Special matchings and Kazhdan-Lusztig polynomials

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Abstract

In 1979 Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials of W, and which have proven to be of importance in several areas of mathematics. In this paper we show that the combinatorial concept of a special matching plays a fundamental role in the computation of these polynomials. Our results also imply, and generalize, the recent one in [12] on the combinatorial invariance of Kazhdan-Lusztig polynomials.

1 Introduction

In their fundamental paper [22] Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials of W (see, e.g., [21], Chap. 7). These polynomials are intimately related to the Bruhat order of W and have proven to be of fundamental importance in several areas of mathematics including representation theory and the geometry and topology of Schubert varieties (see, e.g., [22], [23], [21], [19], [1], [18], [2], [25], and the references cited there).

Our purpose in this paper is to show that the combinatorial concept of a special matching (see §2 for definitions) plays a fundamental role in the computation of these polynomials. Our results also imply the recent one in [12] about the combinatorial invariance of Kazhdan-Lusztig polynomials. More precisely, while the result in [12] is non-constructive and holds for Coxeter systems whose Dynkin diagram is either a tree or affine of type A, our result is constructive and holds for all Coxeter systems.

The organization of the paper is as follows. In the next section we recall some definitions and results that will be used in the rest of this work. In the following three sections (§§3,4,5) we establish some preliminary results on Bruhat order, on the combinatorics of pairs of special matchings, and on general algebraic properties of special matchings of Coxeter systems. In section 6 we study in detail the special matchings of Coxeter systems of rank three. These results are then used in the following section (§7) to obtain the main result of this work. More precisely, we obtain a classification of all the special matchings of any Coxeter system (Theorem 7.6), from which the connection between special matchings and Kazhdan-Lusztig polynomials (Theorem 7.8) follows. In §8 we introduce and study a Hecke algebra naturally associated to the special matchings of any element of any Coxeter system and use it to show that our main result is equivalent to the statement that a certain action of this Hecke algebra on a submodule of the Hecke algebra of W "respects" the canonical involutions (Theorem 8.2). This, in turn, implies that the usual recursion for the Kazhdan-Lusztig polynomials ([22, formula (2.2c)]) holds also when descents are replaced by special matchings (Corollary 8.4). Finally, in the last three sections of this work, we derive some consequences of our main result. These include various closed formulas for both the Kazhdan-Lusztig and R-polynomials. Some of these generalize well-known formulas that have appeared before in the literature.

2 Notation, definitions and preliminaries

In this section we collect some definitions, notation and results that will be used in the rest of this work.

We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$, and $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$. For $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, \ldots, a\}$ (where $[0] \stackrel{\text{def}}{=} \emptyset$) and $[0, a] \stackrel{\text{def}}{=} [a] \cup \{0\}$. We write $S = \{a_1, \ldots, a_r\}_<$ to mean that $S = \{a_1, \ldots, a_r\}$ and $a_1 < \cdots < a_r$. The cardinality of a set A will be denoted by $|A|$ and its power set by $\mathcal{P}(A)$, for $r \in \mathbf{N}$ we let $\binom{A}{r} \stackrel{\text{def}}{=} \{S \subseteq A : |S| = r\}$. Given a polynomial $P(q)$, and $i \in \mathbf{Z}$, we denote by $[q^i](P(q))$ the coefficient of q^i in $P(q)$.

By a *graph* we mean a pair $G = (V, E)$ where V is a set and $E \subseteq \binom{V}{2}$. We call the elements of V *vertices* and those of E *edges*. A *matching* of G is an involution $M : V \to V$ such that $\{v, M(v)\} \in E$ for all $v \in V$.

By a *directed graph* we mean a pair $D = (V, A)$ where V is a set and $A \subseteq V^2$. We call the elements of V *vertices* and those of A *directed edges*. If $(a, b) \in A$ then we also write $a \to b$. A *directed path* (respectively, *undirected path*) in D is a sequence $\Gamma = (a_0, \ldots, a_r)$ of vertices such that $a_{i-1} \to a_i$ (respectively, either $a_{i-1} \to a_i$ or $a_i \to a_{i-1}$) for $i = 1, \ldots, r$. We then say that Γ goes from a_0 to a_r. The *length* of such a path is $\ell(\Gamma) \stackrel{\text{def}}{=} r$. If Γ is a directed path then we also write $\Gamma = (a_0 \to a_1 \to \cdots \to a_r)$. If $U \subseteq V$ then the directed graph *induced* on U by D is $(U, A \cap U^2)$.

Given a set T we let $S(T)$ be the set of all bijections $\pi : T \to T$, and $S_n \stackrel{\text{def}}{=} S([n])$. If $\sigma \in S_n$ then we write $\sigma = \sigma_1 \ldots \sigma_n$ to mean that $\sigma(i) = \sigma_i$, for $i = 1, \ldots, n$. We also write σ in disjoint cycle form, omitting to write the 1-cycles. Given $\sigma, \tau \in S_n$ we let $\sigma\tau \stackrel{\text{def}}{=} \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$.

We follow [24, Chapter 3] for undefined notation and terminology concerning partially ordered sets. In particular, if $(P, \leq)$ is a partially ordered set (or, poset, for short) then two elements $x, y \in P$ are said to be *comparable* if either $x \leq y$ or $y \leq x$, and *incomparable* otherwise. Given $x, y \in P$ we let $[x, y] \stackrel{\text{def}}{=} \{z \in P : x \leq z \leq y\}$ and call this an *interval* of P. If $|[x, y]| = 2$ then we say that y *covers* x and we write $x \lhd y$. An element $z \in [x, y]$ is said to be an *atom* (respectively, a *coatom*) of $[x, y]$ if $x \lhd z$ (respectively, $z \lhd y$). A poset P has a *minimum* (respectively, *maximum*) if there is an element, denoted $\hat{0}$ (respectively, $\hat{1}$), such that $\hat{0} \leq x$ (respectively, $x \leq \hat{1}$) for all $x \in P$. We say that a poset P is *graded* if P has a minimum and there is a function $\rho : P \to \mathbf{N}$ such that $\rho(\hat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ for all $x, y \in P$ with $x \lhd y$. (This definition is slightly different from the one given in [24], but is more convenient for our purposes.) We then call ρ the *rank function* of P. A sequence $(x_0, x_1, \ldots, x_r)$ of elements of P is called a *chain* if $x_0 < x_1 < \ldots < x_r$. We then also say that the chain

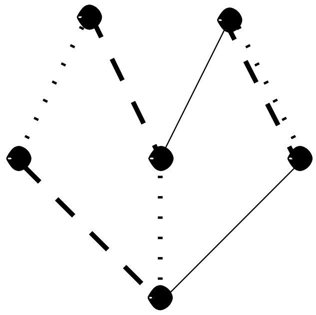

Figure 1: A special and a non-special matching

goes from x_0 to x_r. The integer r is called the *length* of the chain. The *Hasse diagram* of P is the graph $H(P) \stackrel{\mathrm{def}}{=} (P, E)$ where $E \stackrel{\mathrm{def}}{=} \{\{x, y\} \in \binom{P}{2} : \text{either } x \lhd y \text{ or } y \lhd x\}$. Following [7] we say that a matching M of the Hasse diagram of P is *special* if

$$u \lhd v \Longrightarrow M(u) \leq M(v),$$

for all $u, v \in P$ such that $M(u) \neq v$. A different, but equivalent in the case of Eulerian posets, concept has also been introduced in [11].

So, for example, the dotted matching of the poset in Figure 1 is special while the dashed one is not. The following result is easy to prove.

Lemma 2.1 *Let P be a graded poset, M be a special matching of P, and $u, v \in P$ be such that $M(v) \lhd v$ and $M(u) \rhd u$. Then M restricts to a special matching of $[u, v]$.*

Two posets P and Q are *isomorphic* if there exists an order-preserving bijection $f : P \to Q$ such that f^{-1} is also order-preserving. A poset P is a *Boolean algebra* of rank r if there is a set X of cardinality r such that P is isomorphic to $\mathcal{P}(X)$, partially ordered by inclusion.

We assume from now on that all intervals in P are finite. Let $\mathrm{Int}(P) \stackrel{\mathrm{def}}{=} \{(x, y) \in P^2 : x \leq y\}$. Given a commutative ring R the *incidence algebra* of P with coefficients in R, denoted $I(P; R)$, is the set of all functions $f : \mathrm{Int}(P) \to R$ with sum and product defined by

$$(f + g)(x, y) \stackrel{\mathrm{def}}{=} f(x, y) + g(x, y)$$

and

$$(fg)(x, y) \stackrel{\mathrm{def}}{=} \sum_{x \leq z \leq y} f(x, z)\, g(z, y),$$

for all $f, g \in I(P; R)$ and $(x, y) \in \mathrm{Int}(P)$. It is well known (see, e.g., [24], §3.6, and Proposition 3.6.2) that $I(P; R)$ is an associative algebra having δ as identity element (where $\delta(x, y) \stackrel{\mathrm{def}}{=} 1$ if $x = y$, and $\stackrel{\mathrm{def}}{=} 0$ otherwise) and that an element $f \in I(P; R)$ is invertible if and only if $f(x,x)$ is invertible for all $x \in P$. If f is invertible then we denote by f^{-1} its (two-sided) inverse.

By a *composition* of $n \in \mathbf{P}$ we mean a sequence $\alpha \stackrel{\text{def}}{=} (\alpha_1, \ldots, \alpha_s)$ (for some $s \in \mathbf{P}$) of positive integers such that $\alpha_1 + \ldots + \alpha_s = n$. We let $|\alpha| \stackrel{\text{def}}{=} \sum_{i=1}^{s} \alpha_i$, $\ell(\alpha) \stackrel{\text{def}}{=} s$, and $T(\alpha) \stackrel{\text{def}}{=} \{\alpha_s, \alpha_s + \alpha_{s-1}, \ldots, \alpha_s + \ldots + \alpha_2\}$. For $n \in \mathbf{P}$ we let C_n be the set of all compositions of n and $C \stackrel{\text{def}}{=} \bigcup_{n \geq 1} C_n$. Given $(\alpha_1, \ldots, \alpha_s), (\beta_1, \ldots, \beta_t) \in C_n$ we say that $(\alpha_1, \ldots, \alpha_s)$ *refines* $(\beta_1, \ldots, \beta_t)$ if there exist $1 \leq i_1 < i_2 < \cdots < i_{t-1} < s$ such that $\sum_{j=i_{k-1}+1}^{i_k} \alpha_j = \beta_k$ for $k = 1, \ldots, t$ (where $i_0 \stackrel{\text{def}}{=} 0$, $i_t \stackrel{\text{def}}{=} s$). We then write $(\alpha_1, \ldots, \alpha_s) \leq_c (\beta_1, \ldots, \beta_t)$. It is easy to see that the map $\beta \mapsto T(\beta)$ is an isomorphism from $(C_n, \leq_c)$ to the Boolean algebra of subsets of $[n-1]$ ordered by reverse inclusion.

Let $n \in \mathbf{N}$. By a *lattice path* of length n we mean a function $\Gamma : [0,n] \to \mathbf{Z}$ such that $\Gamma(0) = 0$ and $|\Gamma(i) - \Gamma(i-1)| = 1$ for all $i \in [n]$. Given such a lattice path Γ we let

$$N(\Gamma) \stackrel{\text{def}}{=} \{i \in [n-1] : \Gamma(i) < 0\},$$

$$d_+(\Gamma) \stackrel{\text{def}}{=} |\{i \in [0, n-1] : \Gamma(i+1) - \Gamma(i) = 1\}|,$$

$\ell(\Gamma) \stackrel{\text{def}}{=} n$, and $\Gamma_{\geq 0} \stackrel{\text{def}}{=} \ell(\Gamma) - 1 - |N(\Gamma)|$. Note that $n \notin N(\Gamma)$ and that $d_+(\Gamma) = \frac{\Gamma(n)+n}{2}$. Let $\mathcal{L}(n)$ denote the set of all lattice paths of length n. Given $S \subseteq [n-1]$ we let

$$E(S,n) \stackrel{\text{def}}{=} \{\Gamma \in \mathcal{L}(n) : \ N(\Gamma) = S\}.$$

For $\alpha \in C_n$ we define, following [6], a polynomial $\Upsilon_\alpha(q) \in \mathbf{Z}[q]$ by letting

$$\Upsilon_\alpha(q) \stackrel{\text{def}}{=} (-1)^{n-\ell(\alpha)} \sum_{\Gamma \in E(T(\alpha),n)} (-q)^{d_+(\Gamma)}.$$

We follow [21] for undefined Coxeter groups notation and terminology. Given a Coxeter system (W,S) and $w \in W$ we denote by $\ell(w)$ the length of w with respect to S, and we let

$$D_R(w) \stackrel{\text{def}}{=} \{s \in S : \ \ell(w\,s) < \ell(w)\},$$

$D_L(w) \stackrel{\text{def}}{=} \{s \in S : \ \ell(sw) < \ell(w)\} = D_R(w^{-1})$ and $\varepsilon_w \stackrel{\text{def}}{=} (-1)^{\ell(w)}$. We call the elements of $D(w)$ (respectively, $D_L(w)$) the *right* (respectively, *left*) *descents* of w. We denote by e the identity of W, and we let $T \stackrel{\text{def}}{=} \{wsw^{-1} : w \in W, \ s \in S\}$ be the set of *reflections* of W. For $u, v \in W$ we also write $\ell(u,v) \stackrel{\text{def}}{=} \ell(v) - \ell(u)$.

We denote by $B(W)$ the *Bruhat graph* of W. Recall (see, e.g., [21, §8.6], or [14]) that this is the directed graph having W as vertex set and having a directed edge from u to v if and only if $u^{-1}v \in T$ and $\ell(u) < \ell(v)$. The transitive closure of $B(W)$ is a partial order on W that is usually called the *Bruhat order* (see, e.g., [21, §5.9]) and that we denote by $\leq$. Throughout this work, we always assume that W, and its subsets, are partially ordered by $\leq$. There is a well known characterization of Bruhat order on a Coxeter group (usually referred to as the *subword property*) that we will use repeatedly in this work, often without explicit mention. We recall it here for the reader's convenience. By a *subword* of a word $s_1 s_2 \cdots s_q$ we mean a word of the form $s_{i_1} s_{i_2} \cdots s_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq q$.

Theorem 2.2 *Let $u, w \in W$. Then $u \leq w$ if and only if every reduced expression for w has a subword that is a reduced expression for u.*

A proof of the preceding result can be found, e.g., in [21, §5.10]. It is well known that W, partially ordered by Bruhat order, is a graded poset having ℓ as its rank function. Given $v \in W$ and $s \in D_R(v)$ (respectively $s \in D_L(v)$) we define a matching ρ_s (respectively, λ_s) of the Hasse diagram of $[e, v]$ by $\rho_s(u) = us$ (respectively, $\lambda_s(u) = su$) for all $u \leq v$. It then follows easily from the "Lifting Property" (see, e.g., [8, Theorem 1.1], [21, Proposition 5.9] or [4, Proposition 2.2.7]) that ρ_s (resp., λ_s) is a special matching of $[e, v]$. We call a matching M of the Hasse diagram of an interval $[e, v]$ a *multiplication matching* if there exists $s \in S$ such that either $M = \lambda_s$ or $M = \rho_s$.

For $A \subseteq W$ we denote by $< A >$ the subgroup of W generated by A. If $J \subseteq S$ we let $W_J \stackrel{\text{def}}{=} < J >$ and $W^J \stackrel{\text{def}}{=} \{w \in W \,:\, D_R(w) \subseteq S \setminus J\}$. The following result is well known and a proof of it can be found, e.g., in [21].

Proposition 2.3 *Let $J \subseteq S$. Then:*

(i) every $w \in W$ has a unique factorization $w = w^J \cdot w_J$ with $w^J \in W^J$ and $w_J \in W_J$;

(ii) for this factorization: $\ell(w) = \ell(w^J) + \ell(w_J)$.

There are, of course, left versions of the above definitions and results. Namely, if we let

$$^{J}W \stackrel{\text{def}}{=} \{w \in W \,:\, D_L(w) \subseteq S \setminus J\} = (W^J)^{-1}, \tag{1}$$

then every $w \in W$ can be uniquely factorized $w = {}_J w \cdot {}^J w$, where ${}_J w \in W_J$ and ${}^J w \in {}^J W$, and then $\ell(w) = \ell({}_J w) + \ell({}^J w)$. If $J \subseteq S$ and $w \in W$ we let $W_J(w) \stackrel{\text{def}}{=} W_J \cap [e, w]$. It is known (see, e.g., [20, Lemma 7]) that there exists a unique maximal element in $W_J(w)$ that we denote $w[J]$, so that $W_J(w) = [e, w[J]]$.

Let $A \subseteq T$ and $W' \stackrel{\text{def}}{=} < A >$. Following [21], §8.2, we call W' a *reflection subgroup* of W. It is then known (see, e.g., [21], Theorem 8.2) that (W', S') is again a Coxeter system where $S' \stackrel{\text{def}}{=} \{t \in T :\ N(t) \cap W' = \{t\}\}$, and $N(w) \stackrel{\text{def}}{=} \{t \in T :\ \ell(wt) < \ell(w)\}$. We call the elements of S' the *canonical generators* of W'. We say that W' is a *dihedral* *reflection subgroup* if $|S'| = 2$ (i.e., if (W', S') is a dihedral Coxeter system). Following [16] we say that a total ordering $\prec$ of T is a *reflection ordering* if, for any dihedral reflection subgroup W' of W, we have that either $a \prec aba \prec ababa \prec \ldots \prec babab \prec bab \prec b$ or $b \prec bab \prec babab \prec \ldots \prec ababa \prec aba \prec a$ where $\{a, b\} \stackrel{\text{def}}{=} S'$. The existence of reflection orderings is proved in [16], §2. Let $\prec$ be a reflection ordering, and $s \in S$. Define a total ordering $\prec^s$ on T as follows. For $t_1, t_2 \in T$ set $t_1 \prec^s t_2$ if and only if one of the following conditions apply: 1) $t_1 \prec t_2 \prec s$; 2) $t_1, t_2 \succ s$ and $st_1s \prec st_2s$; 3) $t_1 \prec s \prec t_2$; 4) $t_2 = s$. Similarly, we define $\prec_s$ by letting $t_1 \prec_s t_2$ if and only if one of the following conditions is satisfied: 1) $t_1, t_2 \prec s$ and $st_1s \prec st_2s$; 2) $s \prec t_1 \prec t_2$; 3) $t_1 \prec s \prec t_2$; 4) $t_1 = s$. It can be proved (see Proposition 2.5 of [16]) that these orders are still reflection orderings, and that $(\prec_s)^s = \prec^s$.

We denote by $\mathcal{H}(W)$ the *Hecke algebra* associated to W. Recall that this is the free $\mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module having the set $\{T_w : \ w \in W\}$ as a basis and multiplication such that

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + (q-1)T_w, & \text{if } \ell(ws) < \ell(w), \end{cases} \tag{2}$$

for all $w \in W$ and $s \in S$. It is well known that this is an associative algebra having T_e as unity and that each basis element is invertible in $\mathcal{H}(W)$. More precisely, we have the following result (see [21, Proposition 7.4]).

Proposition 2.4 *Let $v \in W$. Then*

$$(T_{v^{-1}})^{-1} = q^{-\ell(v)} \sum_{u \leq v} (-1)^{\ell(u,v)} R_{u,v}(q) \, T_u \,,$$

where $R_{u,v}(q) \in \mathbf{Z}[q]$.

The polynomials $R_{u,v}$ defined by the previous proposition are called the *R-polynomials* of W. It is known that $\deg(R_{u,v}) = \ell(u, v)$, and that $R_{u,u}(q) = 1$, for all $u, v \in W$, $u \leq v$. It is customary to let $R_{u,v}(q) \stackrel{\text{def}}{=} 0$ if $u \not\leq v$. We then have the following fundamental result that follows from (2) and Proposition 2.4 (see [21, §7.5]).

