Enumerative and combinatorial properties of Dyck partitions

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Abstract

The purpose of this paper is to study the combinatorial and enumerative properties of a new class of (skew) integer partitions. This class is closely related to Dyck paths and plays a fundamental role in the computation of certain Kazhdan-Lusztig polynomials of the symmetric group related to Young's lattice. As a consequence of our results, we obtain some new identities for these polynomials.

Keywords: Dyck path, partition, lattice path, Kazhdan-Lusztig polynomial.

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1 Introduction

The purpose of this paper is to study the combinatorial and enumerative properties of a new class of (skew) integer partitions. This class is closely related to Dyck paths and plays a fundamental role in the computation of certain Kazhdan-Lusztig polynomials of the symmetric group related to Young's lattice (see [5], and §6). These, in turn, have applications to representation theory and algebraic geometry (see, e.g., [9], [10], [8], [2]). As a consequence of our results, we obtain some new identities for these polynomials. While [5] deals mainly with the algebraic aspects of Dyck partitions, the present paper focuses almost exclusively on their combinatorial properties.

The organization of the paper is as follows. In the next section we recall some definitions, notation, and results, that will be used in the sequel. In §3 we give some fundamental combinatorial properties of Dyck partitions, including a combinatorial characterization. In §4 we construct a bijection that enables a simple encoding (and construction) of Dyck partitions. This bijection involves a statistic defined over the points of a lattice path which seems to be new. In §5 we study the problem of enumerating Dyck partitions. Using the bijection constructed in §4 we derive a recurrence relation, and a functional equation, for the generating function of Dyck partitions according to three natural statistics. In §6 we apply the results of §4 to the Kazhdan-Lusztig polynomials of the symmetric group, obtaining some new identities for them. Finally, in §7, we discuss several open problems arising from the present work, including possible analogues of Dyck partitions for other Coxeter groups.

2 Notation, Definitions, and preliminaries

In this section we collect some definitions, notation and results that will be used in the rest of this paper. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$, $\mathbf{Z}$ be the set of integers, and $\mathbf{Q}$ be the set of rational numbers; for $a, b \in \mathbf{Z}$, $a \leq b$, we let $[a, b] \stackrel{\text{def}}{=} \{a, a+1, \ldots, b\}$ (where $[a, b] \stackrel{\text{def}}{=} \emptyset$ if $b < a$), and $[a] \stackrel{\text{def}}{=} [1, a]$. The cardinality of a set A will be denoted by $|A|$. Given a polynomial $P(q)$, and $i \in \mathbf{Q}$, we denote by $[q^i](P(q))$ the coefficient of q^i in $P(q)$ (so $[q^i](P(q)) = 0$ unless $i \in \mathbf{N}$). Given a commutative ring R and variables $x_1, x_2, \ldots, x_n$ we denote by $R[x_1, x_2, \ldots, x_n]$

(respectively, $R(x_1, x_2, \ldots, x_n)$, $R[[x_1, x_2, \ldots, x_n]]$) the ring of polynomials (respectively, field of rational functions, ring of formal power series) with coefficients in R in the variables $x_1, x_2, \ldots, x_n$.

Given a set T we let $S(T)$ be the set of all bijections $\pi : T \to T$, and $S_n \stackrel{\text{def}}{=} S([n])$. If $\sigma \in S_n$ then we write $\sigma = \sigma_1 \ldots \sigma_n$ to mean that $\sigma(i) = \sigma_i$, for $i = 1, \ldots, n$. If $\sigma \in S_n$ then we also write σ in *disjoint cycle form* (see, e.g., [13], p.17) and we usually omit to write the 1-cycles of σ. For example, if $\sigma = 365492187$ then we also write $\sigma = (9, 7, 1, 3, 5)(2, 6)$. Given $\sigma, \tau \in S_n$ we let $\sigma\tau \stackrel{\text{def}}{=} \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$. Recall that for $u \in S_n$ the number of *inversions* of u is

$$inv(u) = |\{(i, j) \in [n] \times [n] : \ i < j, \ u(i) > u(j)\}|.$$

We follow Chapter 3 of [13] for poset notation and terminology, and §7.2 of [15] for any undefined notation and terminology concerning partitions.

By an (integer) *partition* we mean a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. We identify a partition λ with its *diagram*,

$$\{(i, j) \in \mathbf{P}^2 : \ 1 \leq i \leq k, \ 1 \leq j \leq \lambda_i\},$$

and consider λ as a poset with the partial ordering induced by $\mathbf{P}^2$ (where $\mathbf{P}^2$ has the product ordering induced by the natural ordering on $\mathbf{P}$). For this reason we draw the diagram of a partition λ rotated counterclockwise by $\frac{3}{4}\pi$ radians with respect to the usual (Anglophone) convention (see, e.g., [13, §1.3]). So, for example, the diagram of $(7, 4, 2, 2)$ is illustrated in Figure 1.

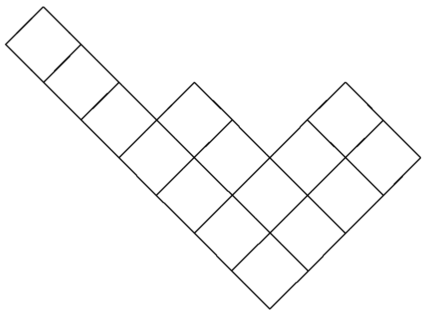

Figure 1: The diagram of $(7, 4, 2, 2)$.

We call the elements of $\mathbf{P}^2$, and hence of λ, cells. Expressions such as "to the left of", or "directly above", always refer to these rotated diagrams. We define the *level* of a cell $(i,j) \in \mathbf{P}^2$ by $lv((i,j)) \stackrel{\text{def}}{=} i+j$. We denote by $\mathcal{P}$ the set of all integer partitions. We will always assume that $\mathcal{P}$ is partially ordered by set inclusion. It is well known, and not hard to see, that this makes $\mathcal{P}$ into a lattice, usually called *Young's lattice* (see, e.g., [15, §7.2]).

Let $\mu, \lambda \in \mathcal{P}$, $\mu \subseteq \lambda$. We then call $\lambda \setminus \mu$ a *skew partition* (or, a *skew shape*). Note that, in poset theoretic language, partitions (respectively, skew-partitions) are the finite order ideals (respectively, finite convex subsets) of $\mathbf{P}^2$. Given a skew partition $\eta \subseteq \mathbf{P}^2$ its *conjugate* is

$$\eta' \stackrel{\text{def}}{=} \{(j,i) \in \mathbf{P}^2 : \ (i,j) \in \eta\}.$$

Let θ be a connected (by which we mean "rookwise connected" so, e.g., $(2,1)\setminus(1)$ is not connected) skew partition, and consider θ as a subposet of $\mathbf{P}^2$. We say that a cell x of θ is an *upper peak* (respectively, *lower valley*) of θ if it is maximal (respectively, minimal). We call an element $x \in \theta$ an *upper valley* of θ if x is covered by exactly two elements of θ whose join is not in θ. Similarly, we define a *lower peak*.

