The intersection cohomology of Schubert varieties is a combinatorial invariant

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Dedicato ad Alain Lascoux

Facesti come quei che va di notte,
che porta il lume dietro e sé non giova,
ma dopo sé fa le persone dotte
Dante, Divina Commedia
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Abstract

We give an explicit and entirely poset-theoretic way to compute, for any permutation v, all the Kazhdan-Lusztig polynomials $P_{x,y}$ for $x, y \leq v$, starting from the Bruhat interval $[e, v]$ as an abstract poset. This proves, in particular, that the intersection cohomology of Schubert varieties depends only on the inclusion relations between the closures of its Schubert cells.

1 Introduction

In their fundamental paper [13] Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials $\{P_{x,y}\}_{x,y \in W}$, indexed by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials of W (see, e.g., [12], Chap. 7). These polynomials are intimately related to the Bruhat order of W (see, e.g., [16]) and to the algebraic geometry of Schubert varieties (see, e.g., [2]), and have proven to be of fundamental importance in representation theory (see, e.g., [1]).

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One of the most outstanding and intriguing open problems, especially from a combinatorial point of view, concerning Kazhdan-Lusztig polynomials is the so-called combinatorial invariance problem. This problem, first posed by Lusztig ([17]) in the early 1980's and then independently by Dyer in [9], asks whether the polynomial $P_{x,y}$ depends only on the Bruhat interval $[x, y]$ as an abstract poset. See, e.g., [5], [8], and [6] for partial results on, and further information about, this problem.

The purpose of this paper is to give an affirmative answer to this problem in the case that W is the symmetric group and $x = e$ (the identity element). More precisely, we give a simple, explicit, entirely poset-theoretic way to compute, for any $v \in S_n$, all the polynomials $\{P_{x,y}\}_{x,y\in[e,v]}$, starting from the Bruhat interval $[e, v]$ as an abstract poset. This proves, in particular, that the intersection cohomology of Schubert varieties depends only on the adjacency relations between its Schubert cells.

The organization of the paper is as follows. In the next section we collect some notation and background that are used in the sequel. In section 3 we prove some preliminary results, mainly on Bruhat intervals, that are needed in later sections. In section 4 we introduce the concept of a special matching of a partially ordered set, and prove some basic results about them, particularly for Bruhat intervals. In section 5 we prove the main result of this work, namely a classification of all the special matchings of a lower Bruhat interval (Theorem 5.1). This implies an entirely poset-theoretic way of computing the Kazhdan-Lusztig and R-polynomials (Theorem 5.2), and hence their combinatorial invariance and so that of the intersection cohomology vector spaces (Corollary 5.3). Finally, in section 6, we give an example of our combinatorial procedure.

2 Notation and Background

In this section we collect some definitions, notation and results that will be used in the rest of this work. We let $\mathbf{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, \ldots\}$, and for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, \ldots, a\}$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). We write $S = \{a_1, \ldots, a_r\}_<$ to mean that $S = \{a_1, \ldots, a_r\}$ and $a_1 < \ldots < a_r$. The cardinality of a set A will be denoted by $|A|$, for $r \in \mathbf{N}$ we let $\binom{A}{r} \stackrel{\text{def}}{=} \{S \subseteq A : |S| = r\}$.

Given a set T we let $S(T)$ be the set of all bijections $\pi : T \to T$, and $S_n \stackrel{\text{def}}{=} S([n])$. If $\sigma \in S_n$ then we write $\sigma = \sigma_1 \ldots \sigma_n$ to mean that $\sigma(i) = \sigma_i$, for $i = 1, \ldots, n$. We will also write σ in *disjoint cycle form* (see, e.g., [19], p.17) and we will usually omit writing the 1-cycles of σ. For example, if $\sigma = 365492187$ then we also write $\sigma = (9, 7, 1, 3, 5)(2, 6)$.

Given $\sigma, \tau \in S_n$ we let $\sigma\tau \stackrel{\rm def}{=} \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$.

We will follow [19], Chap. 3, for notation and terminology concerning partially ordered sets. In particular, given a finite graded poset P and $S \subseteq \mathbf{N}$ we let $P_S \stackrel{\rm def}{=} \{v \in P : \rho(v) \in S\}$, where $\rho : P \to \mathbf{N}$ is the rank function of P, and $P_i \stackrel{\rm def}{=} P_{\{i\}}$ if $i \in \mathbf{N}$. We say that a finite graded poset P as above is *Eulerian* if P has a $\hat{0}$ and $\hat{1}$ and $\mu(v, u) = (-1)^{\rho(u)-\rho(v)}$ for all $v, u \in P$, $v \leq u$. Given $x, y \in P$ we say that x *covers* y if $x \geq y$ and there is no $z \in P \setminus \{x, y\}$ such that $y \leq z \leq x$. We then write $y \lhd x$ (or $x \rhd y$). The *Hasse diagram* of P is the graph $H(P) = (P, E)$ where $E \stackrel{\rm def}{=} \{\{x, y\} \in \binom{P}{2} :$ either $x \lhd y$ or $y \lhd x\}$. Given any graph $G = (V, E)$ a *matching* of G is a subset M of E such that every element of V belongs to exactly one element of M. If $\{x, y\} \in M$ then we also write $M(x) = y$ or $M(y) = x$.

We will follow [12] for general Coxeter groups notation and terminology. Given a Coxeter system (W, S) and $\sigma \in W$ we denote by $l(\sigma)$ the length of σ in W, with respect to S, and we let

$$D(\sigma) \stackrel{\rm def}{=} \{s \in S : \ l(\sigma\, s) < l(\sigma)\},$$

and

$$D_L(\sigma) \stackrel{\rm def}{=} \{s \in S : \ l(s\sigma) < l(\sigma)\} = D(\sigma^{-1}).$$

We call $D(\sigma)$ (respectively, $D_L(\sigma)$) the right (respectively, left) the *descent set* of σ. We denote by e the identity of W, and we let $T \stackrel{\rm def}{=} \{\sigma s \sigma^{-1} : \sigma \in W, \ s \in S\}$ be the set of reflections of W. We will always assume that W is partially ordered by (strong) *Bruhat order.* Recall (see, e.g., [12], §5.9) that this means that $v \leq u$ if and only if there exist $r \in \mathbf{N}$ and $t_1, \ldots, t_r \in T$ such that $t_r \ldots t_1\, v = u$ and $l(t_i \ldots t_1\, v) > l(t_{i-1} \ldots t_1 v)$ for $i = 1, \ldots, r$. There is a well known characterization of Bruhat order on a Coxeter group (usually referred to as the *subword property*) that we will use repeatedly in this work, often without explicit mention. We recall it here for the reader's convenience. By a *subword* of a word $s_1 s_2 \ldots s_q$ we mean a word of the form $s_{i_1} s_{i_2} \ldots s_{i_k}$, where $1 \leq i_1 < \ldots < i_k \leq q$.

Theorem 2.1 *For $u, w \in W$ the following are equivalent:*

(i) $u \leq w$;

(ii) every reduced expression for w has a subword that is a reduced expression for u.

A proof of the preceding result can be found, e.g., in [12, §5.10]. In the case of the symmetric groups there is another characterization of Bruhat order, which we will also use (see, e.g., [18], Chap. 1, for a proof). For $\sigma \in S_n$, and $i \in [n]$, let

$$\{\sigma^{i,1}, \ldots, \sigma^{i,i}\}_< \stackrel{\text{def}}{=} \{\sigma(1), \ldots, \sigma(i)\}\,. \tag{1}$$

Theorem 2.2 *Let $\sigma, \tau \in S_n$. Then $\sigma \leq \tau$ if and only if $\sigma^{i,j} \leq \tau^{i,j}$ for all $1 \leq j \leq i \leq n-1$.*

Given $u, v \in W$ we let $[u, v] \stackrel{\text{def}}{=} \{x \in W : u \leq x \leq v\}$. We consider $[u, v]$ as a poset with the partial ordering induced by W. In particular, we will use notation such as $[u, v]_S$ or $[u, v]_i$ ($S \subseteq \mathbf{N}$, $i \in \mathbf{N}$) to denote the rank-selected subposets of $[u, v]$. It is well known (see, e.g., [3], Corollary 1) that intervals of W (and their duals) are Eulerian posets.