Theorem 2.5 *Let $u, v \in W$ and $s \in D_R(v)$. Then*

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q), & \text{if } s \in D_R(u), \\ qR_{us,vs}(q) + (q-1)R_{u,vs}, & \text{if } s \notin D_R(u). \end{cases} \tag{3}$$

Note that the preceding theorem can be used to inductively compute the R-polynomials since $\ell(vs) < \ell(v)$. There is also a left version of Theorem 2.5. Sometimes it is convenient to use a related family of polynomials with nonnegative integer coefficients. This is introduced in the following, which is a simple consequence of Theorem 2.5.

Proposition 2.6 *Let $u, v \in W$. Then there exists a (necessarily unique) polynomial $\widetilde{R}_{u,v}(q) \in \mathbf{N}[q]$ such that*

$$R_{u,v}(q) = q^{\frac{1}{2}\ell(u,v)}\, \widetilde{R}_{u,v}\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) . \tag{4}$$

We let ι be the canonical involution of $\mathcal{H}(W)$. So for all $P(q) \in \mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ $\iota(P)(q) \stackrel{\text{def}}{=} P(q^{-1})$ and for all $w \in W$ $\iota(T_w) \stackrel{\text{def}}{=} (T_{w^{-1}})^{-1}$. A proof of the following fundamental result can be found, e.g., in [21], Theorem 7.9.

Theorem 2.7 *For each $w \in W$ there exists a unique element $C'_w \in \mathcal{H}(W)$ such that $\iota(C'_w) = C'_w$ and*

$$C'_w = q^{-\frac{\ell(w)}{2}} \sum_{u \leq w} P_{u,w}(q)\, T_u,$$

where $P_{w,w}(q) = 1$ and $P_{u,w}(q) \in \mathbf{Z}[q]$ has degree smaller than $\frac{\ell(u,w)}{2}$ if $u < w$.

We call the basis $\{C'_w \,:\, w \in W\}$ the *Kazhdan-Lusztig basis* of $\mathcal{H}(W)$. The polynomials $P_{u,w}(q)$ defined by the preceding theorem are called the *Kazhdan-Lusztig polynomials* of W. For $u, w \in W$ we let $\overline{\mu}(u,w) \stackrel{\text{def}}{=} [q^{\frac{1}{2}(\ell(u,w)-1)}](P_{u,w}(q))$ if $u < w$ and $\ell(u,w)$ is odd, and $\overline{\mu}(u,w) \stackrel{\text{def}}{=} 0$, otherwise. Kazhdan-Lusztig polynomials have been first defined in [22] and play a prominent role in several branches of mathematics including representation theory (see, e.g., [1], and the references cited there), and algebraic geometry and topology of Schubert varieties (see, e.g., [22], [23], and [2]).

3 A combinatorial property of Bruhat order

In this section we prove a combinatorial property of Bruhat order on a Coxeter group which plays a fundamental role in all that follows. Its proof uses the following lemma which is proved in the same way as Lemma 3.1 of [14], and whose proof we therefore omit.

Lemma 3.1 *Let (W,S) be a Coxeter system and $t_1, \ldots, t_{2n} \in T$ ($n \in \mathbf{P}$) be such that $t_1 t_2 = t_3 t_4 = \ldots = t_{2n-1} t_{2n} \neq e$. Then $W' \stackrel{\text{def}}{=} < \{t_1, \ldots, t_{2n}\} >$ is a dihedral reflection subgroup.*

We can now prove the main result of this section. It immediately implies Proposition 7 of [26].

Theorem 3.2 *Let (W,S) be a Coxeter system and $a, b \in W$ be such that either $|\{w \in W : w \lhd a, w \lhd b\}| \geq 3$ or $|\{w \in W : w \rhd a, w \rhd b\}| \geq 3$. Then $a = b$.*

Proof. We prove the assertion in the first case, the proof for the other one being entirely similar.

Suppose that $a \neq b$ and let $x, y, z \in \{w \in W : w \lhd a, w \lhd b\}$. Let $t_1, \ldots, t_6 \in T$ be such that $at_1 = x$, $at_3 = y$, $at_5 = z$, $bt_2 = x$, $bt_4 = y$, $bt_6 = z$. Then $at_1t_2 = at_3t_4 = at_5t_6 = b$ so $t_1t_2 = t_3t_4 = t_5t_6 \neq e$. This, by Lemma 3.1, implies that $W' \stackrel{\text{def}}{=} < \{t_1, \ldots, t_6\} >$ is a dihedral reflection subgroup. Clearly, $a, b, x, y, z \in aW'$. But, by Theorem 1.4 of [14], the subgraph of the Bruhat graph of W with vertex set aW' is isomorphic, as a directed graph, to the Bruhat graph of W' (considered as an abstract Coxeter system), which is a contradiction since W' is a dihedral Coxeter system, and x, y, z are incomparable. Hence $a = b$, as desired. $\Box$

The following result, though already known (see [12, Theorem 2.4]), is a direct consequence of Theorem 3.2, and will be used in the sequel. We call an interval $[u, v]$ in a poset P *dihedral* if it is isomorphic to a finite Coxeter system of rank ≤ 2 ordered by Bruhat order.

Corollary 3.3 *Let (W, S) be a Coxeter system, and u, $v \in W$. Suppose that $|\{z \in [u, v] : z \lhd v\}| = 2$. Then $[u, v]$ is a dihedral interval.*

Proof. It is well known that, for all $x, y \in W$ such that $y \leq x$ and $\ell(y, x) = 2$, $[y, x]$ is a Boolean algebra of rank 2. Using this and Theorem 3.2 it is easy to prove, by induction on i, that $|\{w \in [u, v] : \ell(w, v) = i\}| = 2$ for all $i \in [\ell(u, v) - 1]$, as desired. $\Box$

4 Pairs of special matchings

In this section we prove some combinatorial properties of pairs of special matchings which are needed in what follows. More precisely, since a matching is an application from the set of vertices of a graph to itself, we can compose special matchings as functions. Given two special matchings, M and N, we look at the structure of the orbits of $\langle M, N \rangle$, the group generated by M and N. Most of the results in this section hold for any graded poset.

For $x \in P$ we denote by $\langle M, N \rangle(x)$ the orbit of x under the action of $\langle M, N \rangle$. We begin with the following simple but fundamental observation.

Lemma 4.1 *Let P be a finite graded poset, and M and N be two special matchings of P. Then the orbit $\langle M, N \rangle(u)$ of any $u \in P$ is a dihedral interval.*

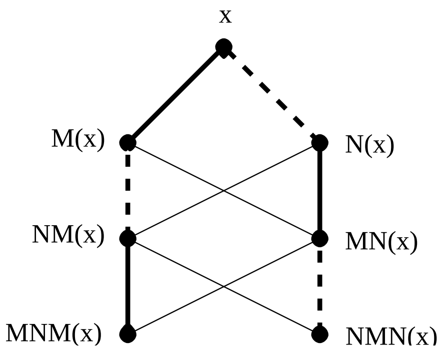

Figure 2: The orbits $\langle M, N\rangle(u)$ are dihedral intervals

Proof. Since P is finite, the orbit $\langle M, N\rangle(u)$ is also finite. Therefore there exists $x \in \langle M, N\rangle(u)$ such that $M(x) \lhd x$ and $N(x) \lhd x$. If $M(x) = N(x)$ then $\langle M, N\rangle(u) = \{x, M(x)\}$ and we are done. Else, by the definition of a special matching we have that $NM(x) \lhd M(x)$, $NM(x) \lhd N(x)$, $MN(x) \lhd N(x)$, and $MN(x) \lhd M(x)$. If $MN(x) = NM(x)$ then $\langle M, N\rangle(u) = \{x, N(x), M(x), NM(x)\}$ and we are done. Otherwise we conclude, similarly, that $MNM(x) \lhd NM(x)$, $MNM(x) \lhd MN(x)$, $NMN(x) \lhd MN(x)$, and $NMN(x) \lhd NM(x)$ (see Figure 2). If $MNM(x) = NMN(x)$ then we are done, else we continue in this way. Since $\langle M, N\rangle(u)$ is finite there exists $l \in \mathbf{P}$ such that $\underbrace{MNM\ldots}_{l}(x) = \underbrace{NMN\ldots}_{l}(x)$ and the result follows. $\square$

We say that a graded poset P *avoids* $K_{3,2}$ if there are no elements $a_1, a_2, a_3, b_1, b_2 \in P$, all distinct, such that either $a_i \lhd b_j$ for all $i \in [3]$, $j \in [2]$ or $a_i \rhd b_j$ for all $i \in [3]$, $j \in [2]$. So, for example, a Coxeter group under Bruhat order avoids $K_{3,2}$ by Theorem 3.2.

Proposition 4.2 *Let P be a finite graded poset that avoids $K_{3,2}$, $v \in P$, and M and N be two special matchings of P such that $M(v) \neq N(v)$. Let $v' \in P \setminus \{M(v), N(v)\}$ and suppose that either*

i) $M(v) \lhd v$, $N(v) \lhd v$ and $v' \lhd v$, or

ii) $M(v) \rhd v$, $N(v) \rhd v$ and $v' \rhd v$.

Then

$$|\langle M, N\rangle(v)| = |\langle M, N\rangle(v')|.$$

Proof. We prove the statement only in case i), case ii) being similar. Suppose that $|\langle M, N\rangle(v)| = 2n$, $|\langle M, N\rangle(v')| = 2m$. Note that, since $v' \notin \{M(v), N(v)\}$, $\langle M, N\rangle(v) \cap \langle M, N\rangle(v') = \emptyset$. Therefore, no element of $\langle M, N\rangle(v)$ is matched by either M or N to an element of $\langle M, N\rangle(v')$. This, by the definition of a special matching, and a simple

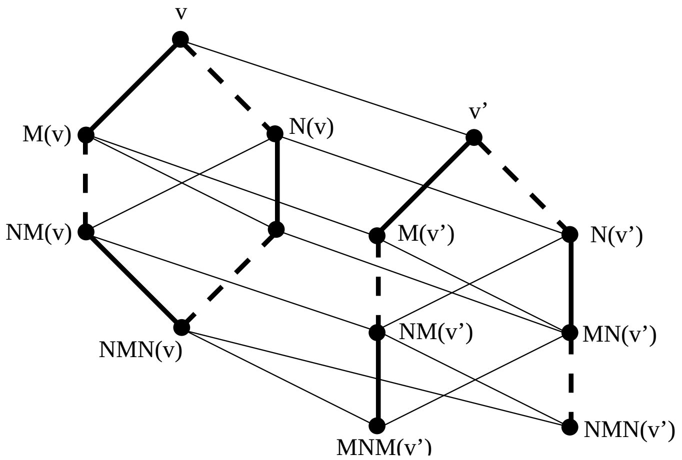

Figure 3: The case $n = 3$ and $m > n$

induction on k, implies that

$$\underbrace{MNM\cdots}_{k}(v') \lhd \underbrace{MNM\cdots}_{k}(v) \quad , \quad \underbrace{MNM\cdots}_{k}(v') \lhd \underbrace{NMN\cdots}_{k-1}(v'),$$

and

$$\underbrace{NMN\cdots}_{k}(v') \lhd \underbrace{NMN\cdots}_{k}(v) \quad , \quad \underbrace{NMN\cdots}_{k}(v') \lhd \underbrace{MNM\cdots}_{k-1}(v'),$$

for all $k \in [n]$. Therefore, $m \geq n$. If $m > n$, then $\underbrace{MNM\cdots}_{n}(v') \neq \underbrace{NMN\cdots}_{n}(v')$. But $\underbrace{MNM\cdots}_{n}(v) = \underbrace{NMN\cdots}_{n}(v)$, and this contradicts the fact that P avoids $K_{3,2}$ (see Figure 3). $\square$

We now restrict our attention to the case where P is an interval of the form $[e, v]$, with $v \in W$. In this case we often refer to a special matching of $[e, v]$ simply as a special matching of v.

The following is the main result of this section.

Lemma 4.3 *Let $u, v \in W$, $u \leq v$ and M and N be two special matchings of v. If $|\langle M, N\rangle(u)| = 2m > 2$, then there exists u' and a dihedral interval I such that $e, M(e), N(e) \in I$, $|\langle M, N\rangle(u')| = 2m$ and $\langle M, N\rangle(u') \subseteq I$. In particular, if $M(e) \neq N(e)$, then $W_{\{M(e),N(e)\}}$ contains an orbit of cardinality 2m.*

Proof. Without loss of generality we may assume that $M(u), N(u) \lhd u$. We claim that we can find a sequence $u = u_1 \rhd u_2 \rhd \cdots \rhd u_k$ such that $M(u_i), N(u_i) \lhd u_i$, $|\langle M, N\rangle(u_i)| = 2m$ for all $i \in [k]$, and $[e, u_k]$ is a dihedral interval. In fact if $\{z \in [e, u] : z \lhd u\} = \{M(u), N(u)\}$ then we are done by Corollary 3.3. Otherwise let $u_2 \in \{z \in [e, u] : z \lhd u\} \setminus \{M(u), N(u)\}$. Then, by Proposition 4.2, $|\langle M, N\rangle(u_2)| = 2m$

and $M(u_2) \lhd u_2$, $N(u_2) \lhd u_2$. If $\{z \in [e, u_2] : z \lhd u_2\} = \{M(u_2), N(u_2)\}$ then our claim is proved. Otherwise let $u_3 \in \{z \in [e, u_2] : z \lhd u_2\} \setminus \{M(u_2), N(u_2)\}$ and continue as above. This proves our claim, and the result follows. $\square$

5 Algebraic properties of special matchings

In this section we establish some algebraic properties of special matchings of Coxeter groups that are needed in the proof of our main result.

Lemma 5.1 *Let $u, w \in W$, $u \leq w$ and M be a special matching of w. Suppose that $u \notin \bigcup_{t \in S} W_{\{t, M(e)\}}$, and that $M(u) \rhd u$. Then*

$$|\{x \in [e, u] : \; x \lhd u \;\; and \;\; M(x) \rhd x\}| \geq 2. \tag{5}$$

Proof. By Lemma 2.1, given an element v with $v \rhd M(v)$, M restricts to a special matching of $[e, v]$. In particular $M(e) \leq v$. Hence, if $M(e) \not\leq u$, then $M(x) \rhd x$ for all $x \in [e, u]$, and the assertion is proved.

If $M(e) \leq u$ then, by our hypotheses, the interval $[e, u]$ is not dihedral and, in particular, $[e, M(u)]$ has at least two coatoms distinct from u, say x_1 and x_2. Then, by the definition of a special matching, $M(x_i) \lhd x_i$ and $M(x_i) \lhd u$ for $i = 1, 2$, and (5) follows. $\square$

The next result is a fundamental tool in our proof.

Lemma 5.2 *Let $u, w \in W$, $u \leq w$ and M be a special matching of w. Suppose that*

$$M(x) = x\, s$$

for all $x \in \bigcup_{t \in S} W_{\{s,t\}}(u)$, where $s \stackrel{\text{def}}{=} M(e)$. Then $M(u) = u\, s$.

Proof. We proceed by induction on $\ell(u)$ the statement being trivial if $\ell(u) = 0$. We may assume that $M(u) \rhd u$, else the statement follows by induction. Furthermore, we may clearly assume that $u \notin \bigcup_{t \in S} W_{\{s,t\}}$. Hence, by Lemma 5.1, there exist two distinct elements u_1 and u_2 such that $u_i \lhd u$ and $M(u_i) \rhd u_i$, for $i = 1, 2$. By our induction hypothesis $M(u_i) = u_i s$, for $i = 1, 2$. Therefore $u\, s$ covers $u, M(u_1)$ and $M(u_2)$ and, by the definition of a special matching, $M(u)$ also covers $u, M(u_1)$ and $M(u_2)$. Hence $M(u) = u\, s$ by Theorem 3.2.$\square$

Note that the reasoning used to prove Lemma 5.2 also proves that if M and N are two special matchings of w and $M(x) = N(x)$ for all $x \in \bigcup_{t \in S} W_{\{s,t\}}(u)$, where $s = M(e)$, then $M(u) = N(u)$.

The next "invariance" property relating special matchings and parabolic subgroups will be used often in the sequel.

Proposition 5.3 *Let $w \in W$ and M be a special matching of w. Then, for all $J \subseteq S$ such that $M(e) \in J$, M stabilizes $W_J(w)$.*

Proof. We prove that $u \in W_J(w)$ implies $M(u) \in W_J(w)$ by induction on $\ell(u)$, this being trivial if $\ell(u) = 0$. We may clearly assume that $M(u) \rhd u$. Let $x \lhd M(u)$, $x \neq u$. Then $M(x) \lhd u$ and by our induction hypothesis $x \in W_J(w)$. Hence all the coatoms of $[e, M(u)]$ are in $W_J(w)$, so $M(u) \in W_J(w)$. $\square$

We conclude this section with a result which shows that if an element $w \in W$ has a special matching which is not a multiplication matching on the atoms of $[e, w]$ then w must satisfy certain constraints.

Lemma 5.4 *Let $w \in W$, M be a special matching of w, $s \stackrel{\text{def}}{=} M(e)$, and $r, t \in S$. Suppose that $M(t) = ts \neq st$ and $M(r) = sr \neq rs$. Then $rst \not\leq w$. Furthermore, if $rt \neq tr$, then $rt \not\leq w$.*

Proof. Suppose $rt \leq w$. Then, by the definition of special matching, $M(rt) \rhd rt$, $M(rt) \rhd ts$ and $M(rt) \rhd sr$. If $rt \neq tr$ there are no such elements and this proves the second part of the statement. If $rt = tr$ then necessarily $M(rt) = tsr$. If $rst \leq w$ then $M(rst)$ would cover both tsr and rst and there are clearly no such elements. $\square$

6 Coxeter systems of rank 3

In this section we study special matchings in Coxeter systems of rank 3. These results are applied in the next section to rank three parabolic subgroups of general Coxeter systems.

Throughout this section (W, S) is a Coxeter system of rank 3, and $S \stackrel{\text{def}}{=} \{s, r, t\}$. We fix $w \in W$, a special matching M of w and we assume that $M(e) = s$.

For $x, y \in S$ we denote by $\cdots xyx$ (respectively $xyx \cdots$) a word given by alternating x and y that ends (respectively begins) with x. Inside any single proof, if the length of such a word is not specified, it is assumed to be arbitrary but fixed. The expressions considered for an element of a Coxeter system are always assumed to be reduced.

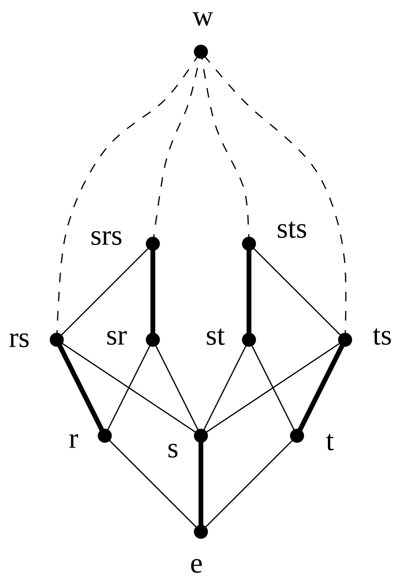

Figure 4: Proof of Lemma 6.1

Lemma 6.1 *If $sr, st \leq w$, $rs \neq sr$, $st \neq ts$, $M(t) = ts$ and $M(r) = rs$, then $M(st) = sts$ and $M(sr) = srs$.*

Proof. By symmetry it suffices to show that $M(st) = sts$. By definition of a special matching $M(st) \rhd st$ and $M(st) \rhd ts$, so $M(st) \in \{sts, tst\}$. Similarly, $M(sr) \in \{srs, rsr\}$. Suppose $M(st) = tst$. If $str \leq w$ then $M(str) \rhd tst$ and $M(str) \rhd M(sr)$. But there are no elements covering both tst and $M(sr)$, so $str \not\leq w$. Similarly $srt \not\leq w$. Now consider a reduced expression for w. Then tst and either srs or rsr are both subwords of it and it is easy to see that these conditions force that either str or srt is also a subword, contradicting the fact that $str \not\leq w$ and $srt \not\leq w$.$\Box$

The next technical result is used repeatedly in what follows.