We say that a skew partition is a *border strip* (also called a *ribbon*) if it contains no 2×2 square of cells. For brevity, we call a connected border strip a *cbs*. Given a cbs θ we will usually number the cells of θ consecutively from left to right, and identify them with their corresponding number. So, for example, 1 is the leftmost cell of θ, and if $x \in \theta$, $x > 1$, then $x-1$ is the cell of θ immediately to the left of x. We also write $[a,b]_\theta$ to mean the set of cells of θ corresponding to the numbers in $[a,b]$.

Let $\lambda, \mu, \nu \in \mathcal{P}$ be such that $\mu \subseteq \nu \subseteq \lambda$. We then say that $\lambda \setminus \nu$ is a *final segment* of $\lambda \setminus \mu$. The *outer border strip* θ of $\lambda \setminus \mu$ is the largest final segment of $\lambda \setminus \mu$ which is a border strip. In other words, a cell of $\lambda \setminus \mu$ is in θ if and only if there is no cell of $\lambda \setminus \mu$ directly above it. Given a cell x in the outer border strip of $\lambda \setminus \mu$ we define $d_{\lambda\setminus\mu}(x)$ to be the number of cells of $\lambda \setminus \mu$ that are directly below x (including x itself). So, for example, the cells of the outer border strip of the skew partition ν illustrated in Figure 2 are numbered from 1 to 15, and we have that $d_\nu(1) = 1$, $d_\nu(4) = 3$, $d_\nu(9) = 1$, and $d_\nu(12) = 2$. In a similar way, we define the *inner border strip* η of $\lambda \setminus \mu$ as the cells of $\lambda \setminus \mu$ which have no cells of $\lambda \setminus \mu$ directly below them. The *descending border strip* (usually called the *horizontal* border strip) of $\lambda \setminus \mu$ is

$$\mathcal{H}(\lambda \setminus \mu) = \{x \in \theta : \ \ lv(x+1) < lv(x)\} \cup \{|\theta|\}.$$

For example, if $\lambda \setminus \mu$ is the skew-partition illustrated in Figure 2, then $\mathcal{H}(\lambda \setminus \mu) = \{4, 5, 7, 8, 11, 13, 14, 15\}$.

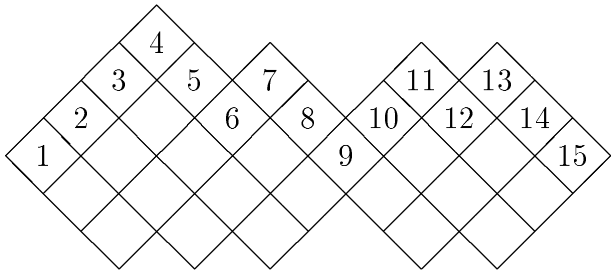

Figure 2: Numbering of the cells of the outer border strip of a skew partition.

Given two skew partitions $\rho, \nu \subset \mathbf{P}^2$ we write $\rho \approx \nu$ if ρ is a translate of ν. We denote by ν^* the (unique, up to translation) skew shape obtained by reflecting ν through some "horizontal line". More formally,

$$\nu^* \stackrel{\text{def}}{=} \{(\ell - j, \ell - i) \in \mathbf{P}^2 : \quad (i, j) \in \nu\}$$

for some $\ell \gg 0$. We say that ν is *self-dual* if $\nu \approx \nu^*$. Note that if ν is a skew shape and θ (respectively, η) is its outer (respectively, inner) border strip, then θ^* (respectively, η^*) is a translate of the inner (respectively, outer) border strip of ν^*. The verification of the following observation is left to the reader.

Proposition 2.1 *Let $\lambda, \mu \in \mathcal{P}$, $\mu \subseteq \lambda$, and θ, η be the outer and inner border strips of $\lambda \setminus \mu$, respectively. Then $(\lambda \setminus \mu) \setminus \theta \approx (\lambda \setminus \mu) \setminus \eta$.*

Let $\theta \subset \mathbf{P}^2$ be a connected border strip. Following [5] we say that θ is a *Dyck cbs* if it is a "Dyck path" (see, e.g., [15, p. 173]). In other words, it is a Dyck cbs if no cell of θ has level strictly less than that of either the leftmost or rightmost of its cells. In particular, in a Dyck cbs the leftmost and rightmost cells have the same level. For example, the cbs's in Figures 3 and 4 are Dyck, while those of Figures 5 and 6 are not. It is clear that a cbs is Dyck if and only if its conjugate is Dyck.

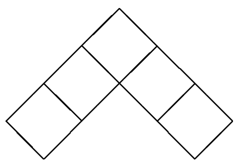

Figure 3: A Dyck cbs.

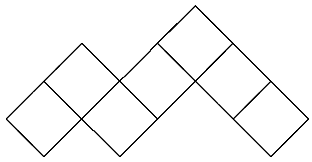

Figure 4: A Dyck cbs.

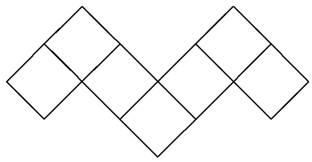

Figure 5 : A cbs that is not Dyck.

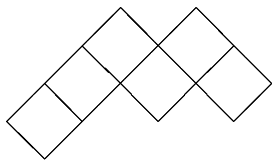

Figure 6: A cbs that is not Dyck.

Given $\lambda, \mu \in \mathcal{P}$, $\mu \subseteq \lambda$, we let $(\lambda \setminus \mu)^{(1)} \stackrel{\text{def}}{=} (\lambda \setminus \mu) \setminus \theta$, where θ is the outer border strip of $\lambda \setminus \mu$. Let $\eta \subset \mathbf{P}^2$ be a skew partition. Following [5] we define η to be *Dyck* in the following inductive way:

i) η is Dyck if and only if each one of its connected components is Dyck;

ii) if η is connected then η is Dyck if and only if:

- **a)** its outer border strip is a Dyck cbs;
- **b)** $\eta^{(1)}$ is Dyck.

Finally, we define $\emptyset$ to be Dyck. So, for example, $(4,4,4,3)$ is not Dyck while $(4,4,4,4)\setminus(1)$ and $(4,4,4,3)\setminus(1)$ are Dyck. Note that it follows immediately from the definitions that η is Dyck if and only if η' is Dyck.

Let $\eta \subset \mathbf{P}^2$ be a skew partition (not necessarily Dyck). We define the *depth* of η, denoted $dp\,(\eta)$, inductively by letting

$$dp\,(\eta) \stackrel{\text{def}}{=} c(\theta) + dp\,(\eta^{(1)})$$

(and $dp\,(\emptyset) \stackrel{\text{def}}{=} 0$), where θ is the outer border strip of η, and $c(\theta)$ denotes the number of connected components of θ. So, for example, $dp\,((4,4,4,3)) = dp\,((4,4,4,4)\setminus(1)) = 3$, while $dp\,((4,4,4,3)\setminus(1)) = 4$. Note that, if η is Dyck, then $dp\,(\eta) \equiv |\eta| \pmod 2$ (since a Dyck cbs has always odd cardinality, so $c(\theta) \equiv |\theta| \pmod 2$).