For $J \subseteq S$ let W_J be the parabolic subgroup of W generated by the set J, and

$$W^J \stackrel{\text{def}}{=} \{w \in W \;:\; D(w) \subseteq S \setminus J\}. \tag{2}$$

The following result is well known and a proof of it can be found, e.g., in [12].

Proposition 2.3 *Let $J \subseteq S$. Then*

(i) Every $w \in W$ has a unique factorization $w = w^J \cdot w_J$ such that $w^J \in W^J$ and $w_J \in W_J$.

(ii) For this factorization: $\ell(w) = \ell(w^J) + \ell(w_J)$.

There is of course a left version of the above definitions and results. Namely, if we let

$${}^J\!W \stackrel{\text{def}}{=} \{w \in W \;:\; D_L(w) \subseteq S \setminus J\} = (W^J)^{-1}, \tag{3}$$

then every $w \in W$ can be uniquely factorized $w = w_J \cdot {}^J\!w$, where $w_J \in W_J$ and ${}^J\!w \in {}^J W$, and then $l(w) = l(w_J) + l({}^J\!w)$. Furthermore, an element w belongs to ${}^J\!W$ if and only if no reduced expression for w begins with a letter from J.

We denote by $\mathcal{H}(W)$ the *Hecke algebra* associated to W. Recall (see, e.g., [12], Chap. 7) that this is the free $\mathbf{Z}[q, q^{-1}]$-module having the set $\{T_w :\; w \in W\}$ as a basis and multiplication such that

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } l(ws) > l(w), \\ qT_{ws} + (q-1)T_w, & \text{if } l(ws) < l(w), \end{cases} \tag{4}$$

for all $w \in W$ and $s \in S$. It is well known that this is an associative algebra having T_e as unity and that each basis element is invertible in $\mathcal{H}(W)$. More precisely, we have the following result (see, [12], Proposition 7.4).

Proposition 2.4 *Let $v \in W$. Then*

$$(T_{v^{-1}})^{-1} = q^{-l(v)} \sum_{u \leq v} (-1)^{l(v)-l(u)} R_{u,v}(q)\, T_u \,,$$

where $R_{u,v}(q) \in \mathbf{Z}[q]$.

The polynomials $R_{u,v}$ defined by the previous proposition are called the *R-polynomials* of W. It is easy to see that $\deg(R_{u,v}) = l(v) - l(u)$, and that $R_{u,u}(q) = 1$, for all $u, v \in W$, $u \leq v$. It is customary to let $R_{u,v}(q) \stackrel{\text{def}}{=} 0$ if $u \not\leq v$. We then have the following fundamental result that follows from (4) and Proposition 2.4 (see [12], §7.5).

Theorem 2.5 *Let $u, v \in W$ and $s \in D(v)$. Then*

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q), & \text{if } s \in D(u), \\ qR_{us,vs}(q) + (q-1)R_{u,vs}, & \text{if } s \notin D(u). \end{cases} \tag{5}$$

Note that the preceding theorem can be used to inductively compute the R-polynomials since $l(vs) < l(v)$. Once again, there is a left version of Theorem 2.5.

The R-polynomials can be used to define the Kazhdan-Lusztig polynomials. The following result is not hard to prove (and, in fact, holds in much greater generality, see [20], Corollary 6.7 and Example 6.9) and a proof can be found, e.g., in [12, §§7.9-11], or [13, §2.2].

Theorem 2.6 *There is a unique family of polynomials $\{P_{u,v}(q)\}_{u,v \in W} \subseteq \mathbf{Z}[q]$, such that, for all $u, v \in W$:*

i) *$P_{u,v}(q) = 0$ if $u \not\leq v$;*

ii) *$P_{u,u}(q) = 1$;*

iii) *$deg(P_{u,v}(q)) < \frac{1}{2}(l(v) - l(u))$, if $u < v$;*

iv)

$$q^{l(v)-l(u)}\, P_{u,v}\left(\frac{1}{q}\right) = \sum_{u \leq z \leq v} R_{u,z}(q)\, P_{z,v}(q)\,, \tag{6}$$

if $u \leq v$.

The polynomials $P_{u,v}(q)$ defined by the preceding theorem are called the *Kazhdan-Lusztig polynomials* of W. Note that parts iii) and iv) of Theorem 2.7 actually yield an inductive procedure to compute the polynomials $P_{u,v}(q)$ for all $u, v \in W$, taking parts i) and ii) as initial conditions.

Kazhdan-Lusztig polynomials have been first defined in [13] and play a prominent role in several branches of mathematics including representation theory (see, e.g., [1], and the references cited there), and the algebraic geometry and topology of Schubert varieties (see, e.g., [13], [14], and [2]). Of interest for us here is mainly their connection to the topology of Schubert varieties. For a permutation $v \in S_{n+1}$ let Ω_v be the Schubert cell indexed by v, and $\overline{\Omega}_v$ (Zariski closure) be the corresponding Schubert variety (we refer the reader to, e.g., [18], [10], or [2] for the definition of, and further information about, Schubert cells and varieties). It is well known (and not hard to see) that $\overline{\Omega}_v = \biguplus_{u \leq v} \Omega_u$ so that $u \leq v$ if and only if $\overline{\Omega}_u \subseteq \overline{\Omega}_v$. We denote by $IH^*(\overline{\Omega}_v, \mathbf{C})_{\Omega_u}$ the (middle perversity) local intersection cohomology of $\overline{\Omega}_v$ at a (equivalently, any) point of Ω_u. This is a graded vector space, and we denote by $IH^i(\overline{\Omega}_v, \mathbf{C})_{\Omega_u}$ ($i \in \mathbf{N}$) its graded pieces (we refer the reader to, e.g., [11], or [15], for further information about intersection (co)homology). The following result was first proved by Kazhdan and Lusztig in [14, Theorem 4.3].

Theorem 2.7 *Let $u, v \in S_{n+1}$, $u \leq v$. Then*

$$P_{u,v}(q) = \sum_{i \geq 0} q^i \ dim_{\mathbf{C}}(IH^{2i}(\overline{\Omega}_v, \mathbf{C})_{\Omega_u}).$$

Note that it is known that $\dim_{\mathbf{C}}(IH^i(\overline{\Omega}_v, \mathbf{C})_{\Omega_u}) = 0$ if $i \equiv 1 \pmod 2$.

3 Preliminary lemmas

In this section we prove some preliminary results, mostly on the Bruhat order of S_n, that are used in the proof of the main theorem, and in the next section.

Our first result is crucial for the understanding of special matchings (defined in the next section) of Bruhat intervals.

Proposition 3.1 *Let $v, u \in S_{n+1}$, $v \neq u$. Then*

$$|\{z \in S_{n+1} : \ z \lhd v, \ z \lhd u\}| \leq 2. \tag{7}$$

Proof. If $\{z \in S_{n+1} : \ z \lhd v, z \lhd u\}$ is empty then the result is clear, so assume that it is not. Then $v(a,b) = u(c,d)$ for some $a < b$ and $c < d$ such that $v(a) > v(b)$ and $u(c) > u(d)$. Note that, since $v \neq u$, $3 \leq |\{a,b,c,d\}| \leq 4$ and $v(i) \neq u(i)$ for $i \in \{a,b,c,d\}$. Now, if $z \lhd v$ and $z \lhd u$ then $z(i) = v(i) = u(i)$ for $i \in [n+1] \setminus \{a,b,c,d\}$ (for if $z(i) \neq v(i) = u(i)$ for some $i \in [n+1] \setminus \{a,b,c,d\}$ then $|\{j \in \{a,b,c,d\} : \ z(j) \neq$

$u(j)\}| \geq |\{j \in \{a,b,c,d\}: \ z(j)=v(j)\}| \geq |\{a,b,c,d\}|-1$ and hence z differs from u in at least three positions, which contradicts $z \lhd u$). This shows that

$$\{z: \ z \lhd v, \ z \lhd u\} \subseteq \{v(a,b), v(a,c), v(a,d), v(b,c), v(b,d), v(c,d)\}. \tag{8}$$

Assume first that $|\{a,b,c,d\}|=3$. Let $\{a_1,a_2,a_3\}_< \stackrel{\text{def}}{=} \{a,b,c,d\}$.Then we have from (8) that

$$\{z: \ z \lhd v, z \lhd u\} \subseteq \{v(a_1,a_2), v(a_2,a_3), v(a_1,a_3)\},$$

and (7) follows in this case since if $v(a_1,a_2), v(a_2,a_3), v(a_1,a_3) \leq v$ then necessarily $v(a_1) > v(a_2) > v(a_3)$ and so $v(a_1,a_3) \not\lhd v$.