Lemma 6.2 *Suppose $M(t) = ts$ and $M(r) = rs$, but $M \neq \rho_s$ on $W_{\{s,t\}}(w)$. Let x_0 be a minimal element of $W_{\{s,t\}}(w)$ such that $M(x_0) \neq x_0 s$. Then*

$$\{u \leq w : \; u \rhd x_0, \; u \notin W_{\{s,t\}}\} \subseteq \begin{cases} \{x_0 r, r x_0\}, & \text{if } sr = rs, \\ \{r x_0\}, & \text{if } sr \neq rs. \end{cases}$$

Proof. Clearly, $s \notin D_R(x_0)$ and $M(x_0) \rhd x_0$. Let $x_0 = \underbrace{\alpha\beta\alpha\cdots tst}_{k}$ where $\alpha = s$ if k is even, $\alpha = t$ if k is odd and $\{\alpha, \beta\} = \{s, t\}$. Since $M \neq \rho_s$ on $W_{\{s,t\}}(w)$ we conclude that $st \leq w$ and $st \neq ts$. Let u be such that $u \leq w$, $u \rhd x_0$, and $u \notin W_{\{s,t\}}$ and assume $u \notin \{x_0 r, r x_0\}$ if $sr = rs$ and $u \neq r x_0$ if $sr \neq rs$. So u is obtained by inserting a letter r in the unique reduced expression of x_0.

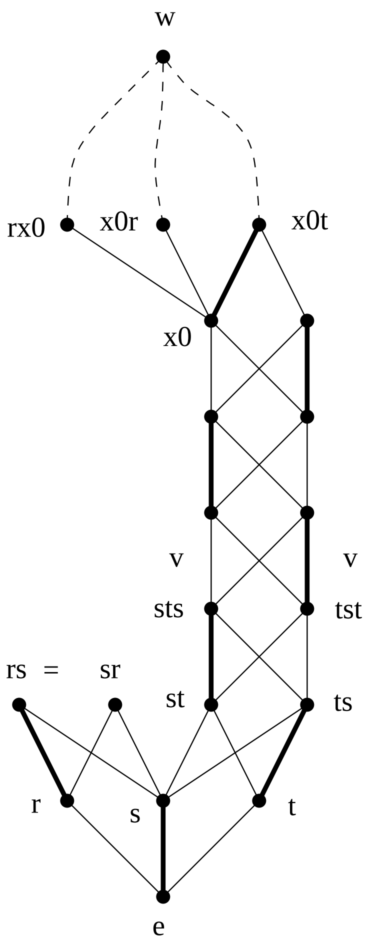

Figure 5: Proof of Lemma 6.2

Let $y \stackrel{\text{def}}{=} \alpha u$. Then $y \lhd u$, hence the elements in $W_{\{s,t\}}(y)$ are all strictly smaller than x_0. Furthermore, the elements in $W_{\{s,r\}}(y)$ are all $\leq srs$ if $sr \neq rs$ or $\leq sr$ if $sr = rs$. Hence, by Lemmas 5.2 and 6.1, $M(y) = ys$. Since x_0 and y are both covered by u, $M(u) \rhd u$, $M(u) \rhd M(x_0) = \underbrace{\beta\alpha\beta \cdots tst}_{k+1} \neq \underbrace{\alpha\beta\alpha \cdots sts}_{k+1}$ and $M(u) \rhd M(y)$. Then it is not difficult to see that these two last conditions force $M(u) = yst$ which is a contradiction since, as one can verify, $yst \not\rhd u$.□

In what follows we will often consider three distinct sets of hypotheses. For convenience and brevity we list them here.

(1) $M(t) = ts \neq st$, $M(r) = rs \neq sr$ and $M \neq \rho_s$ on $W_{\{s,t\}}(w)$.

(2) $M(t) = ts \neq st$, $M(r) = rs = sr$ and $M \neq \rho_s$ on $W_{\{s,t\}}(w)$.

(3) $M(t) = ts \neq st$, $M(r) = sr \neq rs$.

Under hypotheses (1) and (2) we let $x_0 \in W_{\{s,t\}}(w)$ be the unique minimal element of $W_{\{s,t\}}(w)$ such that $M(x_0) \neq x_0 s$ and $\alpha\beta\alpha \ldots tst$ be its unique reduced expression (note that $s \notin D_R(x_0)$).

Proposition 6.3 *Under the hypotheses (1) any element $u \leq w$ has a reduced expression of the form $(\cdots r\beta r)\eta(\alpha\beta\alpha\cdots)$, where $\eta \in \{e, \beta\}$.*

Under the hypotheses (2) any element $u \leq w$ has a reduced expression of the form $(\cdots r\beta r)\eta(\alpha\beta\alpha\cdots)\delta$, where $\eta \in \{e, \beta\}$and $\delta \in \{e, r\}$.

Under the hypotheses (3) any element $u \leq w$ has a reduced expression of the form $(\cdots tst)\varepsilon(rsr\cdots)$, where $\varepsilon \in \{e, s\}$.

Proof. It is clear that in all cases it is enough to prove the statement for $u = w$, the general result following by the subword property.

(1) Let $\alpha\beta\alpha\cdots tst$ be a subword of a reduced expression of w such that $\alpha\beta\alpha\cdots tst = x_0$, with the first α chosen as left as possible and the last t chosen as right as possible. Consider the leftmost r that appears right of the first α of this subword. By Lemma 6.2, no s can appear to the left of this r, and $tr = rt$. Hence we obtain a reduced expression for w where no r appears after the first letter α and the thesis follows.

(2) If $tr = rt$ then the result is clear. If $tr \neq rt$ then reasoning as in the previous case we conclude that either no t appears to the left of this r or no t appears to its right, and the result again follows.

(3) Consider a reduced expression for w and look at the rightmost letter t and at the leftmost letter r of this reduced expression. If this t appears to the left of this r we are done. Otherwise, by Lemma 5.4, there cannot be a letter s between them and $rt = tr$. So these two letters are adjacent and the result follows.□

We can now prove one of the main results of this section. We say that an element $w \in W$ is *dihedral* if the interval $[e, w]$ is a dihedral interval.

Theorem 6.4 *Let (W, S) be a Coxeter system of rank 3, $w \in W$, M be a special matching of w, and $s \stackrel{\text{def}}{=} M(e)$. Then there exists $x \in S \setminus \{s\}$ such that either $M = \lambda_s$ or $M = \rho_s$ on $W_{\{s,x\}}(w)$.*

Proof. We may clearly assume that w is not dihedral, that M is not a multiplication matching and, by Proposition 5.3, that

$$4 \notin \{|W_{\{r,s\}}(w)|, |W_{\{t,s\}}(w)|\}. \tag{6}$$

In particular, $rs \neq sr$ and $ts \neq st$.

Note that the result is true for a special matching M of w if and only if it is true for the special matching $\tilde{M}$ of w^{-1} defined by $\tilde{M}(x) \stackrel{\text{def}}{=} (M(x^{-1}))^{-1}$, for all $x \leq w^{-1}$. If

$M(r) = rs$ and $M(t) = ts$ then, by Lemma 5.2, $M \neq \rho_s$ on $W_{\{s,t\}}(w) \cup W_{\{s,r\}}(w)$ so M satisfies the hypotheses (1) (possibly by exchanging the roles of r and t). If $M(r) = sr$ and $M(t) = st$ then $\tilde{M}$ satisfies the hypotheses (1). If $M(r) = sr$ and $M(t) = ts$ M satisfies the hypotheses (3). If $M(r) = rs$ and $M(t) = st$ then $\tilde{M}$ satisfies the hypotheses (3). So we only need to consider two cases.

If M is in case (1) we have that $\beta = s$ otherwise, by Proposition 6.3, $W_{\{r,s\}}(w) = \{e, s, r, rs\}$ and this is not possible by (6). By contradiction, suppose that $M \neq \rho_s$ on $W_{\{r,s\}}(w)$, and let $y_0 \in W_{\{r,s\}}(w)$ be a minimal element such that $M(y_0) \neq y_0 s$. Then, since w is not dihedral, $y_0 t \leq w$ by Proposition 6.3. This, by Lemma 6.2, implies that $y_0 t = t y_0$, which is a contradiction since $ts \neq st$.

If M is in case (3) we claim that either $M = \rho_s$ on $W_{\{t,s\}}(w)$ or $M = \lambda_s$ on $W_{\{r,s\}}(w)$. We prove this by induction on $\ell(w)$. By Proposition 6.3 $w = (\underbrace{\cdots tst}_{k})\varepsilon(\underbrace{rsr\cdots}_{h})$ (this being a reduced expression) where $\varepsilon \in \{e, s\}$. By (6) we have that $h, k \geq 2$. Let w_1 and w_2 be the two coatoms of $[e, w]$ obtained by deleting, respectively, the first and the last letter of this reduced expression of w. By definition of a special matching, there exists $i \in \{1, 2\}$ such that M restricts to a special matching of $[e, w_i]$. We assume $i = 1$ the case $i = 2$ being similar. By our induction hypothesis either $M = \rho_s$ on $W_{\{t,s\}}(w_1)$ or $M = \lambda_s$ on $W_{\{r,s\}}(w_1)$. In this second case k is odd and we are done since $W_{\{r,s\}}(w_1) = W_{\{r,s\}}(w)$. If $M = \rho_s$ on $W_{\{t,s\}}(w_1)$ then $W_{\{t,s\}}(w) \setminus W_{\{t,s\}}(w_1) = \{\underbrace{\cdots tst}_{k}, \underbrace{\cdots sts}_{k+1}\}$ and since, by Proposition 5.3, M stabilizes $W_{\{t,s\}}(w)$ we necessarily have $M(\underbrace{\cdots tst}_{k}) = \underbrace{\cdots sts}_{k+1}$ and hence $M = \rho_s$ on $W_{\{t,s\}}(w)$.$\square$

The next result describes how M acts on $[e, w]$, under hypotheses (1), (2) and (3).

Proposition 6.5 *Under the hypotheses (1) if $u \leq w$, $u = (\cdots r\beta r)\eta(\alpha\beta\alpha\cdots)$ where $\eta \in \{e, \beta\}$ and $\beta \notin D_R(\cdots r\beta r)$, then $M(u) = (\cdots r\beta r)M(\eta\alpha\beta\alpha\cdots)$.*

Under the hypotheses (2) if $u \leq w$, $u = (\cdots r\beta r)\eta(\alpha\beta\alpha\cdots)\delta$ where $\eta \in \{e, \beta\}$, $\delta \in \{e, r\}$ and $\beta \notin D_R(\cdots r\beta r)$, then $M(u) = (\cdots r\beta r)M(\eta\alpha\beta\alpha\cdots)\delta$.

Under the hypotheses (3) if $u \leq w$, $u = (\cdots tst)\varepsilon(rsr\cdots)$ where $\varepsilon \in \{e, s\}$ and $s \notin D_L(rsr\cdots)$, then $M(u) = M(\cdots tst)\varepsilon(rsr\cdots)$.

Proof. (1) We proceed by induction on $\ell(u)$ the case $\cdots r\beta r = e$ being trivial and the case $\eta\alpha\beta\alpha\cdots = e$ following by Lemma 5.2 if $\beta = t$ and by our hypotheses and Theorem 6.4 if $\beta = s$.

So suppose that $\cdots r\beta r \neq e$ and $\eta\alpha\beta\alpha\cdots \neq e$. If $M(\eta\alpha\beta\alpha\cdots) \lhd \eta\alpha\beta\alpha\cdots$ then $(\cdots r\beta r)M(\eta\alpha\beta\alpha\cdots) \lhd (\cdots r\beta r)\eta\alpha\beta\alpha\cdots$ (since $\beta \notin D_R(\cdots r\beta r)$) hence, by our induction hypothesis, $M(\cdots r\beta r M(\eta\alpha\beta\alpha\cdots)) = (\cdots r\beta r)\eta\alpha\beta\alpha\cdots = u$ and the result follows. So we may assume that $M(\eta\alpha\beta\alpha\cdots) \rhd \eta\alpha\beta\alpha\cdots$. Now let $x \in D_L(\cdots r\beta r)$. Then $xu \lhd u$ and by our induction hypothesis $M(xu) = x(\cdots r\beta r)M(\eta\alpha\beta\alpha\cdots)$, so $xu \lhd M(xu)$, $u \lhd M(u)$ and $M(xu) \lhd M(u)$. Now let v be the unique element such that $v \lhd \eta\alpha\beta\alpha\cdots$ and $M(v) \rhd v$. Then $(\cdots r\beta r)v \lhd u$ and $M(\cdots r\beta rv) = (\cdots r\beta r)M(v)$ by our induction hypothesis. Since $(\cdots r\beta r)M(\eta\alpha\beta\alpha\cdots)$ covers u, $M(xu)$ and $M(\cdots r\beta rv) = (\cdots r\beta r)M(v)$ and these three elements are distinct, we necessarily have $M(u) = (\cdots r\beta r)M(\eta\alpha\beta\alpha\cdots)$.

(2) We proceed by induction on $\ell(u)$. We may again assume that $M(\eta\alpha\beta\alpha\cdots) \rhd \eta\alpha\beta\alpha\cdots$ else the statement follows by induction.

Suppose first that $\cdots r\beta r = e$. Then we may assume $\delta = r$ and $\eta\alpha\beta\alpha\cdots \neq e$ else the result is trivial. So, if we define v as in case (1), we have that $v \lhd vr \lhd u$ and $\eta\alpha\beta\alpha\cdots \lhd u$. Hence $M(v) \lhd M(\eta\alpha\beta\alpha\cdots)$ and $M(vr) \rhd M(v)$. Therefore, by the definition of a special matching, $M(vr), u, M(\eta\alpha\beta\alpha\cdots) \lhd M(u)$. On the other hand, $M(\eta\alpha\beta\alpha\cdots)r \rhd u$, $M(\eta\alpha\beta\alpha\cdots)$, $M(vr)$ (since $M(vr) = M(v)r$ by induction), so $M(u) = M(\eta\alpha\beta\alpha\cdots)r$ by Theorem 3.2.

If $\cdots r\beta r \neq e$ and $\eta\alpha\beta\alpha\cdots = e$ the result follows from Lemma 5.2 and if $\cdots r\beta r \neq e$ and $\eta\alpha\beta\alpha\cdots \neq e$ the proof is similar to case (1).

(3) This is very similar to case (1) and is therefore omitted. □

We can now prove the second main result of this section.

Proposition 6.6 *Under the hypotheses (1) write* $w = (\underbrace{\cdots r\beta r}_{h})\eta(\alpha\beta\alpha\cdots)$, *with* $\eta \in \{e, \beta\}$ *and* $\beta \notin D_R(\cdots r\beta r)$. *If* $h \geq 2$ *and* $\beta \in D_L(w)$, *then* $M\lambda_\beta = \lambda_\beta M$.

Under the hypotheses (2) write $w = (\underbrace{\cdots r\beta r}_{h})\eta(\alpha\beta\alpha\cdots)\delta$, *with* $\eta \in \{e, \beta\}$, $\delta \in \{e, r\}$ *and* $\beta \notin D_R(\cdots r\beta r)$. *If* $h \geq 2$ *and* $\beta \in D_L(w)$, *then* $M\lambda_\beta = \lambda_\beta M$.

Under the hypotheses (3) write $w = (\cdots tst)\varepsilon(\underbrace{rsr\cdots}_{h})$, *with* $\varepsilon \in \{e, s\}$ *and* $s \notin D_L(rsr\cdots)$. *If* $h \geq 2$ *and* $s \in D_R(w)$, *then* $M\rho_s = \rho_s M$.

Proof. By Lemma 4.3, we know that two special matchings M and N of w commute if and only if they do inside the dihedral intervals containing $M(e)$ and $N(e)$.

Since, by Theorem 6.4, $M = \rho_s$ on $W_{\{r,s\}}(w)$ it is clear from Proposition 6.5 that $M\lambda_\beta = \lambda_\beta M$ on $W_{\{r,s\}}(w)$. So we only have to show that $M\lambda_\beta = \lambda_\beta M$ on $W_{\{t,s\}}(w)$.

Let $u \stackrel{\text{def}}{=} \underbrace{\beta\alpha\beta\cdots}_{k} \in W_{\{t,s\}}(w)$. We claim that if $M(u) \rhd u$ then $M(u) = \underbrace{\beta\alpha\beta\cdots}_{k+1}$. In fact, consider $v \stackrel{\text{def}}{=} \beta r \underbrace{\alpha\beta\alpha\cdots}_{k-1}$. It is clear that $u \lhd v \leq w$. By Proposition 6.5

we have that $M(v) = \beta r M(\underbrace{\alpha\beta\alpha\cdots}_{k-1})$. Since, by the definition of a special matching, $M(v) \rhd M(u)$ we necessarily have $M(\underbrace{\alpha\beta\alpha\cdots}_{k-1}) \rhd \underbrace{\alpha\beta\alpha\cdots}_{k-1}$. By Proposition 5.3, $M(\underbrace{\alpha\beta\alpha\cdots}_{k-1}), M(u) \in W_{\{s,t\}}(w)$, so $M(u) = \underbrace{\beta\alpha\beta\cdots}_{k+1}$.

Now consider an orbit of $\langle M, \lambda_\beta \rangle$ inside $W_{\{s,t\}}(w)$ of cardinality greater than 2. Let z be the smallest element of this orbit, say $z = \underbrace{\alpha\beta\alpha\cdots}_{k-1}$. Then $\lambda_\beta(z) = \underbrace{\beta\alpha\beta\cdots}_{k}$, forcing $M(z) = \underbrace{\alpha\beta\alpha\cdots}_{k}$. Then by our claim $M(\lambda_\beta(z)) = \underbrace{\beta\alpha\beta\cdots}_{k+1} = \lambda_\beta(M(z))$, so $|\langle M, \lambda_\beta \rangle(z)| = 4$.

The proof of case (3) is very similar and is therefore omitted. $\Box$

7 Main result

In this section we prove the main result of this work. More precisely, we describe explicitly all the special matchings of any (element of any) Coxeter system and deduce from this that Kazhdan-Lusztig and R-polynomials can be computed using special matchings. Throughout this section (W, S) is a fixed, but arbitrary, Coxeter system.

We begin with the following immediate consequence of Proposition 5.3 and Theorem 6.4.

Lemma 7.1 *Let $w \in W$, M be a special matching of w and $s = M(e)$. Then there exists at most one $x \in S$ such that $M \neq \lambda_s$ and $M \neq \rho_s$ on $W_{\{s,x\}}(w)$.*

Proof. Suppose there are two such elements, say t and r. By Proposition 5.3, M restricts to a special matching of $[e, w[\{s, r, t\}]]$, and this contradicts Theorem 6.4.$\Box$

The next technical lemma is used in the proof of Proposition 7.3.

Lemma 7.2 *Let $w \in W$, M be a special matching of w and $s = M(e)$. Let $t, r \in S$ be such that $M(t) = ts \neq st$ and $M(r) = sr \neq rs$ and let $k_1, \ldots, k_p \in S \setminus \{s\}$ ($p \in \mathbf{N}$) be such that $k_j s = s k_j$ for $j \in [p]$. Suppose that $r k_1 \cdots k_p t \leq w$ and $\ell(r k_1 \cdots k_p t) = p + 2$. Then there exist $h_1, \ldots, h_p \in S$ and $i \in [0, p]$ such that $r k_1 \cdots k_p t = h_1 \cdots h_i t r h_{i+1} \cdots h_p$.*

Proof. By Proposition 5.3 and Lemma 5.4 (applied to the interval $[e, w[\{s, r, t\}]]$), we have that $tr = rt$, so the result holds if $p = 0$.