3 Combinatorial properties

In this section we derive some fundamental combinatorial properties of Dyck partitions. As a consequence of these, we obtain also a combinatorial proof of an interesting recursive characterization of Dyck partitions which was first proved in [5] using the theory of Kazhdan-Lusztig polynomials. Throughout the section $\lambda \setminus \mu$ denotes a skew partition, and θ and η denote its outer and inner border strips, respectively.

We begin by giving a simple characterization of self-dual Dyck partitions. For this we need two preliminary results. The first one is a general observation.

Proposition 3.1 *Let $\lambda \setminus \mu$ be self-dual. Then $(\lambda \setminus \mu)^{(1)}$ is self-dual.*

Proof. We may assume that $(\lambda \setminus \mu)^* = (\lambda \setminus \mu)$. Then $\theta^* = \eta$ and therefore, by Proposition 2.1,

$$((\lambda \setminus \mu)^{(1)})^* = ((\lambda \setminus \mu) \setminus \theta)^* = (\lambda \setminus \mu)^* \setminus \theta^* = (\lambda \setminus \mu) \setminus \eta \approx (\lambda \setminus \mu) \setminus \theta = (\lambda \setminus \mu)^{(1)},$$

as desired. $\diamond$

Proposition 3.2 *Let ν be a skew shape, and x be an upper peak of ν. Suppose that ν has a lower valley, y, directly below x. Then, if ν is not Dyck, both $\nu \setminus \{x, y\}$ and $\nu \setminus \{x\}$ are not Dyck.*

Proof. We proceed by induction on $d_\nu(x)$, the result being clear if $x = y$. Let $x^{(1)}$ be the cell directly below x by one cell. Then $x-1, x, x+1 \in \theta$, and therefore $x^{(1)}$ is an upper peak of $\nu^{(1)}$, and $\nu^{(1)}$ has the lower valley y directly below $x^{(1)}$. Therefore,

$$\nu^{(1)} \setminus \{x^{(1)}\} = (\nu \setminus \{x\})^{(1)}, \tag{1}$$

and

$$\nu^{(1)} \setminus \{x^{(1)}, y\} = (\nu \setminus \{x, y\})^{(1)}. \tag{2}$$

Suppose that θ is not Dyck. Then $(\theta \setminus \{x\}) \cup \{x^{(1)}\}$ is not Dyck (for if it were θ would also be), so $\nu \setminus \{x\}$ and $\nu \setminus \{x, y\}$ are not Dyck.

Suppose now that θ is Dyck. Then $\nu^{(1)}$ is not Dyck. Hence, by induction, $\nu^{(1)} \setminus \{x^{(1)}\}$ and $\nu^{(1)} \setminus \{x^{(1)}, y\}$ are not Dyck. But, by (1) and (2), this implies that $\nu \setminus \{x\}$ and $\nu \setminus \{x, y\}$ are not Dyck. $\diamond$

We can now prove the following simple characterization of self-dual (connected) Dyck skew shapes.

Proposition 3.3 *Let $\lambda \setminus \mu$ be self-dual and connected. Then the following are equivalent:*

i) *$\lambda \setminus \mu$ is Dyck;*

ii) *θ is Dyck.*

Proof. It is true by definition that i) implies ii). Conversely, assume that ii) holds. We prove i) by induction on $|(\lambda \setminus \mu)|$. Let $l \stackrel{\text{def}}{=} lv(1)$ and $x \in \theta$ be such that $lv(x) \geq lv(z)$ for all $z \in \theta$. If $lv(x) = l + 1$ then $(\lambda \setminus \mu)^{(1)}$ consists of $(|\theta| - 1)/2$ disjoint cells and i) clearly holds. So assume that $lv(x) > l+1$. Clearly, x is an upper peak of $\lambda \setminus \mu$, and there is a lower valley, y, directly below x. Let $x^{(1)}$ be the cell directly below x by one cell. Clearly, $(\lambda \setminus \mu) \setminus \{x, y\}$ is self-dual and $(\theta \setminus \{x\}) \cup \{x^{(1)}\}$ is its outer border strip. But $lv(x) > l+1$, so $(\theta \setminus \{x\}) \cup \{x^{(1)}\}$ is a Dyck cbs. Hence, by induction, $(\lambda \setminus \mu) \setminus \{x, y\}$ is Dyck, and this, by Proposition 3.2, implies that $\lambda \setminus \mu$ is Dyck, as desired. $\diamond$

In the rest of this section we give some more combinatorial properties of Dyck partitions, which lead to an interesting recursive characterization of them. We start with a preliminary result.

Lemma 3.4 *Let $\lambda \setminus \mu$ be connected and suppose that θ is Dyck. Let $\mu \subset \nu \subseteq \lambda$ be such that $\nu \setminus \mu$ is a cbs, and $\lambda \setminus \nu$ is Dyck. Then either $\nu \setminus \mu = \eta$ or $\nu \setminus \mu \subseteq (\lambda \setminus \mu)^{(1)}$.*

Proof. Suppose $1 \in \nu$. Then necessarily also $n \in \nu$ (else $\lambda \setminus \nu$ could not be Dyck) So $\nu \setminus \mu = \eta$. Suppose $1 \notin \nu$. Then (by the same reasoning as above) no cell of θ can be in ν. Hence $\nu \setminus \mu \subseteq (\lambda \setminus \mu) \setminus \theta = (\lambda \setminus \mu)^{(1)}$.$\diamond$

Proposition 3.5 *Let $\lambda \setminus \mu$ be Dyck. Then*

$$|\{\mu \subset \nu \subseteq \lambda : \ \nu \setminus \mu \textit{ is a cbs of odd size}, \ \ \lambda \setminus \nu \textit{ is Dyck } \}| >$$

$$|\{\mu \subset \nu \subseteq \lambda : \ \nu \setminus \mu \textit{ is a cbs of even size}, \ \ \lambda \setminus \nu \textit{ is Dyck } \}| = 0.$$

Proof. We proceed by induction on $|\lambda \setminus \mu|$. If $\lambda \setminus \mu$ is not connected then the result follows clearly by induction since each connected component is Dyck.

