matching, then $s_{i-2}s_{i-1}s_i \not\leq v$, by Corollary 3.6. This, since $s_{i-2} \leq v_J$ and $s_{i-1} \in D_R v_J$, implies that $s_i \not\leq {}^J v$, and hence $s_i \not\leq v$. But this contradicts the fact that λ_i is a special matching, hence r_{i-1} is not a special matching. So the only other possible special matching of v indexed by $i-1$ is ρ_{i-1}. Note that the possible neighbors of ρ_{i-1} indexed by $i-2$ are ρ_{i-2} and r_{i-2}. But ρ_{i-2} and r_{i-2} are not special matchings of v because, otherwise, they would be in the same connected component of l_{i-1}, which contradicts Theorem 4.8. $\square$

Lemma 4.10 *Let $i < j$ and $v \in \mathfrak{S}(n)$. If*



is a connected component of G_v and C is another component with a special matching indexed by i, then this special matching is of type ρ or r, also, if it is of type r, C does not contain any special matching with index smaller than i.

Proof. We already know that if $l_i \in S_v$, then $l_i = \lambda_i$. Suppose r_i is a special matching. If l_{i-1} or λ_{i-1} are special matchings, then they would be in the same connected component as λ_i, which is a contradiction; so in the connected component of r_i there are no special matchings with index smaller than i. □
Clearly, a dual statement holds.

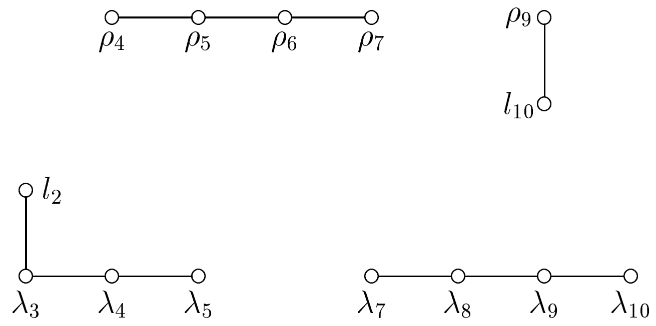
We now introduce an equivalence relation on the connected components of G_v. We say that two connected components C and C' are in the same isotypical component if there exists a sequence $C = C_0, C_1, \ldots, C_t = C'$ of connected components, such that, for all $i \in [t]$, C_{i-1} and C_i contain at least one special matching with the same index. So Lemmas 4.9 and 4.10 tell us that special matchings of type l and r have external indices in isotypical components.

If I is an isotypical component, we denote by $\mathrm{Index}(I)$ the set of indices of the special matchings contained in I. Note that $\mathrm{Index}(I)$ is necessarily an interval of positive integers, for any isotypical component I.

Corollary 4.11 *Let I be an isotypical component of the commutation graph and suppose that* $\mathrm{Index}(I) = [i, j]$. *Then all special matchings of I indexed in* $[i+1, j-1]$ *are of type λ or ρ.*

Proof. This follows directly from Lemmas 4.9 and 4.10. □

This is an example of what an isotypical component looks like.



5 Main result

Given $v \in \mathfrak{S}(n)$, let $S([e,v])$ be the symmetric group on the elements in the Bruhat interval $[e,v]$. We define W_v to be the subgroup of $S([e,v])$ generated by the set S_v of all special matchings of v seen as involutions on $[e,v]$. In this section, we prove the main result of this work, namely we show that the pair (W_v, S_v) is a Coxeter system. This implies new necessary conditions on an abstract poset to be isomorphic to a lower Bruhat interval in the symmetric group, and also answers in the affirmative a question posed in [3].