We proceed by induction on p. Let $u \stackrel{\text{def}}{=} r k_1 \cdots k_p t$. It suffices to show that either $D_L(u) \neq \{r\}$ or $D_R(u) \neq \{t\}$, the result then following by induction on p. It is clear that $k_1 \cdots k_p t \lhd u$. Furthermore, by Lemma 5.2, $M(k_1 \cdots k_p t) = k_1 \cdots k_p ts$. Similarly $M(rk_1 \cdots k_p) = srk_1 \cdots k_p$. Therefore, since M is a special matching, $M(u) \rhd u$, $k_1 \cdots k_p ts$, $srk_1 \cdots k_p$. If r is the unique left descent of u and t is its unique right descent then necessarily either $r \in D_L(M(u))$ or $t \in D_R(M(u))$ (or both). Suppose $r \in D_L(M(u))$ the other case being similar. Since $r \not\leq k_1 \cdots k_p ts$ and $M(u) \rhd k_1 \cdots k_p ts$ we have $M(u) = rk_1 \cdots k_p ts$. Now, since $rk_1 \cdots k_p ts \rhd srk_1 \cdots k_p$ and $t \not\leq srk_1 \cdots k_p$ we have $rk_1 \cdots k_p s = srk_1 \cdots k_p$, which implies $sr = rs$ and this is a contradiction.$\square$

Given $w \in W$, a special matching M of w, and $s \stackrel{\text{def}}{=} M(e)$ we let

$$J \stackrel{\text{def}}{=} \{r \in S' : \ M(r) = sr\}$$

and

$$J' \stackrel{\text{def}}{=} \{r \in J : \ rs \neq sr\},$$

so that

$$S' \setminus J' = \{r \in S' : \ M(r) = rs\},$$

where $S' \stackrel{\text{def}}{=} \{r \in S : r \leq w\}$.

Proposition 7.3 *Let $u \leq w$. Then $u^J \in W_{S \setminus J'}$.*

Proof. Fix a reduced expression of u^J. Suppose, by contradiction, that $\{r \in S : \ r \leq u^J\} \cap J' \neq \emptyset$. Consider the rightmost letter of J' appearing in this expression, say r. Then consider the first letter $t \notin J$ after r. Between r and t there cannot be any s by Lemma 5.4, and there can only be letters commuting with s. By Lemma 7.2 after a finite number of steps we find a reduced expression of u^J that ends with a letter in J which is a contradiction.$\square$

Proposition 7.4 *Let $t \in S$ be such that M is not a multiplication matching on $W_{\{s,t\}}(w)$. Suppose that $M(t) = ts$ and let $x_0 = \alpha\beta\alpha\cdots$ be the minimal element in $W_{\{s,t\}}(w)$ such that $M(x_0) \neq x_0 s$. Then $\alpha \not\leq (u^J)^{\{s,t\}}$ for all $u \leq w$.*

Proof. It is clearly enough to prove the statement for $u = w$ since if $u \leq w$ then $(u^J)^{\{s,t\}} \leq (w^J)^{\{s,t\}}$. Note first that $s \notin D_R(x_0)$, so $x_0 = \alpha\beta\alpha\cdots tst$, and $x_0 = x_0^J \leq w^J$. Consider a reduced expression for w^J and a subword of this expression of the form $\alpha\beta\alpha\cdots tst$, chosen with the leftmost α and the rightmost t. Consider the first letter r which appears after the first α distinct from s and t. Then, by Lemma 6.2, either this letter can be "pushed" to the left of the first α, or it appears after the last t. So we may assume that the first such letter r appears after the last t. By Lemma 6.2, all the letters that appear after the last t necessarily belong to J. So w^J has a reduced expression in which after the first letter α there are only letters s and t and this clearly implies the statement.$\square$

In the next result we use the geometric representation of (W, S) (see, e.g., [21, §5.3]). We denote by α_r the positive root corresponding to an element $r \in T$.

Lemma 7.5 *Let $t \in S$ be such that $M(t) = ts$ but $M \neq \rho_s$ on $W_{\{t,s\}}(w)$, and $u \leq w$. Then*

$$(u^J)^{\{s,t\}} \underbrace{(\cdots tst)}_{k} \in W^J,$$

for all $1 < k < m(s,t)$.

Proof. Let $r \in J$. We wish to show that

$$\ell((u^J)^{\{s,t\}} \cdots tstr) > \ell((u^J)^{\{s,t\}} \cdots tst). \tag{7}$$

If $r = s$ or $r \in J'$ then, by Proposition 7.3, (7) is clear, so assume that $r \in J \setminus (J' \cup \{s\})$. We will prove that $(u^J)^{\{s,t\}}(\cdots tst)(\alpha_r)$ is a positive root, and (7) will follow from well known facts. Since $r \in J \setminus (J' \cup \{s\})$ we have that $r \notin \{s,t\}$ and $rs = sr$. If $rt = tr$ then $(u^J)^{\{s,t\}}(\cdots tst)(\alpha_r) = (u^J)^{\{s,t\}}(\alpha_r) = (u^J)(\alpha_r)$ is a positive root since $r \in J$. If $rt \neq tr$ then a simple induction shows that, for all $1 < k < m(s,t)$,

$$\underbrace{\cdots tst}_{k}(\alpha_r) = \alpha_r + b\alpha_s + c\alpha_t$$

for some $b, c \in \mathbf{R}$, $b, c > 0$. By Proposition 7.4 we know that either $s \not\leq (u^J)^{\{s,t\}}$ or $t \not\leq (u^J)^{\{s,t\}}$. Say $s \not\leq (u^J)^{\{s,t\}}$. Then the coefficient of α_s in $(u^J)^{\{s,t\}}(\alpha_r + b\alpha_s + c\alpha_t)$ is equal to b, so $(u^J)^{\{s,t\}}(\alpha_r + b\alpha_s + c\alpha_t)$ is a positive root, as desired. $\square$

We can now prove one of the main results of this work. It describes explicitly any special matching of any element of any Coxeter group. Note that for any $u \in W$, $J \subseteq S$ and $s, t \in S$ we may write $u = u^J u_J = (u^J)^{\{s,t\}} \, (u^J)_{\{s,t\} \; \{s\}}(u_J)^{\; \{s\}}(u_J)$.

Theorem 7.6 *Let (W, S) be a Coxeter system, $w \in W$, M be a special matching of w and $s = M(e)$.*

(i) If there exists a (necessarily unique) $t \in S$ such that $M(t) = ts$ but $M \neq \rho_s$ on $W_{\{s,t\}}(w)$, then

$$M(u) = (u^J)^{\{s,t\}} \, M\Big((u^J)_{\{s,t\} \; \{s\}}(u_J)\Big)^{\; \{s\}}(u_J),$$

for all $u \leq w$.

(ii) If M is a multiplication matching on $W_{\{x,s\}}(w)$ for all $x \in S$, then

$$M(u) = u^J s u_J,$$

for all $u \leq w$.

Proof. (i) We proceed by induction on $\ell(u)$ the result being clear if $\ell(u) = 0$. Note that, by Proposition 5.3, $M\big((u^J)_{\{s,t\}\ \{s\}}(u_J)\big) \in W_{\{s,t\}}(w)$ and so, if we set

$$v \stackrel{\text{def}}{=} (u^J)^{\{s,t\}}\, M\big((u^J)_{\{s,t\}\ \{s\}}(u_J)\big)\ ^{\{s\}}(u_J),$$

then, by Lemma 7.5, $(v^J)_{\{s,t\}\ \{s\}}(v_J) = M\big((u^J)_{\{s,t\}\ \{s\}}(u_J)\big)$.

If $v \stackrel{\text{def}}{=} M(u) \lhd u$ then by induction $u = M(v) = (v^J)^{\{s,t\}}\, M\big((v^J)_{\{s,t\}\ \{s\}}(v_J)\big)^{\{s\}}(v_J)$ and so by what we just remarked $(u^J)^{\{s,t\}} = (v^J)^{\{s,t\}}$, $(u^J)_{\{s,t\}\ \{s\}}(u_J) = M\big((v^J)_{\{s,t\}\ \{s\}}(v_J)\big)$, and ${}^{\{s\}}(u_J) = {}^{\{s\}}(v_J)$. Hence $M(u) = (v^J)^{\{s,t\}}(v^J)_{\{s,t\}\ \{s\}}(v_J)\ ^{\{s\}}v_J = (u^J)^{\{s,t\}}M\big((u^J)_{\{s,t\}\ \{s\}}(u_J)\big)\ ^{\{s\}}(u_J)$, as desired. We may therefore assume that $M(u) \rhd u$. Similarly, we may assume that $M\big((u^J)_{\{s,t\}\ \{s\}}(u_J)\big) \rhd (u^J)_{\{s,t\}\ \{s\}}(u_J)$.

If $u = (u^J)^{\{s,t\}}$ then, by Proposition 7.4, either $s \not\leq u$ or $t \not\leq u$. Therefore, if $a \in \bigcup_{x \in S} W_{\{x,s\}}(u)$, then either $a \in \{s,t\}$ or, by Proposition 7.3, $a \in W_{\{r,s\}}(u)$ for some $r \in S \setminus J'$, $r \neq t$. Hence, by Lemma 7.1, $M(a) = as$ so $M(u) = us$ by Lemma 5.2 and the result holds in this case. Similarly, the result holds if $u = {}^{\{s\}}(u_J)$, while it is trivial if $u = (u^J)_{\{s,t\}\ \{s\}}(u_J)$.

Now consider the following three definitions:

1. If $(u^J)^{\{s,t\}} \neq e$ let $x_1 \in D_L((u^J)^{\{s,t\}})$ and $u_1 \stackrel{\text{def}}{=} x_1 u$.
2. If $(u^J)_{\{s,t\}\ \{s\}}(u_J) \neq e$ let $v \lhd (u^J)_{\{s,t\}\ \{s\}}(u_J)$ be such that $M(v) \rhd v$ and let $u_2 \stackrel{\text{def}}{=} (u^J)^{\{s,t\}} v\ ^{\{s\}}(u_J)$.
3. If ${}^{\{s\}}(u_J) \neq e$ let $x_3 \in D_R({}^{\{s\}}(u_J))$ and $u_3 \stackrel{\text{def}}{=} u x_3$.

By our last remark we may assume that there exist $i, j \in [3]$, $i \neq j$, such that u_i and u_j can be defined as above. Applying our induction hypothesis to u_i and u_j we have that $M(u_i) \rhd u_i$, $M(u_j) \rhd u_j$, and $(u^J)^{\{s,t\}}\, M\big((u^J)_{\{s,t\}\ \{s\}}(u_J)\big)\ ^{\{s\}}(u_J)$ covers $M(u_i)$ and $M(u_j)$. On the other hand, by the definition of a special matching, $M(u) \rhd M(u_i)$, $M(u_j)$. Since $(u^J)^{\{s,t\}}M\big((u^J)_{\{s,t\}\ \{s\}}(u_J)\big)\ ^{\{s\}}(u_J) \rhd u$ and $M(u) \rhd u$ we conclude from Theorem 3.2 that $M(u) = (u^J)^{\{s,t\}}\, M\big((u^J)_{\{s,t\}\ \{s\}}(u_J)\big)\ ^{\{s\}}(u_J)$, as desired.

(ii) This is similar and simpler than case (i) and is left to the reader.□

The main link between special matchings and Kazhdan-Lusztig polynomials is given by the following result.

Theorem 7.7 *Let (W, S) be a Coxeter system, $w \in W \setminus \{e\}$, w not dihedral, and M be a special matching of w. Then there exists a multiplication matching N of w such that $NM(u) = MN(u)$ for all $u \leq w$, and $N(w) \neq M(w)$.*

Proof. Note first that the result is true for a special matching M if and only if it is true for the special matching $\tilde{M}$ defined in the proof of Theorem 6.4. Hence we may assume that M is in one of the cases of Theorem 7.6.

Suppose M is in case (i). Then, by Lemma 7.1, $M = \rho_s$ on $W_{\{s,y\}}(w)$ for all $y \in S \setminus J'$, $y \neq t$, and $M = \lambda_s$ on $W_{\{s,y\}}(w)$ for all $s \in J'$.

If $(w^J)^{\{s,t\}} \neq e$ let $x \in D_L((w^J)^{\{s,t\}})$. If $x \notin \{s,t\}$ then $M = \rho_s$ on $W_{\{s,x\}}(w)$ so $M\lambda_x = \lambda_x M$ on $W_{\{s,x\}}(w)$ and we are done by Lemma 4.3. If $x \in \{s,t\}$ then, by Proposition 7.4, $x = \beta$ and there exists $r \in S$, $r < (w^J)^{\{s,t\}}$ such that $\beta r \neq r\beta$. Furthermore, by Proposition 7.3, $r \in S \setminus J'$ so $M(r) = rs$. Let $K \stackrel{\text{def}}{=} \{r, s, t\}$, then by Proposition 5.3 M and λ_β restrict to special matchings of $[e, w[K]] = W_K(w)$ and M satisfies either the hypotheses (1) or (2) in §6. Therefore, by Proposition 6.6, $M\lambda_\beta = \lambda_\beta M$ on $[e, w[k]]$ and hence on $W_{\{s,t\}}(w)$ and the thesis follows by Lemma 4.3. Note that $M(w) \neq \lambda_x(w)$ by Theorem 7.6.

If $(w^J)^{\{s,t\}} = e$ then necessarily ${}^{\{s\}}(w_J) \neq e$ (otherwise w is dihedral) and we proceed in a similar way considering a right descent x of ${}^{\{s\}}(u_J)$. In this case M will satisfy the hypotheses (3) in §6 and one concludes that $M\rho_x = \rho_x M$.

If M is in case (ii) the proof is similar and simpler and is left to the reader.□

It is worth noting that the above result does not hold if w is dihedral.

We can now prove the main result of this work, which shows that Kazhdan-Lusztig and R-polynomials can be computed using special matchings. It immediately implies the main result of [12].

Theorem 7.8 *Let (W, S) be a Coxeter system, $w \in W$ and M be a special matching of w. Then*

$$R_{u,w}(q) = q^c R_{M(u),M(w)}(q) + (q^c - 1) R_{u,M(w)}(q)$$

for all $u \leq w$, where $c \stackrel{\text{def}}{=} 1$ if $M(u) \rhd u$ and $c \stackrel{\text{def}}{=} 0$ otherwise.

Proof. We proceed by induction on $\ell(w)$, the result being clearly true if $\ell(w) \leq 2$. So let $\ell(w) \geq 3$. If w is dihedral then the result is easy to check, so suppose that w is not dihedral. Then, by Theorem 7.7, there exists a multiplication matching N of w such that $NM(u) = MN(u)$ for all $u \leq w$, and $N(w) \neq M(w)$.

Fix $u \leq w$. There are four cases to distinguish. We consider only two of them, the other two being exactly similar. Since $M(w) \neq N(w)$, we have that $M(w) \rhd NM(w) = MN(w) \lhd N(w)$ so M restricts to a special matching of $[e, N(w)]$.

a) $N(u) \rhd u$, $M(u) \lhd u$.

Then, since $MN(u) = NM(u)$, $M(u) \lhd MN(u) \lhd N(u)$. Therefore, by Theorem 2.5 and our induction hypothesis,

$$\begin{aligned} R_{u,w} &= q\, R_{N(u),N(w)} + (q-1)R_{u,N(w)} \\ &= q\, R_{MN(u),MN(w)} + (q-1)R_{M(u),MN(w)} \\ &= q\, R_{NM(u),NM(w)} + (q-1)R_{M(u),NM(w)} \\ &= R_{M(u),M(w)}, \end{aligned}$$

as desired.

b) $N(u) \rhd u$, $M(u) \rhd u$.

If $M(u) \neq N(u)$ then $MN(u) \rhd N(u)$ and $MN(u) \rhd M(u)$ so, by Theorem 2.5 and our induction hypothesis

$$\begin{aligned} R_{u,w} &= q\, R_{N(u),N(w)} + (q-1)R_{u,N(w)} \\ &= q(q\, R_{MN(u),MN(w)} + (q-1)R_{N(u),MN(w)}) \\ &\quad +(q-1)(q\, R_{M(u),MN(w)} + (q-1)R_{u,MN(w)}) \\ &= q^2\, R_{NM(u),NM(w)} + q(q-1)R_{N(u),NM(w)} \\ &\quad +q(q-1)\, R_{M(u),NM(w)} + (q-1)^2 R_{u,NM(w)} \\ &= q\, R_{M(u),M(w)} + (q-1)R_{u,M(w)}, \end{aligned}$$

as desired. If $M(u) = N(u)$ then we have similarly that

$$\begin{aligned} R_{u,w} &= q\, R_{N(u),N(w)} + (q-1)R_{u,N(w)} \\ &= q\, R_{MN(u),MN(w)} + (q-1)(qR_{M(u),MN(w)} + (q-1)R_{u,MN(w)}) \\ &= q\, R_{M(u),M(w)} + (q-1)R_{u,M(w)} \end{aligned}$$

and the result again follows. □

8 A Hecke algebra action

In this section we introduce and study, for each $v \in W$, a Hecke algebra naturally associated to the special matchings of v and an action of it on the submodule of the Hecke algebra of W spanned by $\{T_u : u \leq v\}$. This action enables us to reformulate in a very compact way our main result, which turns out to be equivalent to the statement that this action "respects" the canonical involutions ι of these Hecke algebras

(Theorem 8.2). This, in turn, implies that the usual recursion for the Kazhdan-Lusztig polynomials (see, e.g., [21, §7.11]) holds also when descents are replaced by special matchings (Corollary 8.4) thus giving a poset theoretic recursion for the Kazhdan-Lusztig polynomials which does not involve the R-polynomials.

Let $v \in W$ and $\mathcal{S}_v$ be the collection of all the special matchings of v. We denote by $(\widehat{W}_v, \mathcal{S}_v)$ the Coxeter system whose Coxeter generators are the elements of $\mathcal{S}_v$ and whose Coxeter matrix is given by $m(M,N) \stackrel{\rm def}{=} o(MN)$, the period of MN as a permutation of $[e,v]$. We denote by $\widehat{\mathcal{H}}_v$ the Hecke algebra of $\widehat{W}_v$ and by $\mathcal{H}_v$ the submodule of $\mathcal{H}(W)$ defined by

$$\mathcal{H}_v \stackrel{\rm def}{=} \bigoplus_{u \leq v} \mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] T_u.$$

Our first result states what is the action of $\widehat{\mathcal{H}}_v$ on $\mathcal{H}_v$ that we wish to study. It is a natural generalization, and unification, of the left and right multiplication actions of $\mathcal{H}(W_{D_L(v)})$ and $\mathcal{H}(W_{D_R(v)})$ on $\mathcal{H}_v$.

Proposition 8.1 *Let $v \in W$. Then there exists a unique action of $\widehat{\mathcal{H}}_v$ on $\mathcal{H}_v$ such that, for all $u \leq v$ and any $M \in \mathcal{S}_v$,*

$$T_M(T_u) = \begin{cases} T_{M(u)}, & \text{if } M(u) \rhd u, \\ qT_{M(u)} + (q-1)T_u, & \text{otherwise.} \end{cases} \tag{8}$$

Proof. The uniqueness part is trivial. To prove the existence we only have to check that $T_M(T_M(T_u)) = ((q-1)T_M + q)(T_u)$ for all $u \leq v$ and $M \in \mathcal{S}_v$, and that, if $M, N \in \mathcal{S}_v$ and $m \stackrel{\rm def}{=} m(M,N)$, then

$$\underbrace{T_M(T_N(T_M(\cdots}_{m}(T_u)))) = \underbrace{T_N(T_M(T_N(\cdots}_{m}(T_u)))) \tag{9}$$

for all $u \leq v$. The proof of the first part is a simple verification and is left to the reader.