If $\lambda \setminus \mu$ is connected then by Lemma 3.4 we have that

$$\{\mu \subset \nu \subseteq \lambda : \ \nu \setminus \mu \text{ is a cbs}, \ \ \lambda \setminus \nu \text{ is Dyck } \} =$$

$$\{\mu \cup \eta\} \bigcup \{\mu \subset \nu \subseteq \lambda \setminus \theta : \ \nu \setminus \mu \text{ is a cbs}, \ \ (\lambda \setminus \theta) \setminus \nu \text{ is Dyck } \} \tag{3}$$

(since $(\lambda \setminus \mu) \setminus \eta \approx (\lambda \setminus \mu) \setminus \theta = (\lambda \setminus \mu)^{(1)}$ is Dyck, and $(\lambda \setminus \theta) \setminus \nu$ is Dyck if and only if $\lambda \setminus \nu$ is Dyck if $\nu \subseteq \lambda \setminus \theta$). But $|\eta| = |\theta|$ is odd since $\lambda \setminus \mu$ is Dyck, so the result follows by induction. $\diamond$

It is a remarkable fact that the converse of Proposition 3.5 also holds.

Theorem 3.6 *Let $\lambda \setminus \mu$ be not Dyck. Then*

$$|\{\mu \subset \nu \subseteq \lambda : \ \ \nu \setminus \mu \textit{ is a cbs of odd size}, \ \ \lambda \setminus \nu \textit{ is Dyck } \}| =$$

$$|\{\mu \subset \nu \subseteq \lambda : \ \ \nu \setminus \mu \textit{ is a cbs of even size}, \ \ \lambda \setminus \nu \textit{ is Dyck } \}|. \tag{4}$$

Proof. We proceed by induction on $|\lambda \setminus \mu|$, the result being clear if $|\lambda \setminus \mu| \leq 2$.

If $\lambda \setminus \mu$ is not connected then at least one of the connected components of $\lambda \setminus \mu$ is not Dyck. Let $\mu \subset \nu \subseteq \lambda$ be such that $\lambda \setminus \nu$ is Dyck and $\nu \setminus \mu$ is a cbs. Since $\nu \setminus \mu$ is connected, it has to be contained in one of the connected components of $\lambda \setminus \mu$, call it Γ. Since $\lambda \setminus \nu$ is Dyck, and $\lambda \setminus \mu$ is not, this connected component Γ cannot be Dyck, and all the others must be Dyck. Therefore

$$\{\mu \subset \nu \subseteq \lambda : \ \nu \setminus \mu \text{ is a cbs}, \quad \lambda \setminus \nu \text{ is Dyck } \} =$$

$$\{\mu \subset \nu \subseteq (\mu \cup \Gamma) : \ \nu \setminus \mu \text{ is a cbs}, \ \ (\mu \cup \Gamma) \setminus \nu \text{ is Dyck } \},$$

and the result follows by induction since Γ is not Dyck.

So assume that $\lambda \setminus \mu$ is connected. Let y, respectively x, be the rightmost, respectively leftmost, cell of θ such that $lv(y) < lv(n)$ (where $n \stackrel{\text{def}}{=} |\theta|$), respectively $lv(x) < lv(1)$. Note that x and/or y could not exist. There are three cases to consider.

a) x and y do not exist.

Then θ is Dyck, and hence $(\lambda \setminus \mu)^{(1)}$ is not. Therefore by Lemma 3.4, reasoning as in the proof of (3), we conclude that

$$\{\mu \subset \nu \subseteq \lambda : \ \ \nu \setminus \mu \text{ is a cbs}, \ \ \lambda \setminus \nu \text{ is Dyck } \} =$$

$$= \{\mu \subset \nu \subseteq \lambda \setminus \theta : \ \ \nu \setminus \mu \text{ is a cbs}, \ \ (\lambda \setminus \theta) \setminus \nu \text{ is Dyck } \},$$

and the result follows by induction since $(\lambda \setminus \mu)^{(1)}$ is not Dyck.

b) Both x and y exist.

Let $\mu \subset \nu \subseteq \lambda$ be such that $\nu \setminus \mu$ is a cbs and $\lambda \setminus \nu$ is Dyck (if there are no such ν then (4) trivially holds). Then $x, y \in \nu$. Therefore $x, y \in \eta$, and hence $[x, y]_\eta \subseteq \nu$.

Suppose first that $x - 1 \notin \eta$. Then $x - 1 \notin \nu$ and therefore $1 \notin \nu$ (else the outer border strip of $\lambda \setminus \nu$ could not be Dyck). Hence, for the same reason, ν does not intersect θ to the left of cell x. Let $\theta_l \stackrel{\text{def}}{=} [1, x-1]_\theta$. Then θ_l is Dyck (by the definition of x). Therefore $\lambda \setminus \nu$ is Dyck if and only if $(\lambda \setminus \theta_l) \setminus \nu$ is Dyck (since $\nu \cap \theta_l = \emptyset$). But the outer border strip of $(\lambda \setminus \mu) \setminus \theta_l$ is not Dyck (since y exists) so $(\lambda \setminus \mu) \setminus \theta_l$ is not Dyck and hence the result follows by induction.

Similarly, if $y+1 \notin \eta$ then we conclude that $\lambda \setminus \nu$ is Dyck if and only if $(\lambda \setminus \theta_r) \setminus \nu$ is Dyck (where $\theta_r \stackrel{\text{def}}{=} [y+1, n]_\theta$), and the result again follows by induction.

Suppose now that $x - 1 \in \eta$ and $y + 1 \in \eta$. Then if $x - 1 \in \nu$ also $1 \in \nu$. So either $x - 1 \notin \nu$ or $1 \in \nu$. Similarly, either $y + 1 \notin \nu$ or $n \in \nu$. Therefore we conclude that there are four possibilities for $\nu \setminus \mu$, namely $[x, y]_\eta, [x, n]_\eta, [1, y]_\eta, [1, n]_\eta$. Now let Λ_l (respectively, Λ_r) be the leftmost (respectively, rightmost) connected component of $(\lambda \setminus \mu) \setminus [x, y]_\eta$, and θ_l (respectively, θ_r) be the outer border strip of Λ_l (respectively, Λ_r). Then θ_l and θ_r are Dyck (by the definitions of x and y, respectively). Hence Λ_l (respectively, Λ_r) is Dyck if and only if $(\Lambda_l)^{(1)}$ (respectively, $(\Lambda_r)^{(1)}$) is Dyck. But $(\Lambda_l)^{(1)} \approx \Lambda_l \setminus [1, x-1]_\eta$ and $(\Lambda_r)^{(1)} \approx \Lambda_r \setminus [y+1, n]_\eta$. Therefore $(\lambda \setminus \mu) \setminus [x, y]_\eta$

is Dyck if and only if $(\lambda \setminus \mu) \setminus [x,n]_\eta$ is Dyck. Similarly, $(\lambda \setminus \mu) \setminus [x,y]_\eta$ is Dyck if and only if $(\lambda \setminus \mu) \setminus [1,y]_\eta$ is Dyck if and only if $(\lambda \setminus \mu) \setminus [1,n]_\eta$ is Dyck. But $|[1,x-1]_\eta| = |\theta_l|$ is odd since θ_l is Dyck, and similarly $|[y+1,n]_\eta| = |\theta_r|$ is also odd. Therefore $|[1,n]_\eta| \not\equiv |[1,y]_\eta| \pmod 2$ and $|[x,y]_\eta| \not\equiv |[x,n]_\eta| \pmod 2$ and (4) holds.

c) Either x or y exists, but not both.