Assume now that $|\{a,b,c,d\}|=4$. Let $\{a_1,a_2,a_3,a_4\}_< \stackrel{\text{def}}{=} \{a,b,c,d\}$ and σ be the (unique) permutation in S_4 such that

$$v(a_{\sigma(1)}) < v(a_{\sigma(2)}) < v(a_{\sigma(3)}) < v(a_{\sigma(4)}). \tag{9}$$

Let $1 \leq i < j \leq 4$ be such that $v(a_{\sigma(i)}, a_{\sigma(j)}) \lhd v$. Then $a_{\sigma(i)} > a_{\sigma(j)}$ and there is no $k \in [4]$ such that $v(a_{\sigma(i)}) < v(a_{\sigma(k)}) < v(a_{\sigma(j)})$ and $a_{\sigma(j)} < a_{\sigma(k)} < a_{\sigma(i)}$. Therefore $\sigma(i) > \sigma(j)$ and there is no $k \in [4]$ such that $i < k < j$ and $\sigma(j) < \sigma(k) < \sigma(i)$ so $\sigma(i,j) \lhd \sigma$.

Similarly, let τ be the unique permutation in S_4 such that

$$u(a_{\tau(1)}) < u(a_{\tau(2)}) < u(a_{\tau(3)}) < u(a_{\tau(4)}). \tag{10}$$

Then $u(a_{\tau(i)}, a_{\tau(j)}) \lhd u$ implies $\tau(i,j) \lhd \tau$, for all $1 \leq i < j \leq 4$. Note that by (9) and (10), $v(a_{\sigma(r)}) = u(a_{\tau(r)})$ for $r \in [4]$.

We claim that there is an injective map $\{z \in S_{n+1}: \ z \lhd v, \ z \lhd u\} \hookrightarrow \{z \in S_4: z \lhd \sigma, \ z \lhd \tau\}$. In fact, let $z \in S_{n+1}$ be such that $z \lhd v$ and $z \lhd u$. Then $z = v(a_{\sigma(i)}, a_{\sigma(j)})$ for some $1 \leq i < j \leq 4$, and $z = u(a_{\tau(k)}, a_{\tau(l)})$ for some $1 \leq k < l \leq 4$. Hence $\sigma(i,j) \lhd \sigma$ and $\tau(k,l) \lhd \tau$. Let $u \stackrel{\text{def}}{=} \sigma\tau^{-1}$. Then

$$v\tilde{u} = u$$

where $\tilde{u}(a_r) \stackrel{\text{def}}{=} a_{u(r)}$ for $r \in [4]$ and $\tilde{u}(r) \stackrel{\text{def}}{=} r$ for $r \in [n+1] \setminus \{a,b,c,d\}$ (since $v\tilde{u}(a_{\tau(r)}) = v(a_{u(\tau(r))}) = v(a_{\sigma(r)}) = u(a_{\tau(r)})$ for all $r \in [4]$. Therefore $\tilde{u} = (a_{\sigma(i)}, a_{\sigma(j)})\ (a_{\tau(k)}, a_{\tau(l)})$ and hence $u = (\sigma(i), \sigma(j))(\tau(k), \tau(l))$. There follows that $(\sigma(i), \sigma(j))\sigma = (\tau(k), \tau(l))\tau$ and therefore that $\sigma(i,j) = \tau(k,l)$. Note that this map $z \mapsto \sigma(i,j)$ is injective for if $\sigma(i,j) = \sigma(i_1, j_1)$ then $(i,j) = (i_1, j_1)$ and hence $(a_{\sigma(i)}, a_{\sigma(j)}) = (a_{\sigma(i_1)}, a_{\sigma(j_1)})$. Therefore

$$|\{z \in S_{n+1}: \ z \lhd v, \ z \lhd u\}| \leq |\{z \in S_4: \ z \lhd \sigma, \ z \lhd \tau\}|.$$

But it is easy to check that $|\{z \in S_4 : \ z \lhd \sigma, \ z \lhd \tau\}| \leq 2$ for any $\sigma, \tau \in S_4$, $\sigma \neq \tau$, so the result follows. $\square$

The following three technical lemmas are needed in the next section.

For $a, b \in S$ let

$$J(a,b) \stackrel{\text{def}}{=} \{s \in S : \ d(s,a) < d(s,b)\}$$

(where, for $x, y \in S$, $d(x,y)$ is the distance, in graph theoretic terms, between x and y in the Dynkin diagram of S_{n+1}).

Lemma 3.2 *Let $v \in S_{n+1}$, $a, b, c \in S$, $b \neq c$, be such that $m(a,b) = m(a,c) = 3$, $cab \not\leq v$, and $J \stackrel{\text{def}}{=} J(a,c)$. Then $\{s \in S : \ s \leq {}^J v\} \subseteq S \setminus (J \setminus \{a\})$.*

Proof. Let ${}^J v = t_1 \dots t_r$ be reduced and suppose that $\{t_1, \dots, t_r\} \cap (J \setminus \{a\}) \neq \emptyset$. Let t_i be the leftmost letter of $t_1 \dots t_r$ such that $t_i \in J \setminus \{a\}$. If $a \notin \{t_1, \dots, t_{i-1}\}$ then $t_i t_j = t_j t_i$ for all $j \in [i-1]$ and hence $t_i \in D_L({}^J v)$ which is a contradiction since $t_i \in J$. So $a \in \{t_1, \dots, t_{i-1}\}$. Let t_k be the leftmost letter of $t_1 \dots t_{i-1}$ that is equal to a. Then $\{t_1, \dots, t_{k-1}\} \subseteq S \setminus J$. But $t_i = b$ (for if $t_i \in J \setminus \{a, b\}$ then $t_i t_j = t_j t_i$ for all $j \in [i-1]$ and hence $t_i \in D_L({}^J v)$, which is a contradiction). Hence $c \notin \{t_1, \dots, t_{k-1}\}$ (else $cab \leq v$, which is a contradiction). Therefore $\{t_1, \dots, t_{k-1}\} \subseteq S \setminus (J \cup \{c\})$ and hence $t_j a = a t_j$ for all $j \in [k-1]$, which implies that $a \in D_L({}^J v)$ which is a contradiction since $a \in J$. So $\{t_1, \dots, t_r\} \cap (J \setminus \{a\}) = \emptyset$, and the result follows. $\square$

Lemma 3.3 *Let $a, b, c \in S$, $b \neq c$, be such that $m(a,b) = m(a,c) = 3$, and $v = v_1 v_2 \in S_{n+1}$ be such that $\{s \in S : \ s \leq v_1\} \subseteq J(a,c)$, $\{s \in S : \ s \leq v_2\} \subseteq J(a,b)$. Then*

$$v_J = \begin{cases} v_1, & \text{if } a \notin D_L(v_2), \\ v_1 a, & \text{if } a \in D_L(v_2), \end{cases}$$

and

$$^J v = \begin{cases} v_2, & \text{if } a \notin D_L(v_2), \\ a v_2, & \text{if } a \in D_L(v_2), \end{cases}$$

where $J \stackrel{\text{def}}{=} J(a,c)$. In particular, $v_J \, a \, {}^J v = v_1 a v_2$.