To prove the second one let $M, N \in \mathcal{S}_v$ be such that $m(M,N) = m$ and $u \leq v$. If $|\langle M,N \rangle(u)| = 2d$ then necessarily d divides m. Let $(W', \{a,b\})$ be a dihedral Coxeter system of order $2d$. We define a poset isomorphism $\Phi : \langle M,N \rangle(u) \longrightarrow W'$ by

$$\Phi(\underbrace{\cdots MNM}_{k}(u_0)) \stackrel{\rm def}{=} \underbrace{\cdots aba}_{k},$$

for all $k \in [2d]$, where u_0 is the smallest element in $\langle M,N \rangle(u)$, and extend this to a linear map $\Phi : \mathcal{H}(\langle M,N \rangle(u)) \longrightarrow \mathcal{H}(W')$ (where $\mathcal{H}(\langle M,N \rangle(u))$ is the submodule of $\mathcal{H}_v$ spanned by $\{T_x : x \in \langle M,N \rangle(u)\}$) by $\Phi(T_x) \stackrel{\rm def}{=} T_{\Phi(x)}$ for all $x \in \langle M,N \rangle(u)$. Then it is clear that $\Phi(T_M(T_x)) = T_a\Phi(T_x)$ and $\Phi(T_N(T_x)) = T_b\Phi(T_x)$ for all $x \in \langle M, N\rangle(u)$. There follows that

$$\begin{aligned}
\Phi(\underbrace{T_M(T_N(T_M(\cdots}_{d}(T_x)))) &= \underbrace{T_aT_bT_a\cdots}_{d}\Phi(T_x) \\
&= \underbrace{T_bT_aT_b\cdots}_{d}\Phi(T_x) \\
&= \Phi(\underbrace{T_N(T_M(T_N(\cdots}_{d}(T_x)))).
\end{aligned}$$

Hence $\underbrace{T_M(T_N(T_M(\cdots}_{d}(T_x)))) = \underbrace{T_N(T_M(T_N(\cdots}_{d}(T_x))))$ for all $x \in \langle M, N\rangle(u)$ and (9) follows. □

As pointed out by one of the referees, it would be interesting to know if the many (conjectural, in general) nonnegativity properties of structure constants of the Hecke algebra as a left module over itself with respect to various combination of bases (see [17]) extend to properties of the action just defined of $\widehat{\mathcal{H}}_v$ on $\mathcal{H}_v$. Another natural question is to determine when the permutation action of $\widehat{W}_v$ on $[e, v]$ is faithful, or when $\mathcal{H}_v$ is a faithful $\widehat{\mathcal{H}}_v$-module.

We can now state and prove the first main result of this section, which is a compact reformulation of our main result (Theorem 7.8) in terms of the action of $\widehat{\mathcal{H}}_v$ on $\mathcal{H}_v$. Note that, by Proposition 2.4, $\mathcal{H}_v$ is invariant under the involution ι defined on $\mathcal{H}(W)$. For convenience, we use the same symbol ι also for the corresponding involution of the Hecke algebra $\widehat{\mathcal{H}}_v$.

Theorem 8.2 *Let $v \in W$. Then for all $h \in \mathcal{H}_v$, $\hat{h} \in \widehat{\mathcal{H}}_v$*

$$\iota(\hat{h}(h)) = \iota(\hat{h})(\iota(h)).$$

Proof. We may clearly assume that $h = T_u$ for some $u \le v$ and $\hat{h} = T_M$, where M is a special matching of v.

Suppose first that $u \lhd M(u)$. Then, by (8) and Proposition 2.4, we have that

$$\iota(T_M(T_u)) = \iota(T_{M(u)}) = \left(T_{M(u)^{-1}}\right)^{-1} = -\varepsilon_u q^{-\ell(u)-1} \sum_x \varepsilon_x R_{x,M(u)}\, T_x.$$

On the other hand

$$
\begin{aligned}
\iota(T_M)(\iota(T_u)) &= T_M^{-1}(T_{u^{-1}}^{-1}) \\
&= (q^{-1}T_M - (1-q^{-1}))(\varepsilon_u q^{-\ell(u)} \sum_x \varepsilon_x R_{x,u}\, T_x) \\
&= \varepsilon_u q^{-\ell(u)} \Big(\sum_{x \lhd M(x)} (q^{-1}\varepsilon_x R_{x,u}\, T_{M(x)} - (1-q^{-1})\varepsilon_x R_{x,u}\, T_x) \\
&\quad + \sum_{x \rhd M(x)} (q^{-1}\varepsilon_x R_{x,u}(qT_{M(x)} + (q-1)T_x) - (1-q^{-1})\varepsilon_x R_{x,u}\, T_x) \Big) \\
&= \varepsilon_u q^{-\ell(u)} \Big(- \sum_{M(x) \lhd x} q^{-1}\varepsilon_x R_{M(x),u}\, T_x - \sum_{M(x) \rhd x} (1-q^{-1})\varepsilon_x R_{x,u} T_x \\
&\quad - \sum_{M(x) \rhd x} \varepsilon_x R_{M(x),u} T_x \Big) \\
&= \varepsilon_u q^{-\ell(u)} \Big(- \sum_{M(x) \lhd x} q^{-1}\varepsilon_x R_{x,M(u)}\, T_x - \sum_{x \lhd M(x)} q^{-1}\varepsilon_x R_{x,M(u)}\, T_x \Big)
\end{aligned}
$$

by Theorem 7.8 and the assertion follows in this case.

Suppose now that $u \rhd M(u)$. Then applying what we have just proved to $M(u)$ yields that

$$
T_{u^{-1}}^{-1} = \iota(T_u) = \iota(T_M(T_{M(u)})) = \iota(T_M)(\iota(T_{M(u)})) = T_M^{-1}(T_{M(u)^{-1}}^{-1})
$$

Therefore, by Proposition 8.1, $T_M(T_{u^{-1}}^{-1}) = T_{M(u)^{-1}}^{-1}$. Hence

$$
\begin{aligned}
\iota(T_M(T_u)) &= \iota(qT_{M(u)} + (q-1)T_u) \\
&= q^{-1}T_{M(u)^{-1}}^{-1} + (q^{-1}-1)T_{u^{-1}}^{-1} \\
&= q^{-1}T_M(T_{u^{-1}}^{-1}) + (q^{-1}-1)T_{u^{-1}}^{-1} \\
&= (q^{-1}T_M - (1-q^{-1}))(T_{u^{-1}}^{-1}) \\
&= T_M^{-1}(T_{u^{-1}}^{-1}) \\
&= \iota(T_M)(\iota(T_u)),
\end{aligned}
$$

and the result again follows. □

Recall from §2 the definition of the Kazhdan-Lusztig basis $\{C'_v : v \in W\}$ of the Hecke algebra of W.

Theorem 8.3 *Let $v \in W$ and $M \in \mathcal{S}_v$. Then, for all $x \leq v$,*

$$
C'_M(C'_x) = \begin{cases} C'_{M(x)} + \sum_{\{z:\, M(z) \lhd z\}} \overline{\mu}(z,x) C'_z, & \text{if } M(x) \rhd x, \\ (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) C'_x, & \text{if } M(x) \lhd x, \end{cases}
$$

in $\mathcal{H}_v$.

Proof. Suppose that $M(x) \rhd x$. Let, for brevity, $D_{M(x)} \overset{\text{def}}{=} C'_M(C'_x) - \sum_{\{z : M(z) \lhd z\}} \overline{\mu}(z,x) C'_z$. To prove that $D_{M(x)} = C'_{M(x)}$ we use the characterization of the Kazhdan-Lusztig basis given in Theorem 2.7. It is clear from Theorem 8.2 that $\iota(D_{M(x)}) = D_{M(x)}$. So we only need to show that if

$$D_{M(x)} = q^{-\frac{\ell(M(x))}{2}} \sum_{u \le M(x)} \widetilde{P}_{u,M(x)}(q)\, T_u,$$

then $\widetilde{P}_{M(x),M(x)}(q) = 1$ and $\widetilde{P}_{u,M(x)}(q) \in \mathbf{Z}[q]$ has degree $< \frac{1}{2}\ell(u, M(x))$ if $u < M(x)$. We distinguish two cases.

Suppose $u \lhd M(u)$. Then $T_M(C'_x)$ involves T_u with coefficient $q^{-\frac{\ell(x)}{2}} q P_{M(u),x}$. It follows easily that the coefficient of T_u in $C'_M(C'_x)$ is

$$q^{-\frac{\ell(M(x))}{2}} q P_{M(u),x}(q) + q^{-\frac{\ell(M(x))}{2}} P_{u,x}(q).$$

On the other hand, if $u \rhd M(u)$, $T_M(C'_x)$ involves T_u with coefficient $q^{-\frac{\ell(x)}{2}}(P_{M(u),x}(q) + (q-1)P_{u,x}(q))$. Again it follows easily that the coefficient of T_u in $C'_M(C'_x)$ is

$$q^{-\frac{\ell(M(x))}{2}} P_{M(u),x}(q) + q^{-\frac{\ell(M(x))}{2}} q P_{u,x}(q).$$

Finally, the coefficient of T_u in $\sum \overline{\mu}(z,x) C'_z$ is in both cases

$$\sum_{\{z : M(z) \lhd z\}} \overline{\mu}(z,x) q^{-\frac{\ell(z)}{2}} P_{u,z}(q).$$

So, if we set $c = 1$ if $M(u) \lhd u$ and $c = 0$ otherwise, we only have to show that the polynomials

$$q^{1-c} P_{M(u),x}(q) + q^c P_{u,x}(q) - \sum_{\{z \,:\, M(z) \lhd z\}} \overline{\mu}(z,x) q^{\frac{\ell(z,M(x))}{2}} P_{u,z}(q)$$

have the prescribed degree conditions.This is done in exactly the same way as in the proof of [21, Theorem 7.9] (see [21, § 7.11]) and is therefore omitted.

Assume now that $M(x) \lhd x$. We proceed by induction on $\ell(x)$. If $\ell(x) = 1$ then necessarily $x = M(e)$ and the result is easy to verify. So assume $\ell(x) \ge 2$. Then by what we have just proved we have that

$$C'_x = C'_M(C'_{M(x)}) - \sum_{\{z \colon M(z) \lhd z\}} \overline{\mu}(z, M(x)) C'_z. \tag{10}$$

Therefore, since $C'_M C'_M = (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) C'_M$,

$$\begin{aligned} C'_M(C'_x) &= (C'_M C'_M)(C'_{M(x)}) - \sum_{\{z \colon M(z) \lhd z\}} \overline{\mu}(z, M(x)) C'_M\,(C'_z) \\ &= (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) C'_x, \end{aligned}$$

by (10) and our induction hypothesis, as desired. □

Theorem 8.3, and its proof, imply the following poset theoretic recursion for Kazhdan-Lusztig polynomials, which generalizes formula (2.2c) of [22].

Corollary 8.4 *Let $u, v \in W$, $u < v$, and M be a special matching of v. Then*

$$P_{u,v}(q) = q^{1-c} P_{M(u),M(v)}(q) + q^c P_{u,M(v)}(q) - \sum_{\{z\,:\,M(z) \lhd z\}} \overline{\mu}(z, M(v)) q^{\frac{\ell(z,v)}{2}} P_{u,z}(q)$$

where $c = 1$ if $M(u) \lhd u$ and $c = 0$ otherwise. □

We illustrate Corollary 8.4 with an example. Let $v = 3\,4\,2\,1 \in S_4$. One may check that v has 5 distinct special matchings, N, ρ_2, ρ_3, λ_2, λ_1 which are shown in Figure 4. Using Corollary 8.4 for the special matching N we obtain

$$\begin{aligned} P_{e,v} &= qP_{N(e),N(v)} + P_{e,N(v)} - \sum_{\{z:N(z)\lhd z\}} \overline{\mu}(z, N(v)) q^{\frac{\ell(z,v)}{2}} P_{e,z} \\ &= qP_{1324,3412} + P_{e,3412} - (1 \cdot q \cdot P_{e,1432} + 1 \cdot q \cdot P_{e,3214} + 1 \cdot q^2 \cdot P_{e,1324}) \\ &= q(q+1) + (q+1) - q - q - q^2. \end{aligned}$$

Note that using the other 4 special matchings we obtain genuinely different computations for $P_{e,3421}$. Namely,

$$P_{e,3421} = \begin{cases} q + 1 - q & \text{using } \rho_2, \\ q + (1+q) - q - q & \text{using } \rho_3, \\ q + 1 - q & \text{using } \lambda_2, \\ q + (1+q) - q - q & \text{using } \lambda_1. \end{cases}$$

The reason for this is that the special matching N is not isomorphic to any other special matching of $[e, 3421]$. In fact, suppose that Φ is a poset automorphism of $[e, 3421]$ and M is a special matching of $[e, 3421]$ such that $\Phi \circ M = N \circ \Phi$. Then $\Phi(1324) = 1324$ and $\Phi(3412) = 3412$. Therefore $M(e) = 1324$ and $M(3421) = 3412$, but N is the only special matching of $[e, v]$ satisfying these two conditions so $M = N$. Actually, more is true. Namely, let $u \in S_n$ be such that $[e, u] \cong [e, 3421]$ (poset isomorphism). Since $[e, v]$ has only three atoms we deduce that any reduced expression of u contains exactly 3 generators, say s_i, s_j and s_k with $i < j < k$. If these indices are not consecutive we would have at most 4 permutations of length 2 smaller than u so we may assume that $u \in S_4$. But in S_4 there are only 3 permutations of length 5, namely v, v^{-1} and 4231, and $[e, 4231]$ has 4 coatoms. Hence the special matching N of $[e, 3421]$ is not isomorphic to any multiplication matching of any element in any symmetric group.

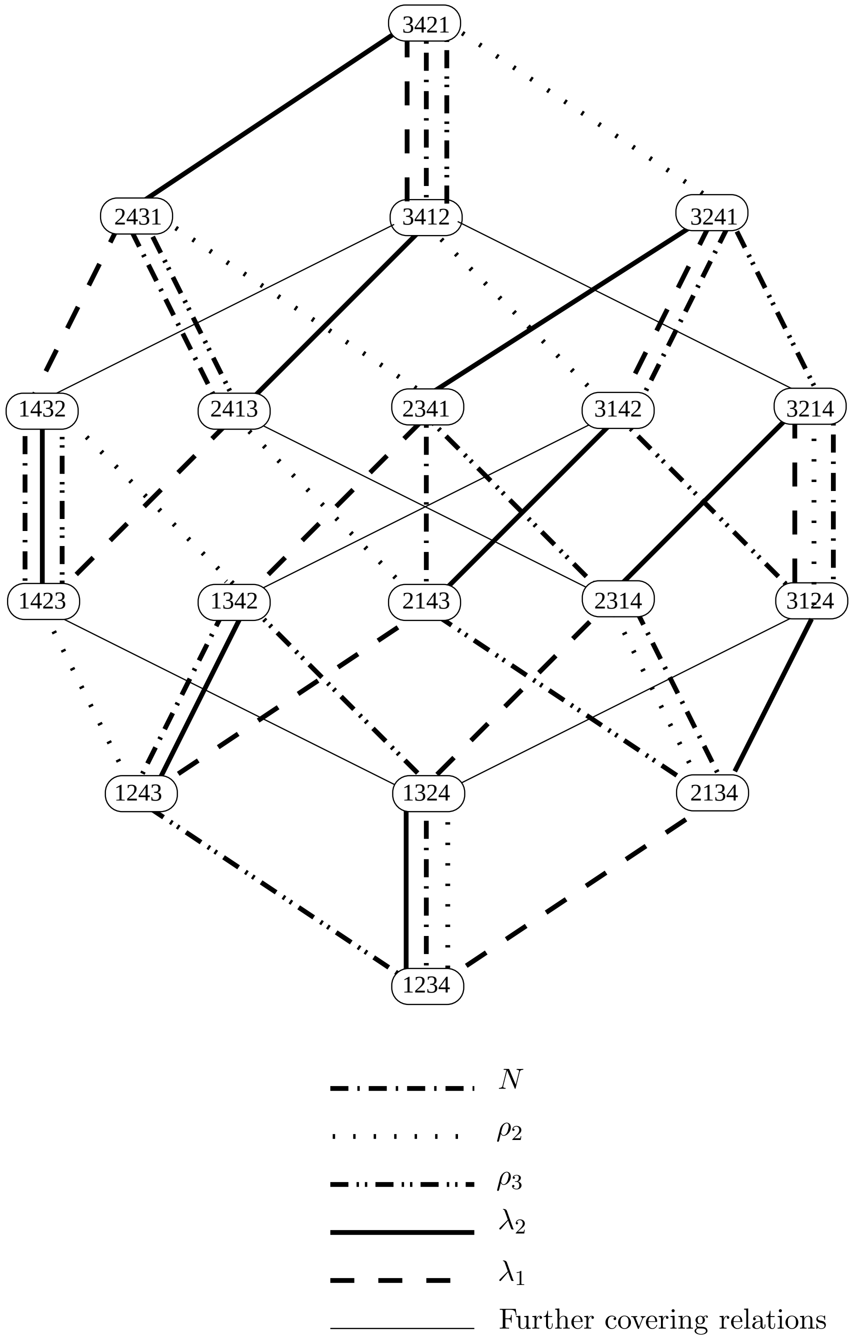

Figure 6: The special matchings of 3421

In fact, with more work one can show that the special matching N of $[e, 3421]$ is not isomorphic to any multiplication matching of any element in any Coxeter system (even infinite). We leave this to the interested reader.

9 Regular sequences

Our purpose in this section is to generalize, using our main result, an algorithm and a closed formula of Deodhar ([10, Algorithm 4.11] and [9, Theorem 1.3]) for the Kazhdan-Lusztig and R-polynomials, respectively.

Definition 9.1 *Let $v \in W$. We say that a sequence $(M_1, \ldots, M_\ell)$ (where $\ell \stackrel{\text{def}}{=} \ell(v)$) is a* regular sequence (of special matchings) *for v if, for all $i \in [\ell]$, M_i is a special matching of $M_{i+1} \cdots M_\ell(v)$.*

Note that, in particular, $M_1 \cdots M_\ell(v) = e$. The *regular chain* associated to a regular sequence $(M_1, \ldots, M_\ell)$ for v is $(v_0, \ldots, v_\ell)$ where $v_i \stackrel{\text{def}}{=} M_{i+1} \cdots M_\ell(v) = M_i \cdots M_1(e)$, for $i = 0, \ldots, \ell$. Clearly, $e = v_0 \lhd v_1 \lhd \cdots \lhd v_\ell = v$ and $M_i(v_{i-1}) = v_i$, for $i = 1, \ldots, \ell$.

For example, if $W = S_4$ and $v = 4231$ then the sequence $(M_1, \ldots, M_5)$ illustrated in Figure 5 is a regular sequence for v. Note that, if $s_1 \cdots s_\ell$ is a reduced expression for v, then $(\lambda_{s_\ell}, \ldots, \lambda_{s_1})$ and $(\rho_{s_1}, \ldots, \rho_{s_\ell})$ are two regular sequences for v. Thus, the concept of a regular sequence is a generalization of that of a reduced expression. We say that a regular sequence $\mathcal{M} = (M_1, \ldots, M_\ell)$ for v comes from a reduced expression if there is a reduced expression $s_1 \cdots s_\ell$ of v such that either $\mathcal{M} = (\lambda_{s_\ell}, \ldots, \lambda_{s_1})$ or $\mathcal{M} = (\rho_{s_1}, \ldots, \rho_{s_\ell})$.

Our first result is the analogue, for any regular sequence, of a well known result for reduced expressions.