Theorem 5.1 *Let $v \in \mathfrak{S}(n)$. Then (W_v, S_v) is a Coxeter system isomorphic to a direct product of symmetric groups.*

Proof. Let p be a word in the alphabet S_v such that $p(u) = u$ for all $u \leq v$ (in other words, p is the identity in $S([e,v])$). We will show that we can obtain the empty word from p using only braid moves either of the form $MNM \leftrightarrow NMN$, (if M and N do not commute), or of the form $MN \leftrightarrow NM$ (if M and N do commute), and nil moves of the form $MM = \emptyset$.
Suppose that I is an isotypical component and that $\mathrm{Index}(I) = [i,j]$. Then, by Corollary 4.11, after commutation of some letters, we may suppose that $p = p_1 p_2 p_3$, where:

- p_1 is a word in $h_i, \lambda_{i+1}, \ldots, \lambda_{j-1}, h_j$, with $h_i \in \{l_i, \lambda_i\}$, and $h_j \in \{r_j, \lambda_j\}$;

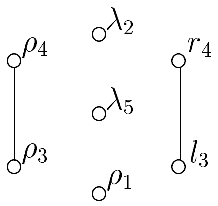
- p_2 is a word in $k_i, \rho_{i+1}, \ldots, \rho_{j-1}, k_j$, with $k_i \in \{r_i, \rho_i\}$, and $k_j \in \{l_j, \rho_j\}$;

- p_3 is a word involving special matchings which are not indexed in $[i, j]$.

It is clear that it is enough to prove our claim for p_1 and p_2, the general result following by induction on the number of isotypical components.
Note that $p_1 p_2 = p_3^{-1}$. In particular we have $p_1 p_2(e) = p_3^{-1}(e)$. But $p_1 p_2(e) \in \mathfrak{S}(n)_{[i,j]}$ and $p_3^{-1}(e) \in \mathfrak{S}(n)_{[1,i-1]\cup[j+1,n-1]}$, so $p_1 p_2(e) = p_3^{-1}(e) = e$. Moreover, for all $s_h \leq v$, we have $p_1 p_2(s_h) \in \mathfrak{S}(n)_{[i,j]\cup\{h\}}$ and $p_3^{-1}(s_h) \in \mathfrak{S}(n)_{[i,i-1]\cup[j+1,n-1]\cup\{h\}}$, and hence $p_1 p_2(s_h) \in \{e, s_h\}$. Since $p_1 p_2$ is a bijection, we have $p_1 p_2(s_h) = s_h$.
We first deal with the case $i = j$. Let $\pi \stackrel{\text{def}}{=} p_1 p_2$. We may clearly assume that π is a subword of $\lambda_i \rho_i l_i r_i$ of even length, since $\pi(e) = e$. If $s_{i-1} \leq v$ and $s_{i+1} \not\leq v$, then $l_i = \rho_i$ and $r_i = \lambda_i$, so π is a subword of $\lambda_i \rho_i$. But $\lambda_i \rho_i(s_{i-1}) \neq s_{i-1}$ and hence π is the empty word. If $s_{i-1} \not\leq v$ and $s_{i+1} \leq v$, the proof is analogous. If $s_{i-1} \not\leq v$ and $s_{i+1} \not\leq v$, there is at most one special matching indexed by i and the result follows. So assume that $s_{i-1} \leq v$ and $s_{i+1} \leq v$. If l_i is a special matching, then $l_i(s_{i-1}s_{i+1}) = s_{i-1}s_i s_{i+1}$ which implies, by Corollary 3.6, that r_i is not a special matching. But the only subword of $\lambda_i \rho_i l_i$ that acts as the identity on both s_{i-1} and s_{i+1} is the empty word and we are done. If $l_i \notin S_v$ then π is a subword of $\lambda_i \rho_i r_i$ of even length and the proof is similar.
So assume $i < j$, and let $P \stackrel{\text{def}}{=} \mathfrak{S}(n)_{[i,j]} \cap [e, v]$. Note that for all $u \in P$ we have $h_i(u) = \lambda_i(u)$, $h_j(u) = \lambda_j(u)$, $k_i(u) = \rho_i(u)$, and $k_j(u) = \rho_j(u)$. Thus, if $s_{i_1} \cdots s_{i_k}$ is a reduced expression of $p_2(e) = p_1^{-1}(e)$, we may obtain, using only braid and nil moves, $p_1^{-1} = \lambda_{i_1} \cdots \lambda_{i_k}$ and $p_2 = \rho_{i_k} \cdots \rho_{i_1}$, so that p_1^{-1} acts on P by multiplying on the left by $u \stackrel{\text{def}}{=} s_{i_1} \cdots s_{i_k}$ (this being a reduced expression) and p_2 acts on P by multiplying on the right by u. Since $p_2(s_h) = p_1^{-1}(s_h)$ for all $h \in [i, j]$ this implies that u belongs to the center of $\mathfrak{S}(n)_{[i,j]}$. The result follows since the center of $\mathfrak{S}(n)_{[i,j]}$ is trivial. □

Theorem 5.1 gives new properties of lower Bruhat intervals in symmetric groups, and so it is a further step towards a combinatorial characterization of such intervals as abstract partially ordered sets.