Proof. Suppose first that $a \notin D_L(v_2)$. Then $D_L(v_2) \subseteq J(a,b) \setminus \{a\} = S \setminus J$ and hence $v_2 \in {}^J(S_{n+1})$.

Since, clearly, $v_1 \in (S_{n+1})_J$, we conclude from the (left version) of Proposition 2.3 that the result holds in this case. If $a \in D_L(v_2)$ then $a \notin D_L(a v_2)$. Hence $D_L(a v_2) \subseteq J(a,b) \setminus \{a\} = S \setminus J$ and hence $a v_2 \in {}^J(S_{n+1})$. But, clearly, $v_1 a \in (S_{n+1})_J$, so the result again follows. $\square$

Lemma 3.4 *Let $v \in S_{n+1}$, $a,b,c \in S$, $b \neq c$, be such that $m(a,b) = m(a,c) = 3$, $cab \not\leq v$, and $J \stackrel{\rm def}{=} J(a,c)$. Suppose $u \lhd v$, and $u_J\, a\, {}^J u \neq v$. Then*

$$u_J\, a\, {}^J u \leq v_J\, a\, {}^J v.$$

Proof. Let $v_J = s_1 \dots s_p$, ${}^J v = t_1 \dots t_r$ be both reduced expressions. Then $\{s_1, \dots, s_p\} \subseteq J$ and by Lemma 3.2 we have that $\{t_1, \dots, t_r\} \subseteq S \setminus (J \setminus \{a\}) = J(a,b)$. Since $u \lhd v$ there follows that $u = u_1 u_2$ with $\{s \in S : s \leq u_1\} \subseteq J$, $\{s \in S :\ s \leq u_2\} \subseteq J(a,b)$ and either $u_1 = v_J$, $u_2 \lhd\ {}^J v$ or $u_1 \lhd v_J$, $u_2 = {}^J v$. Therefore either $u_1 a = v_J a$ and $u_2 \leq {}^J v$, or $u_1 a \leq v_J a$ and $u_2 = {}^J v$ (for if $u_1 a = v_J$ then $u_1 a u_2 = v_J\, {}^J v = v$ and hence, by Lemma 3.3, $u_J\, a\, {}^J u = v$, which is a contradiction). Let $v_J a = r_1 \dots r_k$ be a reduced expression (so $k = p \pm 1$). Since $v_J a \in W_J$ there follows that $r_1 \dots r_k t_1 \dots t_r$ is a reduced expression for $v_J a\ {}^J v$. This implies that either $r_1 \dots r_k u_2 \leq r_1 \dots r_k\ {}^J v$ and $u_1 a = v_J a$ or $u_1 a t_1 \dots t_r \leq v_J a t_1 \dots t_r$ and $u_2 = {}^J v$. Therefore $u_1 a u_2 \leq v_J a\ {}^J v$, and the result follows from Lemma 3.3. $\Box$

We conclude with the following two observations that will be used in the proof of the main theorems (Theorems 5.1 and 5.2).

Lemma 3.5 *Let $v \in W$ and $a,b,c \in S$ be such that $m(a,b) = m(a,c) = 3$, $m(b,c) = 2$, $aba \leq v$, and $aca \leq v$. Then either $abca \leq v$ or $cabac \leq v$.*

Proof. Let $\xi \stackrel{\rm def}{=} (r_1, \dots, r_q)$ be a reduced expression for v. Let r_i (respectively, r_j) be the leftmost (respectively, rightmost) letter of ξ that is equal to a ($1 \leq i \leq j \leq q$). Since $aca \leq v$ there is a letter equal to c to the left of r_j. Similarly, there is a letter equal to c to the right of r_i. Therefore, there is either a letter equal to c between r_i and r_j, or there are (at least) two letters equal to c, one to the left of r_i and one to the right of r_j. Since $aba \leq v$ we conclude similarly that there is either a letter equal to b between r_i and r_j, or (at least) two letters equal to b, one to the left of r_i and one to the right of r_j. Therefore, either $abca = acba \leq v$, or $bacab \leq v$, or $cabac \leq v$, or $bcacb = bcabc = cbacb = cbabc \leq v$, as desired. $\Box$

Lemma 3.6 *Let $\sigma \in S_{n+1}$ and $i \in [n]$. Then $s_{i+1} s_i s_{i-1} \not\leq \sigma$ if and only if $\sigma([i-1]) \subseteq [i+1]$.*

Proof. Let, for brevity, $\tau \stackrel{\rm def}{=} s_{i+1} s_i s_{i-1}$ (so $\tau(i-1) = i+2$, $\tau(j) = j-1$ if $j \in \{i, i+1, i+2\}$, and $\tau(j) = j$ if $j \in [n+1] \setminus [i-1, i+2]$). Let $\{\tau_{j,1}, \dots, \tau_{j,j}\}_< \stackrel{\rm def}{=} \{\tau(1), \dots, \tau(j)\}$ for $j \in [n+1]$. Then $(\tau_{j,1}, \dots, \tau_{j,j}) = (1, 2, \dots, j)$ for $j \in [n+1] \setminus \{i-1, i, i+1\}$ and $(\tau_{j,1}, \dots, \tau_{j,j}) = (1, 2, \dots, j-1, i+2)$ for $j \in \{i-1, i, i+1\}$. Hence $\tau \leq \sigma$ if and only if $\max\{\sigma(1), \dots, \sigma(j)\} \geq i+2$ for $j \in \{i-1, i, i+1\}$, namely if and only if $\max\{\sigma(1), \dots, \sigma(i-1)\} \geq i+2$, and the result follows. $\Box$

4 Special matchings

In this section we introduce the concept of a special matching of a partially ordered set, and study some of its basic properties, particularly in regard to Bruhat intervals.

Let P be a partially ordered set. We say that a matching M of the Hasse diagram of P is a *special matching* if, for all $x, y \in P$, such that $M(x) \neq y$, we have that

$$x \lhd y \Rightarrow M(x) \leq M(y).$$

Note that this implies, in particular, that if $x \lhd y$ and $M(x) \rhd x$ then $M(y) \rhd y$ and $M(y) \rhd M(x)$, and dually that if $x \lhd y$ and $M(y) \lhd y$ then $M(x) \lhd x$ and $M(x) \lhd M(y)$. As pointed out by the referee, a concept equivalent to this one, for Eulerian posets, has been introduced also in [7].

The motivation for this definition is given by the next result.

Proposition 4.1 *Let (W, S) be a Coxeter system, $u, v \in W$, $u \leq v$, and $s \in D(v) \setminus D(u)$. Let*

$$M(x) \stackrel{\text{def}}{=} xs$$

for all $x \in [u, v]$, then M is a special matching of $[u, v]$. In particular, $[e, v]$ has a special matching.

Proof. This follows immediately from the definition of a special matching and the Lifting Lemma (see, e.g., [12, §5.9]). $\Box$

There is of course a left version of the preceding result. Note that the converse of Proposition 4.1 is not true. Namely, there are special matchings which are not given by right or left multiplication by a simple reflection. For example, let (W, S) be a Coxeter system such that $|S| \geq 3$ and there are $a, b, c \in S$ such that $m(a, b), m(b, c) \geq 3$. Then $M = \{\{e, b\}, \{a, ab\}, \{c, bc\}, \{ac, abc\}\}$ is a special matching of $[e, abc]$. Notice that this matching is combinatorially indistinguishable from the two special matchings given by right multiplication by c and left multiplication by a.

Probably the most fundamental property of special matchings is the following one, which is an analogue, for any partially ordered set, of the well-known "Lifting Lemma" which holds for the Bruhat order of a Coxeter group.