Lemma 9.2 *Let $v \in W$, and $(M_1, \ldots, M_\ell)$ be a regular sequence for v. Then for all $u \leq v$ there exist $1 \leq i_1 < \ldots < i_k \leq \ell$ such that $(M_{i_1}, \ldots, M_{i_k})$ is a regular sequence for u.*

Proof. We proceed by induction on ℓ the statement being trivial for $\ell = 1$. So assume that $\ell > 1$. Note that $(M_1, \ldots, M_{\ell-1})$ is a regular sequence for $M_\ell(v)$. Let $u \leq v$. If $M_\ell(u) \lhd u$ then, by Lemma 2.1, $M_\ell(u) \leq M_\ell(v)$ so by induction there exist $1 \leq i_1 < \ldots < i_k \leq \ell - 1$ such that $(M_{i_1}, \ldots, M_{i_k})$ is a regular sequence for $M_\ell(u)$, hence $(M_{i_1}, \ldots, M_{i_k}, M_\ell)$ is a regular sequence for u. If $M_\ell(u) \rhd u$ then, by Lemma 2.1, $u \leq M_\ell(v)$ and we conclude again by induction. $\square$

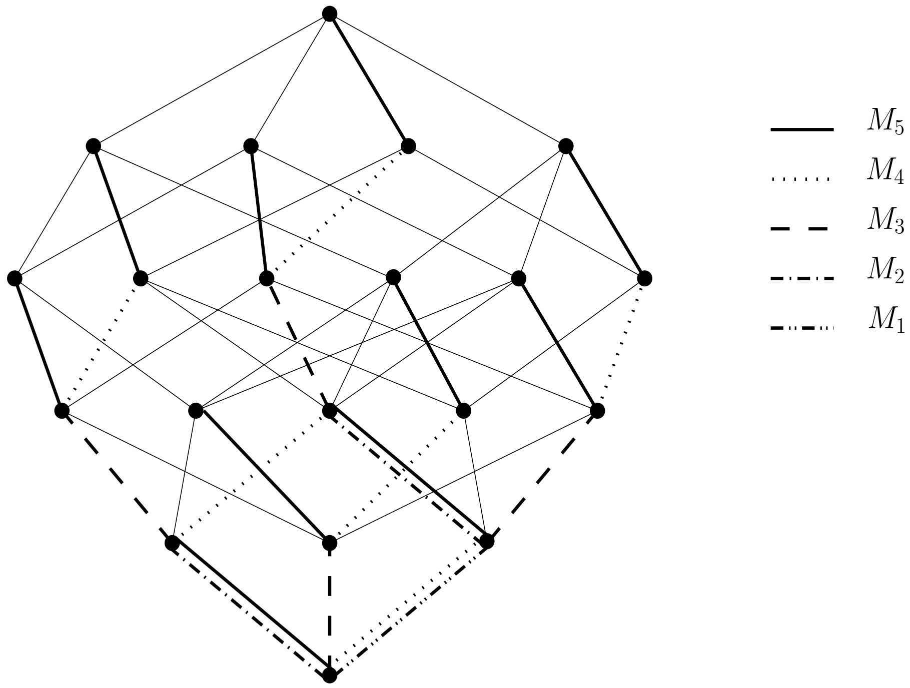

Figure 7: A regular sequence of special matchings

The next result is a sort of converse of the previous one. It is used repeatedly throughout the rest of this work, often without explicit mention.

Lemma 9.3 *Let $v \in W$ and $(M_1, \ldots, M_\ell)$ be a regular sequence for v. Then the composition $M_{i_k} \cdots M_{i_1}(e)$ is defined for any $1 \leq i_1 < i_2 < \cdots < i_k \leq \ell$.*

Proof. Let $(v_0, \ldots, v_\ell)$ be the regular chain associated to $(M_1, \ldots, M_\ell)$. We proceed by induction on k, the claim being clear if $k = 0$. So let $1 \leq i_1 < i_2 < \cdots < i_k \leq \ell$, with $k \geq 1$. By our induction hypothesis $u \stackrel{\text{def}}{=} M_{i_{k-1}} \cdots M_{i_1}(e)$ is defined. Hence $u \leq v_{i_{k-1}} < v_{i_k}$. But, by the definition of a regular sequence, M_{i_k} is a special matching of v_{i_k}. Therefore $M_{i_k}(u)$ is defined, as desired. $\square$

Let $v \in W$ and $\mathcal{M} = (M_1, \ldots, M_\ell)$ be a regular sequence for v (so $\ell = \ell(v)$). Given $S = \{i_1, \ldots, i_k\}_< \subseteq [\ell]$ we let

$$\pi(S) \stackrel{\text{def}}{=} M_{i_k} \cdots M_{i_1}(e)$$

and we define, for each $j \in [\ell]$,

$$\varepsilon_j(S) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } M_j(y) \lhd y, \\ 0, & \text{if } M_j(y) \rhd y, \end{cases}$$

where $y \stackrel{\text{def}}{=} \pi(S \cap [j-1])$. We also let

$$d_1(S, \ell) \stackrel{\text{def}}{=} \sum_{j \in [\ell] \setminus S} \varepsilon_j(S)$$

and

$$d_2(S) \stackrel{\text{def}}{=} \sum_{j \in S} \varepsilon_j(S).$$

Note that $(M_{i_1}, \ldots, M_{i_k})$ is a regular sequence for $M_{i_k} \cdots M_{i_1}(e)$ if and only if $d_2(S) = 0$. Let, for brevity,

$$d(S, \ell) \stackrel{\text{def}}{=} d_1(S, \ell) + d_2(S).$$

We say that S is *distinguished*, with respect to $\mathcal{M}$, if $d_1(S, \ell) = 0$. In the case that $\mathcal{M}$ comes from a reduced expression this concept coincides with the one introduced by Deodhar in [9, Def. 2.3]. We denote by $\mathcal{D}(\mathcal{M})$ the set of all subsets of $[\ell]$ which are distinguished with respect to $\mathcal{M}$, and we let, for $u \in W$,

$$\mathcal{D}(\mathcal{M})_u \stackrel{\text{def}}{=} \{S \in \mathcal{D}(\mathcal{M}) \, : \, \pi(S) = u\}.$$

We can now prove the first main result of this section. It is a combinatorially invariant closed formula for the $\widetilde{R}$-polynomials (and so for the R-polynomials) which generalizes Theorem 1.3 of [9].

Theorem 9.4 *Let $v \in W$ and $\mathcal{M} = (M_1, \ldots, M_\ell)$ be a regular sequence for v. Then*

$$\widetilde{R}_{u,v}(q) = \sum_{S \in \mathcal{D}(\mathcal{M})_u} q^{\ell(v) - |S|},$$

for all $u \in W$.

Proof. Our proof is similar to the one given in [9, §5], but simpler, so we present it here.

The result is clear if $u \not\leq v$, so assume $u \leq v$. We proceed by induction on $\ell \stackrel{\text{def}}{=} \ell(v)$, the result being trivial if $\ell = 0$. So assume $\ell \geq 1$ and let, for convenience, $M \stackrel{\text{def}}{=} M_\ell$. We distinguish two cases.

a) $M(u) \lhd u$.

This implies that if $S \in \mathcal{D}(\mathcal{M})_u$ then $\ell \in S$ by the definition of a distinguished subset. Note that $(M_1, \ldots, M_{\ell-1})$ is a regular sequence for $M(v)$. Define a map

$$\varphi : \mathcal{D}(\mathcal{M})_u \longrightarrow \mathcal{D}(M_1, \ldots, M_{\ell-1})_{M(u)}$$

by letting $\varphi(S) = S \setminus \{\ell\}$ for all $S \in \mathcal{D}(\mathcal{M})_u$. The map φ is well-defined and bijective since $\ell \in S$. Therefore, by Theorem 7.8, Proposition 2.6 and our induction hypothesis

$$\sum_{S\in\mathcal{D}(\mathcal{M})_u} q^{\ell(v)-|S|} = \sum_{S'\in\mathcal{D}(M_1,\ldots,M_{\ell-1})_{M(u)}} q^{\ell(v)-|S'|-1} = \widetilde{R}_{M(u),M(v)}(q) = \widetilde{R}_{u,v}(q).$$

b) $M(u) \triangleright u$.

Let $\mathcal{D}(\mathcal{M})_u^- \stackrel{\mathrm{def}}{=} \{S \in \mathcal{D}(\mathcal{M})_u \,:\, \ell \notin S\}$ and $\mathcal{D}(\mathcal{M})_u^+ \stackrel{\mathrm{def}}{=} \{S \in \mathcal{D}(\mathcal{M})_u \,:\, \ell \in S\}$. Define a map $\varphi : \mathcal{D}(\mathcal{M})_u \longrightarrow \mathcal{D}(M_1,\ldots,M_{\ell-1})_u \cup \mathcal{D}(M_1,\ldots,M_{\ell-1})_{M(u)}$ by letting $\varphi(S) = S \setminus \{\ell\}$ for all $S \in \mathcal{D}(\mathcal{M})_u$.

We claim that φ is a bijection, that $\varphi(\mathcal{D}(\mathcal{M})_u^-) = \mathcal{D}(M_1,\ldots,M_{\ell-1})_u$ and that $\varphi(\mathcal{D}(\mathcal{M})_u^+) = \mathcal{D}(M_1,\ldots,M_{\ell-1})_{M(u)}$. All the verifications are obvious, except for the surjectivity of φ. But if $S' \in \mathcal{D}(M_1,\ldots,M_{\ell-1})_u$ then $S' \in \mathcal{D}(\mathcal{M})_u$ (since $M(u) \triangleright u$), and if $S'' \in \mathcal{D}(M_1,\ldots,M_{\ell-1})_{M(u)}$ then $S'' \cup \{\ell\} \in \mathcal{D}(\mathcal{M})_u$ and this proves the surjectivity. Therefore, by Theorem 7.8, Proposition 2.6 and our induction hypothesis,

$$\begin{aligned}\sum_{S\in\mathcal{D}(\mathcal{M})_u} q^{\ell(v)-|S|} &= \sum_{S'\in\mathcal{D}(M_1,\ldots,M_{\ell-1})_u} q^{\ell(M(v))-|S'|+1} + \sum_{S''\in\mathcal{D}(M_1,\ldots,M_{\ell-1})_{M(u)}} q^{\ell(M(v))-|S''|}\\ &= q\widetilde{R}_{u,M(v)}(q) + \widetilde{R}_{M(u),M(v)}(q)\\ &= \widetilde{R}_{u,v}(q),\end{aligned}$$

as desired. □

The preceding result has the following consequence, which is needed in the rest of this section.

Corollary 9.5 *Let $v \in W$ and $(M_1,\ldots,M_\ell)$ be a regular sequence for v. Then π is a bijection between $\{S \subseteq [\ell] : d_1(S,\ell) = d_2(S) = 0\}$ and $[e,v]$.*

Proof. Clearly, $\pi(S) \in [e,v]$. Furthermore, since $[q^{\ell(u,v)}](\widetilde{R}_{u,v}) = 1$ for all $u \in [e,v]$, we conclude from Theorem 9.4 that for each $u \in [e,v]$ there exists a unique distinguished subset S_u such that $\pi(S_u) = u$ and $|S_u| = \ell(u)$. Since a subset $S \subseteq [\ell]$ is distinguished if and only if $d_1(S,\ell) = 0$, and $\ell(\pi(S)) = |S|$ if and only if $d_2(S) = 0$, the result follows.□

In order to prove the second main result of this section we need some further properties of the action of the Hecke algebra $\widehat{\mathcal{H}}_v$ on the module $\mathcal{H}_v$ defined in §8. The next result is the analogue, for regular sequences, of Proposition 3.5 of [10]. Its proof is similar to that of Proposition 3.5 of [10] and is therefore omitted.

Proposition 9.6 *Let $v \in W$ and $(M_1, \ldots, M_\ell)$ be a regular sequence for v. Then*

$$q^{\frac{\ell}{2}} C'_{M_\ell}(C'_{M_{\ell-1}}(\cdots(C'_{M_1}(T_e)))) = \sum_{S \subseteq [\ell]} q^{d(S,\ell)} T_{\pi(S)}, \tag{11}$$

in $\mathcal{H}_v$.

For brevity, we call a Coxeter system (W, S) *nonnegative* if its Kazhdan-Lusztig polynomials $P_{u,v}$ have nonnegative coefficients for all $u, v \in W$.

Proposition 9.7 *Let (W, S) be a nonnegative Coxeter system, $v \in W$, and $(M_1, \ldots, M_\ell)$ be a regular sequence for v. Then there exist $L_x \in \mathbf{N}[q^{\frac{1}{2}} + q^{-\frac{1}{2}}]$, for each $x \leq v$, such that $L_v = 1$ and*

$$C'_{M_\ell}(C'_{M_{\ell-1}}(\cdots(C'_{M_1}(T_e)))) = \sum_{x \leq v} L_x C'_x. \tag{12}$$

Proof. Let, for brevity, $C'_i \stackrel{\text{def}}{=} C'_{M_i}$ for $i = 1, \ldots, \ell$. We proceed by induction on $\ell \geq 1$, (12) being clear if $\ell = 1$ (with $L_e = 0$).

So let $\ell \geq 2$ and suppose that (12) holds for $\ell - 1$. Then there exist $\tilde{L}_x \in \mathbf{N}[q^{\frac{1}{2}} + q^{-\frac{1}{2}}]$ for each $x \leq M_\ell(v)$ such that

$$C'_{\ell-1}(C'_{\ell-2}(\cdots(C'_1(T_e)))) = \sum_{x \leq M_\ell(v)} \tilde{L}_x C'_x$$

and $\tilde{L}_{M_\ell(v)} = 1$. Therefore, by Theorem 8.3,

$$\begin{aligned} C'_\ell(C'_{\ell-1}(\cdots(C'_1(T_e)))) &= C'_\ell\Big(\sum_{x \leq M_\ell(v)} \tilde{L}_x C'_x\Big) \\ &= \sum_{\{x \leq M_\ell(v):\ M_\ell(x) \triangleright x\}} \tilde{L}_x \Big(C'_{M_\ell(x)} + \sum_{\{z:\ M_\ell(z) \triangleleft z\}} \overline{\mu}(z, x) C'_z\Big) \\ &\quad + \sum_{\{x \leq M_\ell(v):\ M_\ell(x) \triangleleft x\}} (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \tilde{L}_x C'_x, \end{aligned}$$

and the result follows. $\square$

We can now prove the second main result of this section, which plays a fundamental role in the algorithm.

Theorem 9.8 *Let (W, S) be a nonnegative Coxeter system, $v \in W$, $(M_1, \ldots, M_\ell)$ be a regular sequence for v, and $A \subseteq \{x \in [e, v] :\ L_x \neq 0\}$, $v \in A$. Then there esists $\mathcal{E} \subseteq \mathcal{P}([\ell])$ such that*

$$q^{-\frac{\ell}{2}} \sum_{S \in \mathcal{E}} q^{d(S,\ell)} T_{\pi(S)} = \sum_{x \in A} L_x C'_x. \tag{13}$$

Furthermore, for any $y \in A \setminus \{v\}$, y is maximal in $A \setminus \{v\}$ if and only if

$$deg\Big(\sum_{\{S \in \mathcal{E} \colon \pi(S)=y\}} q^{d(S,\ell)} \Big) \geq \frac{\ell(y,v)}{2} \tag{14}$$

and

$$deg\Big(\sum_{\{S \in \mathcal{E} \colon \pi(S)=x\}} q^{d(S,\ell)} \Big) < \frac{\ell(x,v)}{2} \tag{15}$$

for all $y < x < v$. If these conditions are satisfied then

$$L_y = \sum_{\{S \in \mathcal{E} \colon \pi(S)=y,\, d(S,\ell) \geq \frac{\ell(y,v)}{2}\}} q^{d(S,\ell) - \frac{\ell(y,v)}{2}} + \sum_{\{S \in \mathcal{E} \colon \pi(S)=y,\, d(S,\ell) > \frac{\ell(y,v)}{2}\}} q^{\frac{\ell(y,v)}{2} - d(S,\ell)} \tag{16}$$

and

$$P_{y,v} = \sum_{\{S \in \mathcal{E} \colon \pi(S)=y,\, d(S,\ell) < \frac{\ell(y,v)}{2}\}} q^{d(S,\ell)} - \sum_{\{S \in \mathcal{E} \colon \pi(S)=y,\, d(S,\ell) > \frac{\ell(y,v)}{2}\}} q^{\ell(y,v) - d(S,\ell)}. \tag{17}$$

Proof. Let $x \in [e,v]$. The coefficient of T_x in the right-hand side of (13) is $\sum_{y \in A} L_y q^{-\frac{\ell(y)}{2}} P_{x,y}$. Since, by Proposition 9.7 and our hypotheses, L_y and $P_{x,y}$ are Laurent polynomials in $q^{\frac{1}{2}}$ with nonnegative integer coefficients for all $x, y \leq v$, by Propositions 9.6 and 9.7 we have

$$\sum_{y \in A} L_y q^{-\frac{\ell(y)}{2}} P_{x,y} \leq \sum_{y \leq v} L_y q^{-\frac{\ell(y)}{2}} P_{x,y} = q^{-\frac{\ell}{2}} \sum_{\{S \in \mathcal{P}([\ell]) \colon \pi(S)=x\}} q^{d(S,\ell)},$$

where the $\leq$ is coefficientwise, and this implies (13).

Now let y be a maximal element of $A \setminus \{v\}$ and $x \in [e,v]$. Comparing the coefficients of T_x on both sides of (13) we obtain that

$$\begin{aligned} \sum_{\{S \in \mathcal{E} \colon \pi(S)=x\}} q^{d(S,\ell)} &= \sum_{z \in A} L_z q^{\frac{\ell(z,v)}{2}} P_{x,z} && (18)\\ &= \begin{cases} L_y q^{\frac{\ell(y,v)}{2}} + P_{y,v}, & \text{if } x = y, \\ P_{x,v}, & \text{if } y < x < v, \end{cases} && (19) \end{aligned}$$

and (14) and (15) follow since $L_y \neq 0$ and $L_y(q) = L_y(q^{-1})$. Conversely, let $y \in A \setminus \{v\}$ be such that (14) and (15) hold. Then, by (18),

$$\deg \left(\sum_{z \in A} L_z q^{\frac{\ell(z,v)}{2}} P_{x,z} \right) < \frac{\ell(x,v)}{2}$$

for all $y < x < v$. Since L_z and $P_{x,z}$ are Laurent polynomials in $q^{\frac{1}{2}}$ with nonnegative coefficients for all $x, z \leq v$, this implies that $x \notin A$ for all $y < x < v$, so y is maximal in $A \setminus \{v\}$.