Example. Let $v = (3\,1\,6\,4\,2\,5) \in \mathfrak{S}(6)$. Then v admits the following reduced expression $v = s_2 s_3 s_1 s_5 s_4 s_3$ (recall that we consider permutations as functions, and hence the composition should be regarded from right to left). One may check that the interval $[e, v]$ has exactly six distinct special matchings and these are $\lambda_2, \lambda_5, \rho_1, \rho_3, \rho_4, l_3$ and r_4. Then the commutation graph of v is



and the group W_v generated by these special matchings is isomorphic to $\mathfrak{S}(3)^2 \times \mathfrak{S}(2)^3$.

Note that if v is the top element of $\mathfrak{S}(n)$, by Corollary 3.6 all special matchings of v are of type λ or ρ. Since $D_R(v) = D_L(v) = [n-1]$, we have $W_v \cong \mathfrak{S}(n) \times \mathfrak{S}(n)$.

Let $v \in \mathfrak{S}(n)$. For each $M, N \in S_v$ let $o(MN)$ be the order of MN as a permutation of $[e, v]$. Following [3, §8] we let $(\widehat{W}_v, S_v)$ be the Coxeter system whose Coxeter matrix is given by $m(M, N) = o(MN)$ for all $M, N \in S_v$. Clearly, there is a natural permutation action of $\widehat{W}_v$ on $[e, v]$. The proof of the previous result then has the following immediate consequence, which answers in the affirmative a question raised in [3].

Corollary 5.2 *The permutation action of $\widehat{W}_v$ on $[e, v]$ is faithful.*

Remark 5.3 We conclude by showing that Theorem 5.1 cannot be generalized to arbitrary Coxeter groups. Let W be the Coxeter group of rank 2 generated by a and b with $m(a, b) = 4$ and let $w = abab$. Then $[e, w]$ is a dihedral interval of length 4. Consider the special matchings in Figure 3.
Let M_1 be the dashed matching, M_2 the dotted matching, M_3 the dash-dotted matching and M_4 the dash-dot-dotted matching. Then $M_4 M_3 M_2 M_1$ is the identity as application from $[e, w]$ to itself, but it is clearly not a Coxeter relation.

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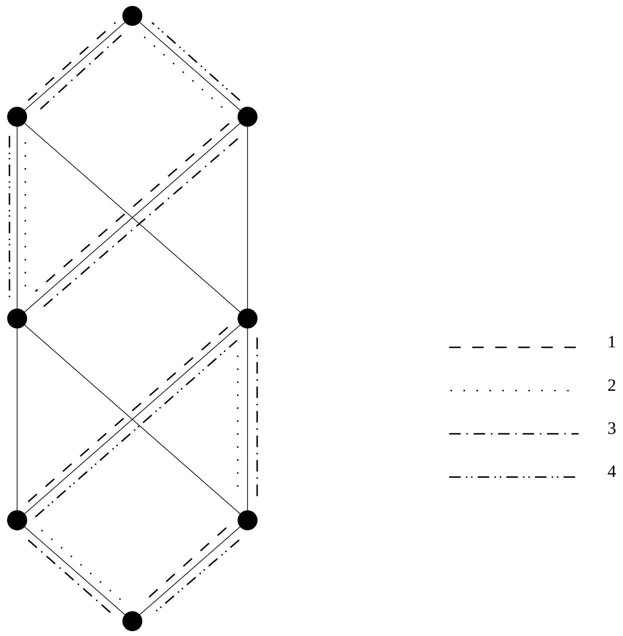

Figure 3: A non-Coxeter relation

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