Lemma 4.2 *Let P be a graded poset, M be a special matching of P, and $x, y \in P$, $x < y$, be such that $M(x) \rhd x$ and $M(y) \lhd y$. Then $M(x) \leq y$ and $x \leq M(y)$.*

Proof. We proceed by induction on $\rho(y) - \rho(x)$, ρ being the rank function of P. If $x \lhd y$ then, by the definition of special matching, $M(x) = y$ and the result holds. So assume $\rho(y) - \rho(x) \geq 2$. Let $z \in P$ be such that $x < z \lhd y$. If $M(z) \neq y$ then, by the definition of special matching, $M(z) \lhd M(y)$ and hence $M(z) \lhd z$. Therefore, by the induction hypothesis, $M(x) \leq z$ and $x \leq M(z)$ and the result follows in this case. We may therefore assume that $z = M(y)$. Let $w \in P$ be such that $x \lhd w < y$. If $M(x) \neq w$ then, by the definition of special matching, $M(x) \lhd M(w)$ and $w \lhd M(w)$, so by induction $M(w) \leq y$ and $w \leq M(y)$, and the result again follows. If $M(x) = w$ then the result trivially holds. □

In the important special case of Eulerian lattices, more can be said.

Proposition 4.3 *Let L be an Eulerian lattice, M be a special matching of L, and $x, y \in L$, $x < y$, be such that $M(x) \rhd x$ and $M(y) \lhd y$. Then $M(x) \not\leq M(y)$.*

Proof. Let ρ be the rank function of L. We proceed by induction on $\rho(y) - \rho(x)$. The result is clear if $\rho(y) - \rho(x) \leq 2$. So suppose $r \stackrel{\text{def}}{=} \rho(y) - \rho(x) \geq 3$, and assume, by contradiction, that $M(x) \leq M(y)$. Let $M(x) = x_1 \lhd x_2 \lhd \ldots \lhd x_{r-1} = M(y)$ be a saturated chain from $M(x)$ to $M(y)$. Since L is Eulerian, there is a unique $y_{r-1} \in L$, $y_{r-1} \neq x_{r-1}$, such that $x_{r-2} \lhd y_{r-1} \lhd y$. This, since M is a special matching, implies that $M(y_{r-1}) \lhd x_{r-1}$ and hence, since L is a lattice, that $M(y_{r-1}) = x_{r-2}$. Therefore $M(y_{r-1}) \geq M(x)$ and this, by our induction hypothesis, is a contradiction. This concludes the induction step and hence the proof. □

Proposition 4.4 *Let P be a graded poset, M be a special matching of P, and $x \in P$ be such that $M(x) \lhd x$. Then M restricts to a special matching of $\{y \in P : \ y \leq x\}$.*

Proof. Let $y \leq x$. It follows immediately from Lemma 4.2 that $M(y) \leq x$, so M restricts to a matching of $\{y \in P : \ y \leq x\}$. It is clear that this is still a special matching. □

We now turn to the study of special matchings of Bruhat intervals. The next three results are fundamental for this work.

Lemma 4.5 *Let $v \in S_{n+1}$ and M, M' be two special matchings of $[e, v]$ such that $M(u) = M'(u)$ for all $u \in [e, v]$ with $l(u) \leq 1$. Then*

$$M(u) = M'(u) \tag{11}$$

for all $u \in [e, v]$.

Proof. We prove (11) by induction on $l(u)$. So let $u \in [e, v]$ be such that $l(u) \geq 2$, and assume that $M(b) = M'(b)$ for all $b \in [e, v]$ such that $l(b) < l(u)$. If $M(u) \lhd u$ then by induction $u = M(M(u)) = M'(M(u))$ and so $M'(u) = M(u)$. Similarly if $M'(u) \lhd u$.

So assume that $M(u) \rhd u$ and $M'(u) \rhd u$. Let $\{c_1, \ldots, c_r\}$ be the elements covered by u. Then $M(c_1), \ldots, M(c_r) \neq u$. We claim that

$$\{M(c_i):\ l(M(c_i)) = l(u)\} = \{z \lhd M(u):\ z \neq u\}.$$

Indeed, it is clear from the fact that M is a special matching that $M(c_i) \lhd M(u)$ if $l(M(c_i)) = l(u)$. Conversely, if $z \lhd M(u)$, $z \neq u$, then, because M is a special matching, $M(z) \lhd z$ and $M(z) \lhd u$. So $M(z)$ is covered by u and $l(M(M(z))) = l(u)$. Of course, the same reasoning holds for M'. Therefore, by our induction hypothesis,

$$\begin{aligned} \{z:\ z \lhd M(u)\} &= \{u\} \biguplus \{M(c_i):\ l(M(c_i)) = l(u)\} \\ &= \{u\} \biguplus \{M'(c_i):\ l(M'(c_i)) = l(u)\} \\ &= \{z:\ z \lhd M'(u)\} \end{aligned}$$

and hence, since $l(M(u)) \geq 3$, we conclude from Proposition 3.1 that $M(u) = M'(u)$, as desired. $\square$

One can show that the preceding lemma actually holds for any (not necessarily lower) Bruhat interval of S_n.

Lemma 4.6 *Let $v \in S_{n+1}$, $a, b, c \in S$ be such that $m(a,b) = m(a,c) = 3$, $cab \not\leq v$, $J \stackrel{\text{def}}{=} J(a,c)$, and M be a special matching of $[e,v]$ such that $M(u) = u_J a\ {}^J u$ for all $u \in [e,v]$ with $l(u) \leq 1$. Then*

$$M(u) = u_J a\ {}^J u$$

for all $u \in [e,v]$.

Proof. We proceed by induction on $l(u)$. Let $u \in [e,v]$ be such that $l(u) \geq 2$. If $M(u) \lhd u$ then by induction we have that $u = M(M(u)) = M(u)_J a\ {}^J M(u)$ Hence $u_J = M(u)_J\, a$, ${}^J u = {}^J M(u)$ and therefore $M(u) = M(u)_J\ {}^J M(u) = u_J a\ {}^J u$. Similarly, if $u_J a\ {}^J u \lhd u$ then by induction $M(u_J a\ {}^J u) = (u_J a) a\ {}^J u = u$ so $M(u) = u_J a\ {}^J u$. So suppose that $M(u) \rhd u$ and $u_J a\ {}^J u \rhd u$. Let $\{c_1, \ldots, c_r\}$ be the elements covered by u. Then $M(c_1), \ldots, M(c_r) \neq u$ and, as we have seen in the proof of Lemma 4.5,

$$\{M(c):\ c \lhd u,\ l(M(c)) = l(u)\} = \{z \lhd M(u):\ z \neq u\}.$$

We claim that

$$\{z \lhd u_J a\ {}^J u:\ z \neq u\} = \{c_J a\ {}^J c:\ c \lhd u, l(c_J a\ {}^J c) = l(u)\}.$$

In fact, it follows from Lemma 3.4 that $c_J a\ {}^J c \leq u_J a\ {}^J u$ (for if $c_J a\ {}^J c = u$ then $u_J a\ {}^J u = (c_J a) a\ {}^J c = c$ which contradicts $u_J a\ {}^J u \rhd u$) and therefore that $c_J a\ {}^J c \lhd u_J a\ {}^J u$

if $l(c_J a\, {}^J c) = l(u)$. Conversely, since $cab \not\leq u_J a\, {}^J u$ (for if $cab \leq u_J a\, {}^J u$ then $cab \leq^J u$ and hence $cab \leq u \leq v$, which is a contradiction) if $z \lhd u_J a\, {}^J u$, $z \neq u$, then by Lemma 3.4 we have that $z_J a\, {}^J z \lhd u_J\, {}^J u = u$ and $z_J a\, {}^J z \lhd z$. So $c \stackrel{\text{def}}{=} z_J a\, {}^J z$ is covered by u and $c_J a\, {}^J c = z$. Therefore, by our induction hypothesis,

$$\begin{aligned} \{z :\ z \lhd M(u)\} &= \{u\} \biguplus \{M(c) :\ c \lhd u,\ l(M(c)) = l(u)\} \\ &= \{u\} \biguplus \{c_J a\, {}^J c :\ c \lhd u,\ l(c_J a\, {}^J c) = l(u)\} \\ &= \{z :\ z \lhd u_J a\, {}^J u\} \end{aligned}$$

and hence, since $l(M(u)) \geq 3$, we conclude from Proposition 3.1 that $M(u) = u_J a\, {}^J u$, as desired. $\Box$

There is, of course, a right version of Lemma 4.6.