Finally, if $y \in A \setminus \{v\}$ satisfies (14) and (15) then by (19) we have

$$\sum_{\{S \in \mathcal{E}:\, \pi(S)=y\}} q^{d(S,\ell)} = L_y q^{\frac{\ell(y,v)}{2}} + P_{y,v},$$

and (16) and (17) follow since $\deg(P_{y,v}) < \frac{\ell(y,v)}{2}$ and $L_y \in \mathbf{N}[q^{\frac{1}{2}} + q^{-\frac{1}{2}}]$. $\square$

Theorem 9.8 yields an inductive, entirely poset theoretic way of computing the Kazhdan-Lusztig polynomials, which generalizes the one given in [10]. In fact, let $v \in W$ and assume that we have already computed the polynomials $P_{x,y}$ for all $x, y < v$. Take a regular sequence for v, and from it compute, for each $x \leq v$, using Propositions 9.6 and 9.7, the coefficient $\mathcal{P}_x$ of T_x in

$$q^{\frac{\ell(v)}{2}} \sum_{x \leq v} L_x C'_x.$$

We apply Theorem 9.8 to the set $A = \{x \in [e,v] : L_x \neq 0\}$. If $\deg(\mathcal{P}_x) < \frac{\ell(x,v)}{2}$ for all $x < v$ then by Theorem 9.8 there are no maximal elements in $A \setminus \{v\}$ so $A = \{v\}$. Hence

$$\sum_{x \in [e,v]} L_x\, C'_x = C'_v$$

and $\mathcal{P}_x = P_{x,v}$ for all $x \leq v$. Otherwise, let $y < v$ be a maximal element such that $\deg(\mathcal{P}_y) \geq \frac{\ell(y,v)}{2}$. Then, by (16),

$$q^{\frac{\ell(y,v)}{2}} L_y = \sum_{i \geq \frac{\ell(y,v)}{2}} a_i q^i + \sum_{i > \frac{\ell(y,v)}{2}} a_i q^{\ell(y,v)-i},$$

where $\sum_{i \geq 0} a_i q^i \stackrel{\text{def}}{=} \mathcal{P}_y$. Since, by induction, we have already computed $P_{x,y}$ for all $x \in [e,v]$ we may compute the differences

$$\mathcal{P}'_x = \mathcal{P}_x - q^{\frac{\ell(y,v)}{2}} L_y P_{x,y} \tag{20}$$

for all $x \in [e,v]$. Clearly, $\mathcal{P}'_x$ is the coefficient of T_x in

$$q^{\frac{\ell(v)}{2}} \sum_{x \in [e,v] \setminus \{y\}} L_x C'_x.$$

If $\deg(\mathcal{P}'_x) < \frac{\ell(x,v)}{2}$ for all $x < v$ then Theorem 9.8 applied to $A \setminus \{y\}$ gives

$$\sum_{x \in [e,v] \setminus \{y\}} L_x C'_x = C'_v$$

and hence $\mathcal{P}'_x = P_{x,v}$ for all $x \leq v$. Otherwise, let $y_1 < v$ be a maximal element such that $\deg(\mathcal{P}'_{y_1}) \geq \frac{\ell(y_1,v)}{2}$, and repeat the above procedure with y_1 in place of y (note that $y_1 \not\geq y$ by (20)). After at most $|[e,v]| - 1$ steps this process will stop.

As an immediate consequence of Theorem 9.8 we obtain the following result which, in the case that the regular sequence comes from a reduced expression, is closely related to Theorem 4.12 of [10].

Corollary 9.9 *Let (W, S) be a nonnegative Coxeter system, $v \in W$, and $(M_1, \ldots, M_\ell)$ be a regular sequence for v. Then there exists $\mathcal{E} \subseteq \mathcal{P}([\ell])$ such that*

$$P_{u,v}(q) = \sum_{\{S \in \mathcal{E} : \pi(S) = u\}} q^{d(S,\ell)},$$

for all $u \leq v$.

Proof. This follows immediately by taking $A = \{v\}$ in Theorem 9.8. $\square$

10 A bijection

Our purpose in this section is to establish a bijection between subsequences of certain regular sequences and certain paths in an appropriate directed graph. This bijection has several nice properties, and transforms the concepts and statistics used in the previous section into familiar ones on paths.

Let $v \in W$ and $\mathcal{M} \stackrel{\text{def}}{=} (M_1, \ldots, M_\ell)$ be a regular sequence for v.

Definition 10.1 *We say that $\mathcal{M}$ is* B-regular *if*

$$M_i(x) \neq M_{i+1} M_{i+2} \cdots M_{i+k} \cdots M_{i+2} M_{i+1}(x)$$

for all $i \in [\ell]$, $k \in [\ell - i]$, and for all $x \in [e, v]$ for which both sides are defined.

Note that $\mathcal{M}$ is B-regular if and only if

$$M_i(x) \neq M_{i-1} M_{i-2} \cdots M_{i-k} \cdots M_{i-2} M_{i-1}(x)$$

for all $i \in [\ell]$, $k \in [i - 1]$, and for all $x \in [e, v]$ for which both sides are defined.

Let $v \in W$ and $\mathcal{M} \stackrel{\text{def}}{=} (M_1, \ldots, M_\ell)$ be a B-regular sequence for v.

Definition 10.2 *The* B-graph *of v, with respect to $\mathcal{M}$, is the directed graph having $[e, v]$ as vertex set and where, for any $x, y \in [e, v]$, $x \to y$ if and only if $\ell(x) < \ell(y)$ and there exists $i \in [\ell]$ such that*

$$y = M_\ell M_{\ell-1} \cdots M_{i+1} M_i M_{i+1} \cdots M_{\ell-1} M_\ell(x).$$

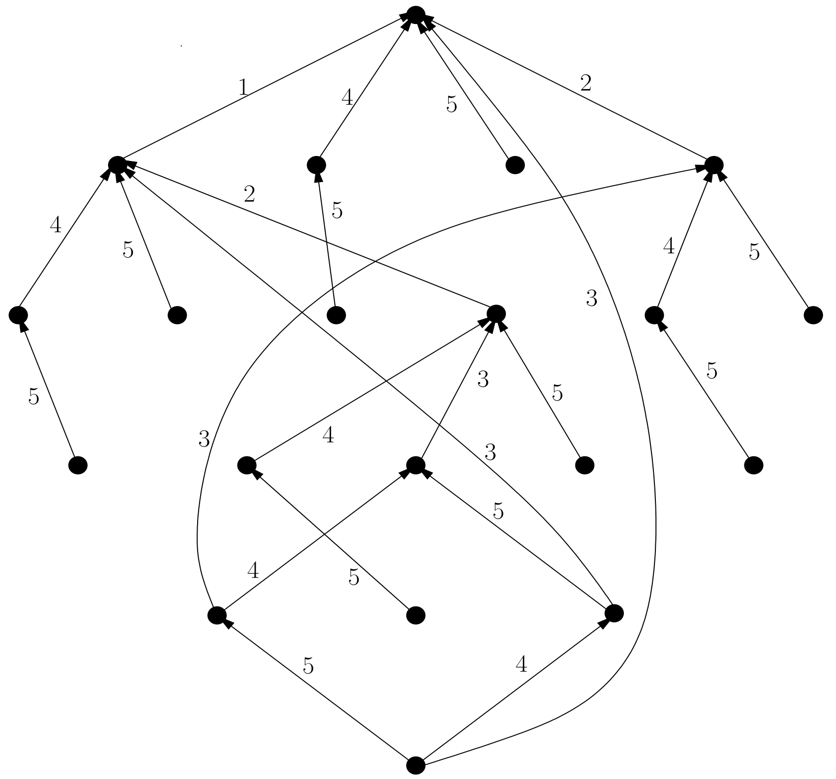

Figure 8: The B-graph corresponding to the B-regular sequence of Figure 5

Note that, if $x \longrightarrow y$, then there is a unique $i \in [\ell]$ such that $y = M_\ell \cdots M_i \cdots M_\ell(x)$ (for if $M_\ell \cdots M_i \cdots M_\ell(x) = M_\ell \cdots M_j \cdots M_\ell(x)$ for some $1 \leq i < j \leq \ell$ then $M_j(\tilde{x}) = M_{j-1} \cdots M_i \cdots M_{j-1}(\tilde{x})$ where $\tilde{x} \stackrel{\text{def}}{=} M_j \cdots M_\ell(x)$, which contradicts the fact that $\mathcal{M}$ is B-regular). We therefore define

$$\lambda(x, y) \stackrel{\text{def}}{=} \lambda(y, x) \stackrel{\text{def}}{=} i.$$

For example, one may easily check that the regular sequence in Figure 5 is actually B-regular. The corresponding B-graph is shown in Figure 6, where we have labeled all edges $x \longrightarrow y$ with $\lambda(x, y)$, and we have kept all vertices in the same place for clarity.

Note that B-regular sequences always exist. In fact, given any reduced expression $s_1 s_2 \cdots s_n$ of v, the sequences $(\lambda_{s_n}, \lambda_{s_{n-1}}, \ldots, \lambda_{s_1})$ and $(\rho_{s_1}, \rho_{s_2}, \ldots, \rho_{s_n})$ are B-regular, as it is easy to check. Therefore, the concept of a B-regular sequence is a generalization of that of a reduced expression.

One of the crucial properties of the B-graphs is that they are always directed subgraphs of the Bruhat graph. This hinges on the following result. Recall that we denote by T the set of reflections of a Coxeter system (W, S).

Theorem 10.3 *Let $v \in W$, and M be a special matching of v. Suppose $x, y \in [e, v]$ are such that $x^{-1}y \in T$. Then*

$$M(x)^{-1}M(y) \;\in\; T. \tag{21}$$

Proof. We assume that $\ell(x) < \ell(y)$ and we proceed by induction on $\ell(x, y) \geq 1$.

If $\ell(x, y) = 1$ then $x \lhd y$. If either $M(x) \rhd x$ or $M(y) \lhd y$, then (21) follows immediately from the definition of a special matching. If $M(x) \lhd x \lhd y \lhd M(y)$ then, by Lemma 2.1, M restricts to a special matching of $[M(x), M(y)]$. But it is well known (see, e.g., [3], (4.7)) that a Bruhat interval of rank 3 is isomorphic to either S_3 or to the lattice of faces of a k-gon, P_k, for some $k \geq 3$. On the other hand, it is easy to see that P_k has no special matchings if $k \geq 4$, while P_3 has no special matching M satisfying $M(\hat{0}) < M(\hat{1})$. Hence $[M(x), M(y)]$ is isomorphic to S_3, and it is known (see the proof of Proposition 3.3 of [14]) that this implies that $M(x)^{-1}M(y) \in T$.

Suppose now that $\ell(x, y) \geq 3$. From our hypotheses and (the proof of) Proposition 3.3 of [14], we have that there exist $a, b, c, d \in [x, y]$, all distinct, such that $\ell(x) < \ell(a) < \ell(c) < \ell(y)$, $\ell(x) < \ell(b) < \ell(d) < \ell(y)$, and $\{x^{-1}a, a^{-1}c, c^{-1}y, x^{-1}b, b^{-1}d, d^{-1}y, a^{-1}d, b^{-1}c\} \subseteq T$. Therefore, from our induction hypothesis, we conclude that

$$\begin{aligned}\{M(x)^{-1}M(a), M(a)^{-1}M(c), M(c)^{-1}M(y), M(x)^{-1}M(b), M(b)^{-1}M(d),\\ M(d)^{-1}M(y), M(a)^{-1}M(d), M(b)^{-1}M(c)\} \;\subseteq T.\end{aligned} \tag{22}$$

But $(M(x)^{-1}M(a))(M(a)^{-1}M(c)) = (M(x)^{-1}M(b))(M(b)^{-1}M(c)) \neq e$. Hence, by Lemma 3.1,

$$W_{x,a,b,c} \stackrel{\text{def}}{=} \langle M(x)^{-1}M(a), M(a)^{-1}M(c), M(x)^{-1}M(b), M(b)^{-1}M(c)\rangle$$

is a dihedral reflection subgroup of W. Similarly,

$$W_{x,a,b,d} \stackrel{\text{def}}{=} \langle M(x)^{-1}M(a), M(a)^{-1}M(d), M(x)^{-1}M(b), M(b)^{-1}M(d)\rangle$$

and

$$W_{b,c,d,y} \stackrel{\text{def}}{=} \langle M(b)^{-1}M(c), M(c)^{-1}M(y), M(b)^{-1}M(d), M(d)^{-1}M(y)\rangle$$

are dihedral reflection subgroups of W. But $W_{x,a,b,c} \cap W_{x,a,b,d} \supseteq \langle M(x)^{-1}M(a), M(x)^{-1}M(b)\rangle$. Therefore, by Remark 3.2 of [14], there exists a dihedral reflection subgroup W' of W such that $W' \supseteq W_{x,a,b,c} \cup W_{x,a,b,d}$. Similarly, $W' \cap W_{b,c,d,y} \supseteq \langle M(b)^{-1}M(c), M(b)^{-1}M(d)\rangle$, so there exists a dihedral reflection subgroup W'' of W such that $W'' \supseteq W' \cup W_{b,c,d,y}$. This implies that

$$\{M(x), M(a), M(b), M(c), M(d), M(y)\} \subseteq M(x)W''.$$

By Theorem 1.4 of [14], there is an isomorphism of directed graphs ϕ from the directed graph induced on $M(x)W''$ by the Bruhat graph of W to the Bruhat graph of W'' (considered as an abstract Coxeter system). Hence, by (22), in the Bruhat graph of W'' there are edges connecting $\phi(M(x))$ with $\phi(M(a))$, $\phi(M(a))$ with $\phi(M(c))$, and $\phi(M(c))$ with $\phi(M(y))$. But W'' is a dihedral Coxeter group, hence for any $u, w \in W''$ there is an edge in the Bruhat graph of W'' connecting u with w if and only if $\ell''(u,w) \equiv 1 \pmod 2$, where ℓ'' is the length function of W'' with respect to its set of canonical generators. Therefore $\ell''(\phi(M(x)), \phi(M(a))) \equiv \ell''(\phi(M(a)), \phi(M(c))) \equiv \ell''(\phi(M(c)), \phi(M(y))) \equiv 1 \pmod 2$, which implies that $\ell''(\phi(M(x)), \phi(M(y))) \equiv 1 \pmod 2$, and hence that there is an edge, in the Bruhat graph of W'', connecting $\phi(M(x))$ with $\phi(M(y))$. But ϕ is an isomorphism of directed graphs, so there is an edge in the Bruhat graph of W connecting $M(x)$ with $M(y)$, and (21) follows. □

We can now prove that the B-graphs are always directed subgraphs of the Bruhat graph.

Corollary 10.4 *Let $v_1, \ldots, v_r \in W$ and N_i be a special matching of v_i for $i = 1, \ldots, r$. Let $x \in W$ be such that $N_r N_{r-1} \cdots N_2 N_1 N_2 \cdots N_{r-1} N_r(x)$ is defined. Then*

$$x^{-1} N_r N_{r-1} \cdots N_2 N_1 N_2 \cdots N_{r-1} N_r(x) \quad \in \quad T. \tag{23}$$

Proof. We proceed by induction on $r \geq 1$, the result being clear if $r = 1$. So assume that $r \geq 2$. From our hypothesis we have that $N_{r-1} \cdots N_2 N_1 N_2 \cdots N_{r-1}(N_r(x))$ is defined. Hence, by our induction hypothesis, $N_r(x)^{-1} N_{r-1} \cdots N_2 N_1 N_2 \cdots N_{r-1}(N_r(x)) \in T$. Therefore, by Theorem 10.3, $x^{-1} N_r N_{r-1} \cdots N_2 N_1 N_2 \cdots N_{r-1} N_r(x) \in T$. □

An important consequence of Corollary 10.4 is the following result, which in the case that the B-regular sequence comes from a reduced expression is a consequence of the Exchange Condition.

Proposition 10.5 *Let $v \in W$, $(M_1, \ldots, M_\ell)$ be a B-regular sequence for v, and $y \leq v$, $j \in [\ell]$ be such that $M_j(y)$ is defined. Then the following are equivalent:*

i) $M_j(y) \rhd y$;

ii) $M_\ell \cdots M_j(y) > M_\ell \cdots M_{j+1}(y)$.

Proof. Assume first that i) holds. We will prove, by induction on k, that

$$M_{j+k} \cdots M_j(y) > M_{j+k} \cdots M_{j+1}(y) \tag{24}$$

for $k = 0, \ldots, \ell - j$. If $k = 0$ then (24) is true by our hypothesis i). So let $k \geq 1$ and assume, by induction, that

$$a \stackrel{\text{def}}{=} M_{j+k-1} \cdots M_j(y) > M_{j+k-1} \cdots M_{j+1}(y) \stackrel{\text{def}}{=} b. \tag{25}$$

Note that

$$M_{j+k}(a) = M_{j+k} \cdots M_{j+1} M_j M_{j+1} \cdots M_{j+k}(M_{j+k}(b)).$$

Therefore, by Corollary 10.4, $M_{j+k}(a)$ and $M_{j+k}(b)$ are comparable in the Bruhat order. Hence, to prove (24), it is enough to show that

$$\ell(M_{j+k}(a)) \geq \ell(M_{j+k}(b)). \tag{26}$$

Suppose, by contradiction, that

$$\ell(M_{j+k}(a)) < \ell(M_{j+k}(b)). \tag{27}$$

From (25) we have that $\ell(a) > \ell(b)$. This, together with (27), forces that $b \lhd a$ and this implies that $M_{j+k}(b) = a$, since M_{j+k} is a special matching. Therefore

$$M_{j+k}(b) = M_{j+k-1} \cdots M_{j+1} M_j M_{j+1} \cdots M_{j+k-1}(b)$$

and this contradicts the hypothesis that $(M_1, \ldots, M_\ell)$ is a B-regular sequence. This proves (26) and hence (24) and concludes the induction step.

Assume now that i) doesn't hold, i.e. $M_j(y) \lhd y$. Then $M_j(M_j(y)) \rhd M_j(y)$. Hence, by what we have just proved

$$M_\ell \cdots M_j M_j(y) > M_\ell \cdots M_{j+1} M_j(y)$$

so ii) doesn't hold. $\square$

Note that the above proposition does not hold if $(M_1, \ldots, M_\ell)$ is not B-regular. For example, let $W = S_5$, $v = 32154$, $(M_1, \ldots, M_4) = (\rho_{(2,3)}, \rho_{(1,2)}, \rho_{(4,5)}, \lambda_{(1,2)})$, $y = e$, and $j = 2$. Then $(M_1, \ldots, M_4)$ is a regular sequence for v and $M_2(e) \rhd e$ but $M_4 M_3 M_2(e) = 12354 \not\geq 21354 = M_4 M_3(e)$.

We can now prove the main result of this section, which gives a bijection between subsequences of a B-regular sequence and certain paths in the B-graph of v. The result is new even in the case that the B-regular sequence comes from a reduced expression. Recall the definition of π, $d_1(S, \ell)$ and $d_2(S)$ from §9.

Theorem 10.6 *Let $v \in W$ and $(M_1, \ldots, M_\ell)$ be a B-regular sequence for v. Then there is a bijection between subsets S of $[\ell]$ and undirected paths $\Delta = (x_0, x_1, \ldots, x_s)$ in the B-graph of v such that $x_0 = v$ and $\lambda(x_0, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_{s-1}, x_s)$. Furthermore:*

i) $\ell(\Delta) = \ell - |S|$;

ii) $x_s = \pi(S)$;

iii) $d_1(S, \ell) = |\{i \in [s] : \ x_{i-1} < x_i\}|$;

iv) $d_2(S) = \frac{1}{2}(\ell - \ell(x_s) - \ell(\Delta))$.

Proof. For $S = \{i_1, \ldots, i_k\}_< \subseteq [\ell]$ let $\{j_1, \ldots, j_s\}_< \stackrel{\text{def}}{=} [\ell] \setminus S$ and

$$x_i \stackrel{\text{def}}{=} R_{j_i} \cdots R_{j_2} R_{j_1}(v)$$

for $i = 0, \ldots, s$, where $R_i \stackrel{\text{def}}{=} M_\ell \cdots M_i \cdots M_\ell$ for $i \in [\ell]$. Then $x_i = R_{j_i}(x_{i-1})$ and hence $\lambda(x_{i-1}, x_i) = j_i$ for $i \in [s]$. Clearly $s = \ell - k$ and

$$\begin{aligned} x_i &= R_{j_i} \cdots R_{j_2} R_{j_1} M_\ell \cdots M_1(e) \\ &= M_\ell \cdots \widehat{M}_{j_i} \cdots \widehat{M}_{j_2} \cdots \widehat{M}_{j_1} \cdots M_1(e) \\ &= M_\ell \cdots M_{j_i+1}(y), \end{aligned}$$

where $y = \pi(S \cap [j_i - 1])$, for each $i \in [s]$, and $\widehat{\ }$ means that the corresponding factor is omitted. Hence $x_s = \pi(S)$ and, for $i \in [s]$, $x_{i-1} < x_i$ if and only if

$$R_{j_i}(x_i) = M_\ell \cdots M_{j_i}(y) < M_\ell \cdots M_{j_i+1}(y) = x_i$$

which, by Proposition 10.5, happens if and only if $M_{j_i}(y) \lhd y$ namely if and only if $\varepsilon_{j_i}(S) = 1$. This proves iii).