We may clearly assume that y exists, but x doesn't. Let $\mu \subset \nu \subseteq \lambda$ be such that $\nu \setminus \mu$ is a cbs and $\lambda \setminus \nu$ is Dyck. Then $y \in \nu$ and $1 \in \nu$ (since x does not exist) and hence $[1,y]_\eta \subseteq \nu$.

Suppose that $y+1 \notin \eta$. Then $y+1 \notin \nu$ and hence $n \notin \nu$ (else the rightmost connected component of $\lambda \setminus \nu$ could not be Dyck). Hence ν does not intersect θ to the right of cell y. Let $\theta_r \stackrel{\mathrm{def}}{=} [y+1,n]_\theta$. Then θ_r is Dyck, by the definition of y. Hence $\lambda \setminus \nu$ is Dyck if and only if $(\lambda \setminus \theta_r) \setminus \nu$ is Dyck (since $\nu \cap \theta_r = \emptyset$). But the outer border strip of $(\lambda \setminus \mu) \setminus \theta_r$ is not Dyck (since $y+1 \notin \eta$) so the result follows by induction.

Suppose now that $y+1 \in \eta$. Then if $y+1 \in \nu$ also $n \in \nu$ (else the rightmost connected component of $\lambda \setminus \nu$ could not be Dyck). So either $y+1 \notin \nu$ or $n \in \nu$. Therefore either $\nu \setminus \mu = [1,y]_\eta$ or $\nu \setminus \mu = [1,n]_\eta$. Let Λ_r be the rightmost connected component of $(\lambda \setminus \mu) \setminus [1,y]_\eta$, and θ_r be its outer border strip. Then θ_r is Dyck (by the definition of y). Hence $(\lambda \setminus \mu) \setminus [1,y]_\eta$ is Dyck if and only if $(\lambda \setminus \mu) \setminus [1,n]_\eta$ is Dyck. But $|[y+1,n]_\eta| = |\theta_r|$ is odd since θ_r is a Dyck cbs. Hence $|[1,y]_\eta| \not\equiv |[1,n]_\eta| \pmod 2$ and (4) holds. $\diamond$

Putting together the last two results, we obtain the following interesting recursive characterization of Dyck partitions. This characterization has also been proved in [5] (see Proposition 6.7) using the theory of Kazhdan-Lusztig polynomials. While that proof is hardly combinatorial, and does not really explain the result, our proof here is entirely self-contained and combinatorial.

Corollary 3.7 *Let $\lambda \setminus \mu$ be a skew partition. Then $\lambda \setminus \mu$ is Dyck if and only if*

$$|\{\mu \subset \nu \subseteq \lambda : \ \nu \setminus \mu \textit{ is a cbs of odd size}, \ \lambda \setminus \nu \textit{ is Dyck }\}| >$$

$$|\{\mu \subset \nu \subseteq \lambda : \ \nu \setminus \mu \textit{ is a cbs of even size}, \ \lambda \setminus \nu \textit{ is Dyck }\}| = 0.$$

4 A bijection

In this section we construct a bijection that yields a simple way to produce, given a partition λ, all partitions $\mu \subseteq \lambda$ such that $\lambda \setminus \mu$ is Dyck. This bijection is used in §5 to enumerate Dyck partitions, and in §6 to prove some new identities for Kazhdan-Lusztig polynomials.

Let ν be a connected skew partition and θ be its outer border strip. We call a cell $x \in \theta$ a *left-to-right minimum* (or *lrm*, for short) of ν if:

i) $lv(y) \geq lv(x)$ for all $y \in \theta$, $y \leq x$;

ii) $lv(x) < lv(1)$.

So 1, in particular, is never a lrm. We let

$$\mathcal{H}^*(\nu) = \{x \in \mathcal{H}(\nu) : \ x \text{ is not an lrm of } \ \nu\},$$

(where $\mathcal{H}(\nu)$ denotes the descending border strip of ν, defined in §2). For $x \in \mathcal{H}^*(\nu)$ we let $\theta(x)$ be the unique Dyck final cbs of ν that has x as its rightmost cell ($\theta(x)$ exists because x is not an lrm). So, for example, if ν is the partition depicted in Figure 1 then its lrm's are $\{2, 3, 4, 6, 7\}$, $\mathcal{H}^*(\nu) = \{1, 5, 9, 10\}$, and $|\theta(1)| = |\theta(5)| = |\theta(9)| = 1$ and $|\theta(10)| = 3$.

We can now state and prove the main result of this section.

Theorem 4.1 *Let $\lambda \in \mathcal{P}$, and θ be its outer border strip. There is an explicit bijection between $\{S : \ S \subseteq \mathcal{H}^*(\lambda)\}$ and $\{\mu \subseteq \lambda : \ \ \lambda \setminus \mu$ is Dyck $\}$. Furthermore, if S and μ correspond under this bijection then:*

i) $|S| = dp(\lambda \setminus \mu)$ *;*

ii) $\sum_{x \in S} |\theta(x)| = |\lambda \setminus \mu|$ *;*

iii) $d_{\lambda \setminus \mu}(y) = |\{x \in S : \ y \in \theta(x)\}|$*, for all $y \in \theta$.*

Proof. Let S be a subset of $\mathcal{H}^*(\lambda)$. We associate to S a Dyck final segment of λ as follows. Let x_1 be the rightmost cell of S, and $\theta_{1,1} \stackrel{\text{def}}{=} \theta(x_1)$. Note that $lv(x) > lv(x_1)$ for all $x \in (S \cap \theta_{1,1}) \setminus \{x_1\}$. Now let x_2 be the rightmost cell of $S \setminus \theta_{1,1}$, and $\theta_{1,2} \stackrel{\text{def}}{=} \theta(x_2)$. Note that $lv(x) > lv(x_2)$ for all $x \in (S \cap \theta_{1,2}) \setminus \{x_2\}$. Then let x_3 be the rightmost cell of $S \setminus (\theta_{1,1} \cup \theta_{1,2})$, and $\theta_{1,3} \stackrel{\text{def}}{=} \theta(x_3)$, etc .... We continue in this way until we reach an r_1 such that $S \setminus (\theta_{1,1} \cup \ldots \cup \theta_{1,r_1}) = \emptyset$. Note that $\theta_{1,1}, \ldots, \theta_{1,r_1}$ are the connected components of $\theta_{1,1} \cup \ldots \cup \theta_{1,r_1}$, and that $y \in \theta_{1,1} \cup \ldots \cup \theta_{1,r_1}$ if and only if $|\{i \in [r_1] : \ y \in \theta(x_i)\}| = 1$, for all $y \in \theta$.