The same reasoning used to prove the previous lemma also proves the following result.

Lemma 4.7 *Let $v \in S_{n+1}$ and M be a special matching of $[e, v]$ such that $M(u) = uM(e)$ for all $u \in [e, v]$ with $l(u) \leq 1$. Then*

$$M(u) = u\, M(e) \tag{12}$$

for all $u \in [e, v]$. $\Box$

Note that Lemma 4.7 is not a special case of Lemma 4.5 since in Lemma 4.7 we do not assume that $M'(u) \stackrel{\text{def}}{=} uM(e)$ is a special matching of $[e, v]$. Once again, there is of course a left version of Lemma 4.7.

We conclude this section by mentioning one more property of special matchings of lower intervals of symmetric groups. Namely that they (seen as involutions on the elements of the interval) generate a Coxeter group, having them as Coxeter generators. This surprising fact, which is a non-trivial consequence of one of the main results of this paper (Theorem 5.1), will be proved elsewhere.

5 Main Results

In this section we prove the main results of this work. Namely, we classify all the possible special matchings of lower intervals in the Bruhat order of the symmetric groups. This implies a poset-theoretic way of computing the Kazhdan-Lusztig and R-polynomials, and hence their combinatorial invariance and so that of the local intersection cohomology of Schubert varieties.

Theorem 5.1 *Let $v \in S_{n+1}$, and M be a special matching of $[e, v]$. Then either*

$$M(u) = ua \tag{13}$$

for all $u \in [e, v]$, or

$$M(u) = au \tag{14}$$

for all $u \in [e, v]$, or

$$M(u) = u_J \, a \, {}^J u \tag{15}$$

for all $u \in [e, v]$, where $a \stackrel{\text{def}}{=} M(e)$, b is a neighbor of a in the Dynkin diagram of S_{n+1}, and $J \stackrel{\text{def}}{=} J(a, b)$.

Proof. Let b be a neighbor of a in the Dynkin diagram of S_{n+1}.

If there is no $c \in [e, v]_1 \setminus \{b\}$ such that $m(c, a) = 3$ then $ca = ac$ for all $c \in [e, v]_1 \setminus \{b\}$ and therefore $M(c) = ca = ac$ for all $c \in [e, v]_1 \setminus \{b\}$. Hence either (14) holds for all $u \in [e, v]$ with $l(u) \leq 1$ (if $M(b) = ab$, or if $b \not\leq v$) or (13) holds for all $u \in [e, v]$ with $l(u) \leq 1$ (if $M(b) = ba$, or if $b \not\leq v$). This, by Lemma 4.7, implies that either (14) or (13) holds, as desired.

We may therefore assume that there is a (necessarily unique) $c \in [e, v]_1 \setminus \{b\}$ such that $m(a, c) = 3$, and that $b \leq v$. Then $m(b, c) = 2$ so $bc = cb \leq v$.

Since M is a special matching we have that

$$M(b) \in \{ab, ba\}$$

and

$$M(c) \in \{ac, ca\}.$$

Assume first that $cab \leq v$ and $bac \leq v$. We claim that then either $M(c) = ca$ and $M(b) = ba$, or $M(c) = ac$ and $M(b) = ab$. In fact, suppose that $M(b) = ba$ and $M(c) = ac$. Then, since M is a special matching, $M(ab) \triangleright ab$ and $M(ab) \triangleright M(b) = ba$ which implies that $M(ab) = aba$. Similarly, $M(ca) = aca$. But then, since M is a special matching, $M(bc) \triangleright M(b) = ba$ and $M(bc) \triangleright M(c) = ac$, which implies that $M(bc) = bac$. But then, since M is a special matching, $M(cab) \triangleright M(ca) = aca$, $M(cab) \triangleright M(cb) = bac$, and $M(cab) \triangleright M(ab) = aba$, and this, by Lemma 3.5, is impossible. The case $M(b) = ab$, $M(c) = ca$ is exactly analogous. This proves our claim. Hence either $M(u) = ua$ for all $u \in [e, v]$, with $l(u) \leq 1$, or $M(u) = au$ for all $u \in [e, v]$, with $l(u) \leq 1$, and this, by Lemma 4.7, implies that either (13) or (14) holds as desired.

Assume now that either $cab \not\leq v$ or $bac \not\leq v$. Suppose that $bac \not\leq v$. If either $M(c) = ca$ and $M(b) = ba$, or $M(c) = ac$ and $M(b) = ab$, then either $M(u) = ua$

for all $u \in [e,v]$, with $l(u) \leq 1$, or $M(u) = au$ for all $u \in [e,v]$, with $l(u) \leq 1$, and this, by Lemma 4.7, implies that either (13) or (14) holds as desired. If $M(c) = ac$ and $M(b) = ba$ then, since M is a special matching, $M(bc) \rhd M(b) = ba$ and $M(bc) \rhd M(c) = ac$, which implies that $M(bc) = bac$, and this is a contradiction since $bac \not\leq v$. If $M(c) = ca$ and $M(b) = ab$ then $M(u) = u_J a \, {}^J u$ for all $u \in [e,v]$ with $l(u) \leq 1$ and hence, by Lemma 4.6, (15) holds. Similarly, if $cab \not\leq v$ then we conclude that either (13), (14), or (15) holds, as desired. $\Box$

Using the preceding result we can show that the Kazhdan-Lusztig R-polynomials satisfy the following poset-theoretic recursion.

Theorem 5.2 *Let $v \in S_{n+1}$, and M be a special matching of $[e,v]$. Then*

$$R_{u,v}(q) = q^c R_{M(u),M(v)} + (q^c - 1)\, R_{u,M(v)}, \tag{16}$$

for all $u \in [e,v]$, where $c \stackrel{\mathrm{def}}{=} 0$ if $M(u) \lhd u$ and $c \stackrel{\mathrm{def}}{=} 1$ if $M(u) \rhd u$.

Proof. Let $a \stackrel{\mathrm{def}}{=} M(e)$. We proceed by induction on $l(v)$, the result being easy to check if $l(v) \leq 2$. So assume that $l(v) \geq 3$. Since M is a special matching of $[e,v]$ it follows from Theorem 5.1 that there are three possibilities for M. If $M(u) = ua$ for all $u \in [e,v]$ or $M(u) = au$ for all $u \in [e,v]$ then (16) clearly holds.

Suppose that $M(u) = u_J \, a \, {}^J u$ for all $u \in [e,v]$, where b is a neighbor of a in the Dynkin diagram of S_{n+1}, and $J \stackrel{\mathrm{def}}{=} J(a,b)$. Note that it follows from the proof of Theorem 5.1 that this can only happen if there is a (necessarily unique) $c \in [e,v]_1 \setminus \{b\}$ such that $m(a,c) = 3$, $b \leq v$, and $bac \not\leq v$.

Let $s_i \stackrel{\mathrm{def}}{=} a$ ($i \in [2, n-1]$), and suppose that $b = s_{i+1}$ and $c = s_{i-1}$. Let $u \leq v$. Then, by Lemma 3.6, $u([i-1]) \subseteq [i+1]$. There follows that there are (unique) indices $j, k \in [i, n+1]$, $j < k$, such that $u(j), u(k) \in [i+1]$, and it is easy to check that

$$M(u) = u_J s_i \, {}^J u = u(j,k). \tag{17}$$

Let $l, m \in [i, n+1]$ be such that $M(v) = v(l,m)$ (note that we do not assume that $l < m$).