Finally, by ii),

$$\begin{aligned} \ell(x_s) &= k - 2|\{a \in [k] : \ \ M_{i_a} M_{i_{a-1}} \cdots M_{i_1}(e) \lhd M_{i_{a-1}} \cdots M_{i_1}(e)\}| \\ &= k - 2 \sum_{a \in [k]} \varepsilon_{i_a}(S) \\ &= k - 2 d_2(S). \end{aligned}$$

It is clear that this map $S \mapsto (x_0, x_1, \ldots, x_s)$ is a bijection. $\square$

Combining Theorems 10.6 and 9.4 we obtain the following result.

Corollary 10.7 *Let $v \in W$, and $(M_1, \ldots, M_\ell)$ be a B-regular sequence for v. Then, for all $u \leq v$,*

$$\widetilde{R}_{u,v}(q) = \sum_{\Delta} q^{\ell(\Delta)}$$

where Δ runs over all the directed paths $u = x_s \to \ldots \to x_2 \to x_1 \to x_0 = v$ in the B-graph of v such that $\lambda(x_0, x_1) < \lambda(x_1, x_2) < \ldots < \lambda(x_{s-1}, x_s)$. $\square$

In the case that the B-regular sequence comes from a reduced expression Corollary 10.7 is equivalent to Corollary 3.4 of [16] for any of a certain family of corresponding reflection orders.

We illustrate Corollary 10.7 with an example. Consider the B-regular sequence $(M_1, \ldots, M_5)$ illustrated in Figure 5. Then by Corollary 10.7 we can "read off" from the corresponding B-graph (Figure 6) that, for example,

$$\widetilde{R}_{e,v}(q) = q^5 + 2q^3 + q,$$

corresponding to the directed paths from e to v having sequences of labels $(5, 4, 3, 2, 1)$, $(5, 3, 2)$, $(4, 3, 1)$ and (3).

Combining Theorem 10.6 with Corollary 9.9 we obtain the following result, which appears to be new even in the case that the B-regular sequence comes from a reduced expression.

Corollary 10.8 *Let (W, S) be a nonnegative Coxeter system, $v \in W$, and $(M_1, \ldots, M_\ell)$ be a B-regular sequence for v. Then there is a subset $\mathcal{E}$ of the set of undirected paths $\Delta = (x_0, x_1, \ldots, x_{\ell(\Delta)})$ in the B-graph of v satisfying $x_0 = v$ and $\lambda(x_0, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_{\ell(\Delta)-1}, x_{\ell(\Delta)})$, such that*

$$P_{u,v}(q) = \sum_{\{\Delta \in \mathcal{E}:\, x_{\ell(\Delta)} = u\}} q^{\frac{1}{2}(\ell(u,v) + \ell(\Delta) - 2d(\Delta))}$$

for all $u \leq v$, where $d(\Delta) = |\{i \in [\ell(\Delta)] :\ x_{i-1} > x_i\}|$. $\square$

Note that the subset $\mathcal{E}$ can be determined using the algorithm in §9 and Theorem 9.8.

11 R-regular sequences

In this section we generalize, using our main result, what is probably the most explicit closed formula known for the Kazhdan-Lusztig polynomials which holds in complete generality, namely Theorem 7.3 of [6].

Let $v \in W$, and $\mathcal{M} \stackrel{\text{def}}{=} (M_1, \ldots, M_\ell)$ be a regular sequence for v. We denote by $P_{\mathcal{M}}$ the set of palindromes in the alphabet $\{M_1, \ldots, M_\ell\}$, i.e. words of the form $M_{i_1} \cdots M_{i_{k-1}} M_{i_k} M_{i_{k-1}} \cdots M_{i_1}$ with $i_1, \ldots, i_k \in [\ell]$.

Definition 11.1 *We say that $\mathcal{M}$ is a* reflection regular sequence*, or simply an* R-regular sequence*, for v, if:*

i) for $p_1, p_2 \in P_{\mathcal{M}}$, if $p_1(u_0) = p_2(u_0)$ for some $u_0 \leq v$ then $p_1(u) = p_2(u)$ for all $u \leq v$ for which both sides are defined;

ii) for $p_1, p_2, \ldots, p_n \in P_{\mathcal{M}}$, if p_i and p_{i+1} coincide on a point, for each $i = 1, \ldots, n-1$, then p_1 and p_n coincide where they are both defined;

iii) $\mathcal{M}$ admits a reflection labeling.

We now define reflection labelings. Define an equivalence relation $\sim$ on $P_{\mathcal{M}}$ by letting $p_1 \sim p_2$ if there exists $u_0 \in [e, v]$ such that $p_1(u_0) = p_2(u_0)$ and taking the transitive closure. Note that this is stronger than requiring that $p_1(u) = p_2(u)$ for all $u \leq v$ for which both sides are defined. We denote by $R_{\mathcal{M}} \stackrel{\text{def}}{=} P_{\mathcal{M}}/\sim$ the quotient set. If $p \in P_{\mathcal{M}}$ we let $\overline{p}$ be the corresponding class in $R_{\mathcal{M}}$. Note that, for each $i, j \in [\ell]$, $\overline{M_i} = \overline{M_j}$ if and only if $M_i(e) = M_j(e)$. Therefore, by Lemma 9.2, we may identify $\{\overline{M_i} : i \in [\ell]\}$ with the set of atoms of $[e, v]$. We say that an element $r \in R_{\mathcal{M}}$ is defined on some $u \leq v$ if $p(u)$ is defined for some $p \in r$. In this case we write $r(u) \stackrel{\text{def}}{=} p(u)$. Now let (W', S') be another Coxeter system and T' be its set of reflections. A *reflection labeling* of $R_{\mathcal{M}}$ in (W', S') is a map $L : R_{\mathcal{M}} \to T'$ such that:

a) $\{L(\overline{M_i}) : i \in [\ell]\} = S'$;

b) $L(\overline{M_{i_1} \cdots M_{i_k} \cdots M_{i_1}}) = L(\overline{M_{i_1}}) \cdots L(\overline{M_{i_k}}) \cdots L(\overline{M_{i_1}})$ for all $i_1, \ldots, i_k \in [\ell]$;

c) If $r_1, r_2 \in R_{\mathcal{M}}$, $r_1 \neq r_2$, are both defined on some $u \leq v$ then $L(r_1) \neq L(r_2)$.

In particular, $|S'|$ equals the number of atoms of $[e, v]$.

It is not hard to see that R-regular sequences always exist. In fact, if $v = s_1 \cdots s_\ell$ is a reduced expression for v then $\mathcal{M} \stackrel{\text{def}}{=} (\rho_{s_1}, \ldots, \rho_{s_\ell})$ is clearly a regular sequence for v satisfying i) and ii). If we denote by W' the parabolic subgroup of W generated by $\{s_i : i \in [\ell]\}$ and by T' its set of reflections, then the map $L : P_{\mathcal{M}} \longrightarrow T'$ defined by $\rho_{s_{i_1}} \cdots \rho_{s_{i_k}} \cdots \rho_{s_{i_1}} \mapsto s_{i_1} \cdots s_{i_k} \cdots s_{i_1}$ clearly factors through $R_{\mathcal{M}}$ to a reflection labeling. Similarly for $(\lambda_{s_\ell}, \ldots, \lambda_{s_1})$. Thus, the concept of an R-regular sequence is a generalization of that of a reduced expression. On the other hand, one can show that there are R-regular sequences which don't come from reduced expressions.

Although this is not obvious from the definition, an R-regular sequence is also B-regular.

Proposition 11.2 *Let $v \in W$ and $\mathcal{M}$ be an R-regular sequence for v. Then $\mathcal{M}$ is B-regular.*

Proof. Let $\mathcal{M} \stackrel{\text{def}}{=} (M_1, \ldots, M_\ell)$ and fix $i \in [\ell]$. We will show that

$$M_i(x) \neq M_{i-1} \cdots M_{i-k} \cdots M_{i-1}(x)$$

for all $k \in [i-1]$ and all $x \in [e, v]$ for which both sides are defined, and the result will follow from the remarks following the definition of a B-regular sequence in §10.

Suppose, by contradiction, that there are $x \in [e, v]$ and $k \in [i-1]$ such that $M_i(x) = M_{i-1} \cdots M_{i-k} \cdots M_{i-1}(x)$. Since $\mathcal{M}$ is R-regular this implies, by condition i), that $M_i(y) = M_{i-1} \cdots M_{i-k} \cdots M_{i-1}(y)$ for all $y \in [e, v]$ for which both sides are defined. Let $(v_0, \ldots, v_\ell)$ be the regular chain associated to $\mathcal{M}$. Then, in particular,

$$v_i = M_i(v_{i-1}) = M_{i-1} \cdots M_{i-k} \cdots M_{i-1}(v_{i-1}) = M_{i-1} \cdots M_{i-k+1}(v_{i-k-1}).$$

Therefore

$$i = \ell(v_i) = \ell(M_{i-1} \cdots M_{i-k+1}(v_{i-k-1})) \leq \ell(v_{i-k-1}) + k - 1 = i - 2,$$

which is a contradiction. $\square$

Note that the converse of the above proposition is not true. For example, let $W = S_4$ and $v = 3421$. Then it is easy to check that $\mathcal{M} \stackrel{\text{def}}{=} (\rho_{(2,3)}, \rho_{(3,4)}, \rho_{(2,3)}, \lambda_{(1,2)}, \lambda_{(2,3)})$ is a B-regular sequence for v. However, $\mathcal{M}$ is not R-regular since $\rho_{(2,3)}(e) = \lambda_{(2,3)}(e)$ but $\rho_{(2,3)}(1243) \neq \lambda_{(2,3)}(1243)$, so condition i) does not hold.

Let $v \in W$, $\mathcal{M}$ an R-regular sequence for v, and $L : R_{\mathcal{M}} \to T'$ be a reflection labeling.

Definition 11.3 *We define a labeled directed graph, that we call the R-*graph *of v with respect to $\mathcal{M}$, as follows. The R-graph has $[e, v]$ as vertex set and, for any $x, y \in [e, v]$, $x \xrightarrow{r} y$ if and only if $\ell(y) > \ell(x)$ and $y = r(x)$, for some $r \in R_{\mathcal{M}}$.*

Note that the B-graph is a directed subgraph of the R-graph and, by Corollary 10.4, the R-graph is a directed subgraph of the Bruhat graph.

If $\Delta = (x_0 \xrightarrow{r_1} x_1 \xrightarrow{r_2} \cdots \xrightarrow{r_k} x_k)$ is a directed path in the R-graph we write $E(\Delta) \stackrel{\text{def}}{=} \{r_1, \ldots, r_k\}$ and if $\prec$ is a reflection ordering on T' we let

$$D(\Delta, L, \prec) \stackrel{\text{def}}{=} \{i \in [k-1] \,:\, L(r_i) \succ L(r_{i+1})\}. \tag{28}$$

Finally, we define an element $R_\prec$ in the incidence algebra of $[e, v]$ by letting

$$R_\prec(x, y) \stackrel{\text{def}}{=} \sum_{\{\Delta \in B(x,y) : D(\Delta, L, \prec) = \emptyset\}} q^{\ell(\Delta)}$$

where $B(x, y)$ denotes the set of all directed paths in the R-graph from x to y.

We can now state the first main result of this section. It is a "global version" of Corollary 10.7 and generalizes Corollary 3.4 of [16]. The proof follows the lines of the ones given in [13], [15] and [4, Theorem 5.3.4], and is therefore omitted.

Theorem 11.4 *Let $v \in W$, $\mathcal{M} = (M_1, \ldots, M_\ell)$ be an R-regular sequence for v, $L : R_{\mathcal{M}} \to T'$ be a reflection labeling and $\prec$ a reflection ordering on T'. Then*

$$\widetilde{R}_{x,y}(q) = R_{\prec}(x, y)$$

for all $x \leq y \leq v$.

Now fix $v \in W$, an R-regular sequence $\mathcal{M}$ for v, a reflection labeling $L : R_M \to T'$ and a reflection ordering $\prec$ on T'. Let $\Delta \in B(x, y)$, where $x \leq y \leq v$. We define the *descent composition* of Δ with respect to $\prec$ to be the unique composition $\mathcal{C}(\Delta, L, \prec) \stackrel{\text{def}}{=} (b_1, \ldots, b_j)$ such that $b_1 + \ldots + b_j = \ell(\Delta)$ and $D(\Delta, L, \prec) = \{b_1, b_1 + b_2, \ldots, b_1 + \ldots + b_{j-1}\}$. For $x, y \leq v$, and $\alpha \in C$, we let

$$c_\alpha(x, y) \stackrel{\text{def}}{=} |\{\Delta \in B(x, y) : \ \ell(\Delta) = |\alpha| \text{ and } \mathcal{C}(\Delta, L, \prec) \geq_c \alpha\}|. \tag{29}$$

Using Theorem 11.4 one can prove the following result. Its proof is analogous to that of Proposition 4.4 of [5] and is therefore omitted.

Proposition 11.5 *Let $x \leq y \leq v$, and $\alpha \in C$. Then*

$$c_\alpha(x, y) = \sum_{(x_0, \ldots, x_r) \in C_r(x,y)} \prod_{j=1}^{r} [q^{\alpha_j}](\widetilde{R}_{x_{j-1}, x_j})$$

where $C_r(x, y)$ denotes the set of all chains of length r from x to y, and $r \stackrel{def}{=} \ell(\alpha)$.

We can now state the second main result of this section, which generalizes the main result of [6] (Theorem 7.2). Its proof is the same as that of Theorem 7.2 of [6] (except that, for a path $\Delta \in B(x, y)$, its descent set is now defined using the reflection labeling L, see (28)) and is therefore omitted. Recall the definition of the polynomials $\Upsilon_\alpha(q)$ from §2.

Theorem 11.6 *Let $v \in W$, $\mathcal{M}$ be an R-regular sequence for v, $L : R_{\mathcal{M}} \to T'$ be a reflection labeling and $\prec$ be a reflection ordering on T'. Then*

$$P_{x,y}(q) - q^{\ell(x,y)} P_{x,y}\left(\frac{1}{q}\right) = \sum_{\Delta \in B(x,y)} q^{\frac{\ell(x,y) - \ell(\Delta)}{2}} \Upsilon_{\mathcal{C}(\Delta, L, \prec)}(q), \tag{30}$$

for all $x \leq y \leq v$.

In the same way as Theorem 7.3 is deduced from Theorem 7.2 in [6] one obtains the following result from Theorem 11.6 . Given $n \in \mathbf{Z}$ and $A \subseteq \mathbf{Z}$ we let $n - A \stackrel{\text{def}}{=} \{n - a : \ a \in A\}$. Recall our notations concerning lattice paths from §2.

Theorem 11.7 *Let $v \in W$, $\mathcal{M}$ be an R-regular sequence for v, $L : R_{\mathcal{M}} \to T'$ be a reflection labeling and $\prec$ be a reflection ordering on T'. Then, for all $x \leq y \leq v$,*

$$P_{x,y}(q) = \sum_{(\Gamma,\Delta)} (-1)^{\Gamma_{\geq 0} + d_+(\Gamma)} q^{\frac{\ell(x,y)+\Gamma(\ell(\Gamma))}{2}}$$

where the sum is over all pairs (Γ, Δ) such that Γ is a lattice path, $\Delta \in B(x,y)$, $\ell(\Gamma) = \ell(\Delta)$, $N(\Gamma) = \ell(\Delta) - D(\Delta, L, \prec)$, and $\Gamma(\ell(\Gamma)) < 0$. $\square$

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References

[1] H. H. Andersen, *The irreducible characters for semi-simple algebraic groups and for quantum groups*, Proceedings of the International Congress of Mathematicians, Zürich, 1994, 732-743, Birkhäuser, Basel, Switzerland, 1995.

[2] S. Billey, V. Lakshmibai, *Singular loci of Schubert varieties*, Progress in Math., **182**, Birkhäuser, Boston, MA, 2000.

[3] A. Björner, *Orderings of Coxeter groups*, Combinatorics and Algebra, Contemporary Math. vol. 34, Amer. Math. Soc. 1984, 175-195.

[4] A. Björner, F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics, Springer-Verlag, New York, NY, 2005, to appear.

[5] F. Brenti, *Combinatorial expansions of Kazhdan-Lusztig polynomials*, J. London Math. Soc., **55** (1997), 448-472.

[6] F. Brenti, *Lattice paths and Kazhdan-Lusztig polynomials*, J. Amer. Math. Soc. **11** (1998), 229-259.

[7] F. Brenti, *The intersection cohomology of Schubert varieties is a combinatorial invariant*, Special issue in honor of Alain Lascoux's 60th birthday, Europ. J. Combinatorics, **25** (2004), 1151-1167.

[8] V. V. Deodhar, *Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function*, Invent. Math., **39** (1977), 187-198.

[9] V. Deodhar, *On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells*, Invent. Math., **79** (1985), 499-511.

[10] V. Deodhar, *A combinatorial setting for questions in Kazhdan-Lusztig theory*, Geometriae Dedicata, **36** (1990), 95-119.

[11] F. Du Cloux, *An abstract model for Bruhat intervals*, Europ. J. Combinatorics, **21** (2000), 197-222.

[12] F. Du Cloux, *Rigidity of Schubert closures and invariance of Kazhdan-Lusztig polynomials*, Advances in Math., **180** (2003), 146-175.

[13] M. J. Dyer, *Hecke algebras and reflections in Coxeter groups*, Ph. D. Thesis, University of Sydney, 1987.

[14] M. Dyer, *On the Bruhat graph of a Coxeter system*, Compositio Math., **78** (1991), 185-191.

[15] M. Dyer, *Hecke algebras and shellings of Bruhat intervals. II. Twisted Bruhat orders*, Contemp. Math., **139** (1992), 141-165.

[16] M. Dyer, *Hecke algebras and shellings of Bruhat intervals*, Compositio Math., **89** (1993), 91-115.

[17] M. Dyer, G. Lehrer, *On positivity in Hecke algebras*, Geom. Dedicata, **35** (1990), 115-125.

[18] I. Frenkel, M. Khovanov, A. Kirillov, *Kazhdan-Lusztig polynomials and canonical basis*, Transform. Groups, **3** (1998), 321-336.

[19] M. Haiman, *Hecke algebra characters and immanant conjectures*, J. Amer. Math. Soc., **6** (1993), 569-595.

[20] A. van den Hombergh, *About the automorphisms of the Bruhat-ordering in a Coxeter group*, Indag. Math., **36** (1974), 125-131.

[21] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, no.29, Cambridge Univ. Press, Cambridge, 1990.

[22] D. Kazhdan, G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165-184.

[23] D. Kazhdan, G. Lusztig, *Schubert varieties and Poincaré duality*, Geometry of the Laplace operator, Proc. Sympos. Pure Math. 34, Amer. Math. Soc., Providence, RI, 1980, pp. 185-203.

[24] R. P. Stanley, *Enumerative Combinatorics* , vol.1, Wadsworth and Brooks/Cole, Monterey, CA, 1986.

[25] D. Uglov, *Canonical bases of higher level q-deformed Fock spaces and Kazhdan-Lusztig polynomials*, Physical Combinatorics, Progress in Math., **191**, Birkhäuser, Boston, MA, 2000, 249-299.

[26] W. C. Waterhouse, *Automorphisms of the Bruhat order on Coxeter groups*, Bull. London Math. Soc., **21** (1989), 243-248.