Let S_1 be the set of cells that are directly below the elements of $S \setminus \{x_1, \ldots, x_{r_1}\}$ by one cell, $x^{(1)} \in S_1$, and $x \in S \setminus \{x_1, \ldots, x_{r_1}\}$ be the cell directly above $x^{(1)}$ by one cell. Then there exists $i \in [r_1]$ such that $lv(x) > lv(x_i)$. Since x_i is not an lrm of λ (because $x_i \in S$) we conclude that either $lv(x_i) \geq lv(1)$ or there exists $y \in \theta$, $y < x_i$, such that $lv(y) < lv(x_i)$. In the first case the leftmost cell of θ' (where θ' is the outer border strip of $\lambda^{(1)}$) is at level $\leq lv(1) - 1$, and $lv(x^{(1)}) \geq lv(x_i) - 1 \geq lv(1) - 1$. In the second case, if y is in the first row of λ then the leftmost cell of θ' is at level $lv(y) - 1$ and $lv(x^{(1)}) \geq lv(x_i) - 1 > lv(y) - 1$. If y is not in the first row of λ then the cell directly below y by one cell, call it $y^{(1)}$, is in λ (and hence in θ') and $lv(y^{(1)}) = lv(y) - 2 < lv(x_i) - 2 < lv(x^{(1)})$. So in all cases we conclude that $x^{(1)}$ is not an *lrm* of $\lambda^{(1)}$. Furthermore, the leftmost cell of $\theta(x)$ is not the leftmost cell of θ (since $x \in \theta(x_i)$). Hence $x^{(1)} \in \mathcal{H}^*(\lambda^{(1)})$, and $|\theta(x)| = |\theta'(x^{(1)})|$, for all $x^{(1)} \in S_1$.

Now proceed as above with S_1 in place of S, and $\lambda^{(1)}$ in place of λ. We obtain a sequence $\theta_{2,1}, \ldots, \theta_{2,r_2}$ of Dyck final cbs's of $\lambda^{(1)}$ such that $S_1 \setminus (\theta_{2,1} \cup \ldots \cup \theta_{2,r_2}) = \emptyset$. Now let S_2 be the set of cells that are directly below the cells of $S_1 \setminus \{y_1, \ldots, y_{r_2}\}$ by one cell (where $y_1, \ldots, y_{r_2}$ are the rightmost cells of $\theta_{2,1}, \ldots, \theta_{2,r_2}$, respectively). We continue in this way until we reach an h such that $S_h = \emptyset$. Then $\nu \stackrel{\text{def}}{=} \bigcup_{1 \leq i \leq h;\, 1 \leq j \leq r_i} \theta_{i,j}$ is a Dyck final segment of λ, and it is clear from our construction that $dp(\nu) = |S|$, $|\nu| = \sum_{x \in S} |\theta(x)|$, and

$$d_\nu(y) = |\{x \in S : \quad y \in \theta(x)\}| \tag{5}$$

for all $y \in \theta$.

We show that the map $S \mapsto \nu$ just constructed is a bijection by explicitly constructing its inverse. Namely, let ν be a Dyck final segment of λ. We define $S \stackrel{\text{def}}{=} \{x \in \theta : \quad d_\nu(x+1) < d_\nu(x)\}$ (where $d_\nu(|\theta| + 1) \stackrel{\text{def}}{=} 0$). It is clear that $S \subseteq \mathcal{H}^*(\lambda)$ (since $S \subseteq \nu$ and ν is a Dyck final segment), and it is not hard to check, using (5), that this map is the inverse of the one constructed above. $\diamond$

We illustrate the bijection constructed in Theorem 4.1 with an example. Let λ be the partition depicted in Figure 7, and $S = \{4, 6, 7, 18, 19, 24, 25, 31\}$. Then the corresponding Dyck final segment ν of λ is given by the shaded area.

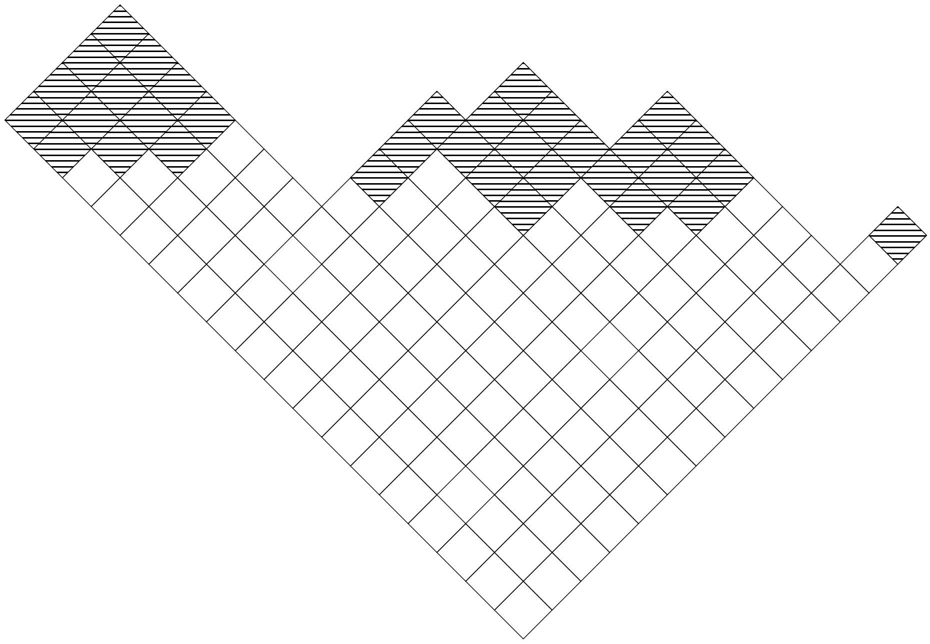

Figure 7: Illustration of the bijection of Theorem 4.1.

The preceding theorem implies the following result, which in turn generalizes Theorem 6.4 of [5] (the case $q = t = 1$).

Corollary 4.2 *Let $\lambda \in \mathcal{P}$, and θ be its outer border strip. Then*

$$\sum_{\{\mu \subseteq \lambda \colon \lambda \backslash \mu \ \ is \ \ Dyck\}} q^{|\lambda \backslash \mu|} \ t^{dp(\lambda \backslash \mu)} = \prod_{x \in \mathcal{H}^*(\lambda)} (1 + tq^{|\theta(x)|}).$$

Proof. By Theorem 4.1 we have that

$$\begin{aligned} \sum_{\{\mu \subseteq \lambda \colon \lambda \backslash \mu \ \ is \ \ Dyck\}} q^{|\lambda \backslash \mu|} \ t^{dp(\lambda \backslash \mu)} &= \sum_{S \subseteq \mathcal{H}^*(\lambda)} t^{|S|} q^{\sum_{x \in S} |\theta(x)|} \\ &= \sum_{S \subseteq \mathcal{H}^*(\lambda)} \prod_{x \in S} tq^{|\theta(x)|} \\ &= \prod_{x \in \mathcal{H}^*(\lambda)} (1 + tq^{|\theta(x)|}). \diamond \end{aligned}$$

5 Enumeration

In this section we enumerate Dyck partitions according to three natural statistics. Our main results are a recurrence relation, and a functional equation, for these generating functions.