If $\{l,m\} = \{i, i+1\}$ then $v([i+1]) = [i+1]$ and therefore, by Theorem 2.2, $u([i+1]) = [i+1]$ for all $u \leq v$. This, by (17), forces $M(u) = us_i$ for all $u \leq v$, so (16) clearly holds in this case.

We may therefore assume, without loss of generality, that $m \geq i+1$ and $m - 1 \neq l$. Let $s \stackrel{\mathrm{def}}{=} (m-1, m)$. Then $s \in D(v)$, $s \in D(M(v))$, and $M(vs) = M(v)s$, so $M(vs) \lhd vs$. In particular, by Proposition 4.4, M restricts to a special matching of $[e, vs]$. Let $u \leq v$, and let j, k have the same meaning as in (17). Note that $M(us) = M(u)s$. There are now two cases to distinguish.

Suppose first that $M(u) \lhd u$. Then, by (17), $u(j) > u(k)$. If $s \in D(u)$ and $s \in D(M(u))$ then $M(us) \lhd us$. So by our induction hypothesis we have that

$$R_{u,v} = R_{us,vs} = R_{M(us),M(vs)} = R_{M(u)s,M(v)s} = R_{M(u),M(v)}$$

as desired. If $s \in D(u)$ and $s \notin D(M(u))$ then necessarily $s = (j, k)$. So $M(u) = us$ and $M(us) \rhd us$ and we obtain, using induction, that

$$\begin{aligned} R_{u,v} = R_{us,vs} &= qR_{M(us),M(vs)} + (q-1)R_{us,M(vs)} \\ &= qR_{M(u)s,M(v)s} + (q-1)R_{us,M(v)s} \\ &= qR_{M(u)s,M(v)s} + (q-1)R_{M(u),M(v)s} \\ &= R_{M(u),M(v)} \end{aligned}$$

and (16) follows. If $s \notin D(u)$ then, since $u(j) > u(k)$, $s \notin D(M(u))$ and hence $M(us) \lhd us$ so, by induction and Lemma 4.2,

$$\begin{aligned} R_{u,v} &= qR_{us,vs} + (q-1)R_{u,vs} \\ &= qR_{M(us),M(vs)} + (q-1)R_{M(u),M(vs)} \\ &= qR_{M(u)s,M(v)s} + (q-1)R_{M(u),M(v)s} \\ &= R_{M(u),M(v)}, \end{aligned}$$

(note that, if $us \not\leq vs$, then by Lemma 4.2 $M(us) \not\leq M(vs)$, so $R_{us,vs} = R_{M(us),M(vs)}$ also in this case) and (16) again follows.

Suppose now that $M(u) \rhd u$. Then, by (17), $u(j) < u(k)$. Therefore, if $s \in D(u)$ then $s \in D(M(u))$ and hence $M(us) \rhd us$, so we obtain

$$\begin{aligned} R_{u,v} = R_{us,vs} &= qR_{M(us),M(vs)} + (q-1)R_{us,M(vs)} \\ &= qR_{M(u)s,M(v)s} + (q-1)R_{us,M(v)s} \\ &= qR_{M(u),M(v)} + (q-1)R_{u,M(v)}, \end{aligned}$$

as desired. If $s \notin D(u)$ and $s \notin D(M(u))$ then $M(us) \rhd us$ and we obtain that

$$\begin{aligned} R_{u,v} &= qR_{us,vs} + (q-1)R_{u,vs} \\ &= q(qR_{M(us),M(vs)} + (q-1)R_{us,M(vs)}) + (q-1)(qR_{M(u),M(vs)} + (q-1)R_{u,M(vs)}) \\ &= q(qR_{M(u)s,M(v)s} + (q-1)R_{us,M(v)s}) + (q-1)(qR_{M(u),M(v)s} + (q-1)R_{u,M(v)s}) \\ &= qR_{M(u),M(v)} + (q-1)R_{u,M(v)}, \end{aligned}$$

so (16) again holds. If $s \notin D(u)$ and $s \in D(M(u))$ then $s = (j,k)$. Therefore $M(us) = u \lhd us$ and we conclude that

$$\begin{aligned} R_{u,v} &= qR_{us,vs} + (q-1)R_{u,vs} \\ &= qR_{M(us),M(vs)} + (q-1)(qR_{M(u),M(vs)} + (q-1)R_{u,M(vs)}) \\ &= qR_{M(u)s,M(v)s} + (q-1)(qR_{us,M(v)s} + (q-1)R_{u,M(v)s}) \\ &= qR_{M(u),M(v)} + (q-1)R_{u,M(v)}, \end{aligned}$$

and (16) holds.

The case $c = s_{i+1}$, $b = s_{i-1}$ is analogous. $\Box$

One may conjecture that Theorem 5.2 holds for any Coxeter group, and any Bruhat interval (not necessarily lower).

Theorem 5.2 immediately implies, by Theorems 2.6 and 2.7, the combinatorial invariance of the Kazhdan-Lusztig and R-polynomials, and hence of the local intersection cohomology spaces of Schubert varieties.

Corollary 5.3 *Let $v \in S_{n+1}$ and $v' \in S_{m+1}$ ($n, m \in \mathbf{P}$) be such that $[e,v] \cong [e,v']$ (as posets). Then*

$$R_{u,z} = R_{f(u),f(z)},$$

$$P_{u,z} = P_{f(u),f(z)},$$

and

$$dim_{\mathbf{C}}(IH^i(\overline{\Omega}_z, \mathbf{C})_{\Omega_u}) = dim_{\mathbf{C}}(IH^i(\overline{\Omega}_{f(z)}, \mathbf{C})_{\Omega_{f(u)}})$$

for all $u, z \in [e,v]$, $i \in \mathbf{N}$, and all poset isomorphisms $f : [e,v] \to [e,v']$. $\Box$

The preceding result has also been proved, independently, by F. Du Cloux in [8]. Du Cloux's proof is different from ours: his method is nonconstructive (i.e., does not yield an algorithm for computing the polynomials from the poset $[e,v]$, see [8, §1.3]) but applies to a more general class of Coxeter groups.

6 An example

We illustrate Theorem 5.2 with an example. Let $P = [e,w]$ be the lower interval whose Hasse diagram is depicted in Figure 1 [2]. Identify, for convenience, the elements of P with the integers from 1 to 18 by numbering the elements of P from bottom to

[2]We do not give w since the point of the example is exactly to show how one can compute the polynomials from the poset rather than from the permutation.

top and from left to right (so, for example, 12 is the third element, from the left, of rank three of P). According to Theorem 5.2, we need to find a special matching M of P. Suppose $M(1) = 2$. We have two possible choices for $M(3)$, namely 7 and 8, and two for $M(4)$, namely 5 and 6. Suppose we choose $M(3) = 7$ and $M(4) = 5$. According to Lemma 4.5, if there is a special matching M of P such that $M(1) = 2$, $M(3) = 7$, and $M(4) = 5$ then it is unique. Indeed, in this case these choices force $M = \{\{1,2\},\{3,7\},\{4,5\},\{6,11\},\{8,10\},\{9,12\},\{13,15\},\{14,16\},\{17,18\}\}$.

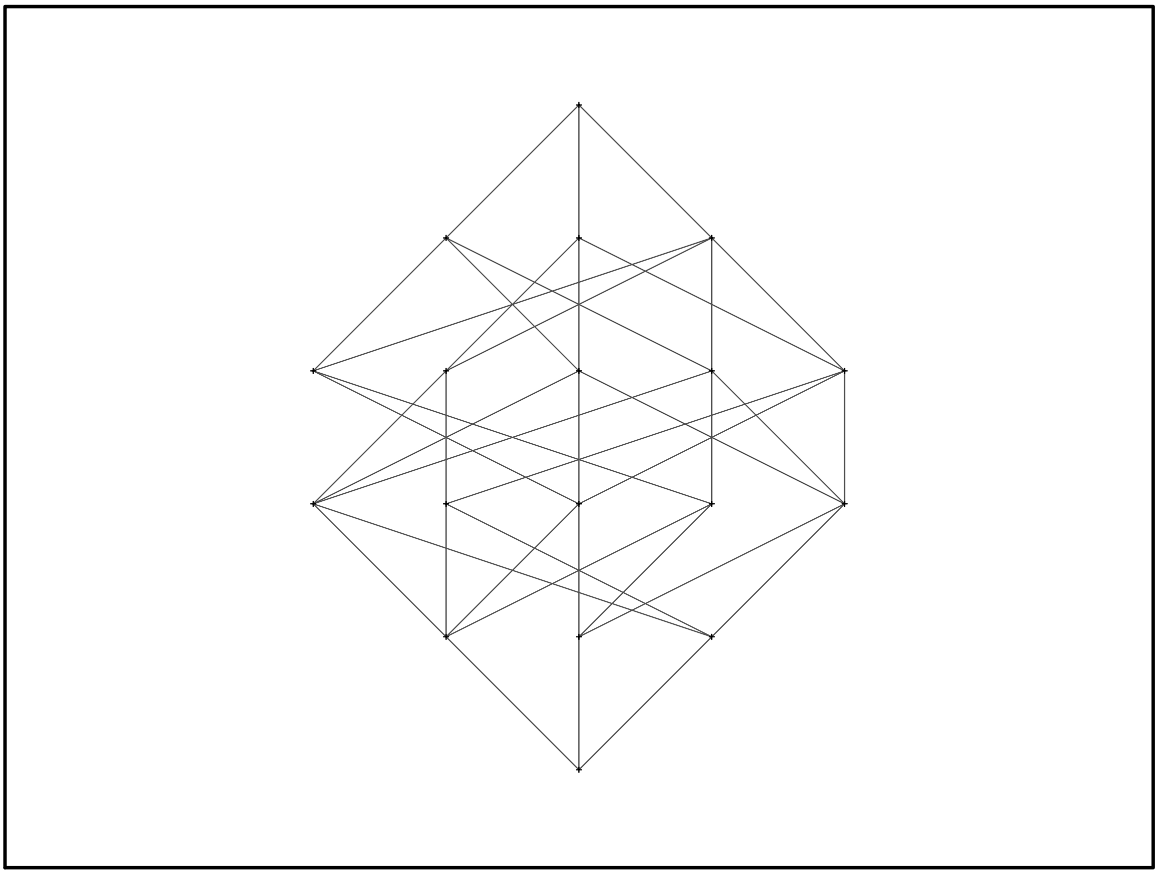

Figure 1.

Applying Theorem 5.2 we obtain that

$$R_{1,18} = q\, R_{M(1),M(18)} + (q-1)R_{1,M(18)} = qR_{2,17} + (q-1)R_{1,17}$$

(as well as $R_{2,18} = R_{1,17}$, $R_{3,18} = qR_{7,17} + (q-1)R_{3,17}$, $R_{4,18} = qR_{5,17} + (q-1)R_{4,17}$, $R_{5,18} = R_{4,17}$, $R_{6,18} = qR_{11,17} + (q-1)R_{6,17}$, $R_{7,18} = R_{3,17}$, $R_{8,18} = qR_{10,17} + (q-1)R_{8,17}$, $R_{9,18} = qR_{12,17} + (q-1)R_{9,17} = (q-1)R_{9,17}$, and similarly $R_{10,18} = R_{8,17}$, $R_{11,18} = R_{6,17}$, $R_{12,18} = R_{9,17}$, $R_{13,18} = (q-1)R_{13,17}$, $R_{14,18} = (q-1)R_{14,17}$, $R_{15,18} =$

$R_{13,17}$, $R_{16,18} = R_{14,17}$, $R_{17,18} = (q-1)R_{17,17}$). We therefore need to compute the polynomials $R_{u,17}$ for all $u \leq 17$. Since M does not restrict to a special matching of $[1, 17]$ ($= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 17\}$ we need to repeat the above procedure to find a special matching, N, of $[1, 17]$. Suppose that $N(1) = 2$. This forces $N(3) \in \{7, 8\}$ and $N(4) \in \{5, 6\}$. Suppose we choose $N(3) = 7$ and $N(4) = 6$. Then by Lemma 4.5 the rest of N is uniquely determined and indeed our choices force $N = \{\{1, 2\}, \{3, 7\}, \{4, 6\}, \{5, 11\}, \{8, 10\}, \{9, 14\}, \{13, 17\}\}$. Applying Theorem 5.2 we get

$$\begin{aligned} R_{1,17} &= q\, R_{2,13} + (q-1)R_{1,13}, \\ R_{2,17} &= R_{1,13}, \end{aligned}$$

(as well as $R_{3,17} = (q-1)R_{3,13}$, $R_{4,17} = (q-1)R_{4,13}$, $R_{5,17} = (q-1)R_{5,13}$, $R_{6,17} = R_{4,13}$, $R_{7,17} = R_{3,13}$, $R_{8,17} = (q-1)R_{8,13}$, $R_{9,17} = (q-1)R_{9,13}$, $R_{10,17} = R_{8,13}$, $R_{11,17} = R_{5,13}$, $R_{13,17} = (q-1)R_{13,13}$, $R_{14,17} = R_{9,13}$). We now need to compute the polynomials $R_{u,13}$ for all $u \in [1, 13]$ ($= \{1, 2, 3, 4, 5, 8, 9, 13\}$). The poset $[1, 13]$ is a Boolean algebra of rank 3, so $R_{u,13} = (q-1)^{l(u,13)}$ for all $u \leq 13$ (e.g., by Theorem 6.3 of [5], and also directly in this case). However, since no outside result is needed by the procedure in Theorem 5.2, and for completeness, we conclude the example using only Theorem 5.2. We need a special matching, L, of $[1, 13]$. Suppose $L(1) = 4$, then this forces $L = \{\{1, 4\}, \{2, 5\}, \{3, 9\}, \{8, 13\}\}$. So by Theorem 5.2

$$\begin{aligned} R_{1,13} &= (q-1)R_{1,8}, \\ R_{2,13} &= (q-1)R_{2,8}, \end{aligned}$$

(as well as $R_{3,13} = (q-1)R_{3,8}$, $R_{4,13} = R_{1,8}$, $R_{5,13} = R_{2,8}$, $R_{8,13} = (q-1)R_{8,8}$, $R_{9,13} = R_{3,8}$). Now, a special matching of $[1, 8]$ ($= \{1, 2, 3, 8\}$) is $\{\{1, 2\}, \{3, 8\}\}$, and from Theorem 5.2 we get

$$R_{1,8} = (q-1)R_{1,3}, \quad R_{2,8} = R_{1,3}, \quad R_{3,8} = (q-1)R_{3,3},$$

and $\{\{1, 3\}\}$ is a special matching of $[1, 3]$ ($= \{1, 3\}$) and so again by Theorem 5.2 we obtain $R_{1,3} = (q-1)R_{1,1}$. Putting all these relations together we then get

$$\begin{aligned} R_{1,18} &= q\, R_{2,17} + (q-1)R_{1,17} \\ &= q\, R_{1,13} + (q-1)(q\, R_{2,13} + (q-1)R_{1,13}) \\ &= q(q-1)R_{1,8} + q(q-1)^2 R_{2,8} + (q-1)^3 R_{1,8} \\ &= q(q-1)^2 R_{1,3} + q(q-1)^2 R_{1,3} + (q-1)^4 R_{1,3} \\ &= 2q(q-1)^3 + (q-1)^5, \end{aligned}$$

and similarly for all the other polynomials $R_{u,18}$. Clearly, in the same way (and in fact without much additional effort since we already have a special matching of $[e, v]$ for all $v \in P$) we may compute all the polynomials $R_{u,v}$ for $u, v \in P$, $u \leq v$. The computation of the Kazhdan-Lusztig polynomials $P_{u,v}$ for $u, v \in P$, $u \leq v$, now proceeds using Theorem 2.6 and induction on $l(v) - l(u)$.

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