Given a connected Dyck skew shape ν define the *width* of ν , denoted $w(\nu)$, to be the least positive integer n such that ν is contained in an $n \times n$ square. So, for example, the width of the Dyck skew shape depicted in Figure 2 is 8. Let

$$\mathcal{F}(t,q,x) \stackrel{\text{def}}{=} \sum_{\nu} t^{dp(\nu)} q^{|\nu|}\, x^{w(\nu)}$$

where the sum is over all (up to translation) the connected Dyck skew shapes $\nu \neq \emptyset$, and write

$$\mathcal{F}(t,q,x) = \sum_{n\geq 1} D_n(t,q) x^n, \tag{6}$$

so that

$$D_n(t,q) \stackrel{\text{def}}{=} \sum_{\nu} t^{dp(\nu)} q^{|\nu|} \tag{7}$$

where ν runs over all (up to translation) the connected Dyck skew shapes of width n. So, for example, $D_1(t,q) = tq$, $D_2(t,q) = tq^3 + t^2q^4$, and $D_3(t,q) = tq^5(2 + 3tq + tq^3 + t^2q^2 + t^2q^4)$.

Lemma 5.1 *Let* $n \in \mathbf{P}$. *Then*

$$D_n(t,q) = \frac{tq^{2n-1}}{1 + tq^{2n-1}} \sum_{\theta} \prod_{x\in\mathcal{H}(\theta)} (1 + tq^{|\theta(x)|}),$$

where the sum is over all (up to translation) the Dyck cbs's θ *of width* n.

Proof. We may clearly assume that all the skew shapes appearing in the sum on the RHS of (7) have $(1,n)$ as their leftmost cell, and $(n,1)$ as their rightmost one. Let $\lambda \setminus \mu$ be such a shape, and θ be its outer border strip. Note that θ is also the outer border strip of λ, and that λ is uniquely determined by θ. Since $\lambda \setminus \mu$ is Dyck, θ is a Dyck cbs and $(\lambda \setminus \mu)^{(1)}$ is a Dyck final segment of $\lambda \setminus \theta$. Conversely, given a Dyck cbs θ whose leftmost cell is $(1,n)$ and whose rightmost cell is $(n,1)$, and a Dyck final segment ν of $\lambda \setminus \theta$ (where λ is the unique partition whose outer border strip is θ), then $\nu \cup \theta$ is a connected Dyck skew shape whose leftmost cell is $(1,n)$ and whose rightmost cell is $(n,1)$. Therefore

$$D_n(t,q) = \sum_{\theta} \sum_{\{\rho\subseteq\lambda^{(1)}:\ \lambda^{(1)}\setminus\rho\ \ is\ \ Dyck\}} t^{dp(\lambda^{(1)}\setminus\rho)+1} q^{|\lambda^{(1)}\setminus\rho|+2n-1} \tag{8}$$

where θ runs over all the Dyck cbs's whose leftmost cell is $(1,n)$ and whose rightmost cell is $(n,1)$. Now, by Corollary 4.2, we have that

$$\sum_{\{\rho\subseteq\lambda^{(1)}:\ \lambda^{(1)}\setminus\rho\ \ is\ \ Dyck\}} t^{dp(\lambda^{(1)}\setminus\rho)} q^{|\lambda^{(1)}\setminus\rho|} = \prod_{x\in\mathcal{H}^*(\lambda^{(1)})} (1 + tq^{|\theta'(x)|}), \tag{9}$$

where θ' is the outer border strip of $\lambda^{(1)}$. We claim that $\mathcal{H}^*(\lambda^{(1)}) = \mathcal{H}(\lambda^{(1)})$. In fact, since θ is Dyck, $lv(x) \geq lv(1,n) = n+1$ for all $x \in \theta$. Therefore, each cell of θ' is at level $\geq n$ except possibly for some upper valley of $\lambda^{(1)}$ which could be at level $n-1$. Since the leftmost cell of θ' is at level n this implies that the only lrm's of $\lambda^{(1)}$ are upper valleys of $\lambda^{(1)}$. Hence $\mathcal{H}^*(\lambda^{(1)}) = \mathcal{H}(\lambda^{(1)})$, as claimed. Therefore from (8) and (9) we conclude that

$$D_n(t,q) = tq^{2n-1} \sum_{\theta} \prod_{x \in \mathcal{H}(\lambda^{(1)})} (1 + tq^{|\theta'(x)|}). \tag{10}$$

It follows that to complete the proof it is enough to show that

$$(1 + tq^{2n-1}) \prod_{x \in \mathcal{H}(\theta')} (1 + tq^{|\theta'(x)|}) = \prod_{x \in \mathcal{H}(\theta)} (1 + tq^{|\theta(x)|}) \tag{11}$$

for any Dyck cbs θ of width n.

For $x \in \mathcal{H}(\theta) \setminus \{2n-1\}$ let $x^{(1)}$ be the cell of θ' directly below x by one cell. Note that the map $x \mapsto x^{(1)}$ is a bijection between $\mathcal{H}(\theta) \setminus \{2n-1\}$ and $\mathcal{H}(\theta')$. But if $y \in \mathcal{H}(\theta) \setminus \{2n-1\}$ then, since θ is Dyck, $lv(y) > lv(1)$ and therefore the map $x \mapsto x^{(1)}$ is a bijection between $\theta(y)$ and $\theta'(y^{(1)})$ (note that $\mathcal{H}(\theta) = \mathcal{H}^*(\theta)$ since θ is Dyck, so $\theta(y)$ is well defined). Hence $|\theta(y)| = |\theta'(y^{(1)})|$, and (11) follows. $\diamond$

We can now prove the first main result of this section.

Theorem 5.2 *Let $n \in \mathbf{P}$. Then we have that*

$$D_n(t,q) = \frac{q}{t} \sum_{k=1}^{n-1} (1 + tq^{2k-1}) D_k(t,q) D_{n-k}(t,q)$$

for $n \geq 2$, and $D_1(t,q) = tq$.

Proof. Let, for convenience,

$$C_n(t,q) \stackrel{\text{def}}{=} \sum_{\theta} E(\theta) \tag{12}$$

where θ runs over all, up to translation, the Dyck cbs's of width n (equivalently, over all the Dyck cbs's whose leftmost cell is $(1,n)$ and whose rightmost cell is $(n,1)$) and where

$$E(\theta) \stackrel{\text{def}}{=} \prod_{x \in \mathcal{H}(\theta)} (1 + tq^{|\theta(x)|}).$$

So, by Lemma 5.1,

$$D_n(t,q) = \frac{tq^{2n-1}}{1+tq^{2n-1}}\, C_n(t,q). \tag{13}$$

Fix $n \geq 2$. Consider a Dyck cbs θ appearing in the sum on the RHS of (12). If $lv(x) > n+1$ for all $x \in [2, 2n-2]_\theta$ then $[2, 2n-2]_\theta$ is a Dyck cbs of width $n-1$ and $E(\theta) = (1+tq^{2n-1})\, E([2, 2n-2]_\theta)$. Furthermore, all Dyck cbs's of width $n-1$ arise in this way. If $lv(x) = n+1$ for some $x \in [2, 2n-2]_\theta$, then let $2k-1$ be the leftmost cell of θ which is at level $n+1$, $\theta' \stackrel{\text{def}}{=} [2, 2k-2]_\theta$, and $\theta'' \stackrel{\text{def}}{=} [2k-1, 2n-1]_\theta$. Then θ' is a Dyck cbs of width $k-1$, and θ'' is a Dyck cbs of width $n-k+1$, and this is clearly a bijection. Furthermore, $\mathcal{H}(\theta) = \mathcal{H}(\theta') \cup \mathcal{H}(\theta'')$ and $|\theta(x)| = |\theta'(x)|$ for all $x \in \mathcal{H}(\theta')$, $|\theta(x)| = |\theta''(x)|$ for all $x \in \mathcal{H}(\theta'') \setminus \{2n-1\}$ (since $lv(2k-1) = n+1$). Therefore

$$E(\theta) = \frac{1+tq^{2n-1}}{1+tq^{2(n-k)+1}}\, E(\theta')E(\theta'').$$

Thus we conclude from (12) that

$$\begin{aligned}
C_n(t,q) &= (1+tq^{2n-1})C_{n-1}(t,q) + \sum_{k=2}^{n-1} \frac{1+tq^{2n-1}}{1+tq^{2(n-k)+1}} C_{n-k+1}(t,q)C_{k-1}(t,q) \\
&= \sum_{k=1}^{n-1} \frac{1+tq^{2n-1}}{1+tq^{2(n-k)-1}} C_k(t,q)\, C_{n-k}(t,q),
\end{aligned}$$

and the result follows from (13) after routine algebra. $\diamond$

The preceding result yields the following functional equation for the generating function $\mathcal{F}(t,q,x)$.

Corollary 5.3 *We have that*

$$\mathcal{F}(t,q,x) = xtq + \frac{q}{t}\mathcal{F}^2(t,q,x) + \mathcal{F}(t,q,xq^2)\mathcal{F}(t,q,x), \tag{14}$$

in $\mathbf{Z}(t,q)[[x]]$.

Proof. We have from (6) and Theorem 5.2 that

$$\begin{aligned}
\mathcal{F}(t,q,x) &= xtq + \sum_{n\geq 2} \frac{q}{t} \sum_{k=1}^{n-1} (1+tq^{2k-1}) D_k(t,q) D_{n-k}(t,q) x^n \\
&= xtq + \frac{q}{t}\mathcal{F}^2(t,q,x) + \sum_{n\geq 2}\sum_{k=1}^{n-1} q^{2k} D_k(t,q)D_{n-k}(t,q)x^n \\
&= xtq + \frac{q}{t}\mathcal{F}^2(t,q,x) + \mathcal{F}(t,q,xq^2)\mathcal{F}(t,q,x),
\end{aligned}$$

as desired. $\diamond$

In general, it seems to be difficult to "solve" the functional equation (14), or even to get a continued fraction expansion from it. However, this is easy to do if $q = 1$. For $n \in \mathbf{P}$ let C_n be the n-th Catalan number, so $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proposition 5.4 *We have that* $D_n(t, 1) = t(1+t)^{n-1}C_{n-1}$ *for all* $n \in \mathbf{P}$. *In particular,*

$$\mathcal{F}(t, 1, x) = \frac{t}{2(1+t)}(1 - \sqrt{1 - 4(1+t)x}).$$

Proof. The first statement follows immediately from Theorem 5.2. Using this, we have from (6) that

$$\mathcal{F}(t, 1, x) = t \sum_{n \geq 1} \frac{1}{(1+t)} C_{n-1}((1+t)x)^n = \frac{t}{2(1+t)}(1 - \sqrt{1 - 4(1+t)x})$$

using a well known result on Catalan numbers (see, e.g., [6]). $\diamond$

It is interesting to compare the functional equation (14), for $t = 1$, with the corresponding functional equation for all the connected skew shapes. In fact, let

$$\mathcal{L}(q, x) \stackrel{\text{def}}{=} \sum_{\nu} q^{|\nu|} x^{\frac{hp(\nu)}{2}}$$

where ν runs over all the connected skew shapes, and $hp(\nu)$ denotes the half-perimeter of ν (i.e., the smallest $n \in \mathbf{P}$ such that there exists an $a \times b$ rectangle, with $a + b = n$, that contains ν). Note that $\mathcal{L}(q, x) \in \mathbf{Z}[[q, \sqrt{x}]]$ and $hp(\nu) = 2w(\nu)$ if ν is Dyck. Then it is known (see, e.g., [3, p. 214]) that

$$\mathcal{L}(q, x) = xq + 2q\sqrt{x}\,\mathcal{L}(q, x) + \mathcal{L}(q, x)\,\mathcal{L}(q, xq^2).$$

This functional equation is remarkably similar to (14), and can be solved explicitly (see, e.g., [12, §4]).

Probably the biggest obstacle in trying to obtain a formula for $D_n(t, q)$ or $\mathcal{F}(t, q, x)$ is the fact that the statistic associated to q, namely size, corresponds, under the bijection of Theorem 4.1, to a fairly mysterious statistic on the elements of $\mathcal{H}^*(\theta)$, namely $|\theta(x)|$. For this reason, it is of interest to try to understand these numbers better, and possibly relate them to some more familiar statistics on lattice paths. We have been able to do this only in the case that the outer border strip θ of λ is Dyck.

Let θ be a Dyck cbs, $n \stackrel{\text{def}}{=} w(\theta)$, and ρ be a square of size n^2 that contains θ. Let $A(\theta)$ be the number of cells of ρ that have no cells of θ weakly above them. This statistic is well known in the theory of Dyck paths, and is usually called the *area* of θ (see, e.g., [15, Ex. 6.34, p.235]). For example, if θ is the outer border strip of the skew partition illustrated in Figure 2 then $A(\theta) = 15$.

Proposition 5.5 *Let θ be a Dyck cbs of size $2n-1$. Then*

$$\sum_{x \in \mathcal{H}(\theta)} |\theta(x)| = n^2 - 2A(\theta).$$

Proof. Let ν be the unique (up to translation) self-dual skew shape whose outer border strip is θ. Then, by Proposition 3.3 and our hypothesis, ν is Dyck. Therefore ν is a Dyck final segment of λ (where λ is the unique partition whose outer border strip is θ). By the proof of Theorem 4.1 we know that ν corresponds, under the bijection of §4, to the subset

$$\{x \in \theta : \; d_\nu(x+1) < d_\nu(x)\}$$

of $\mathcal{H}(\theta)$. Since ν is self-dual, $d_\nu(x+1) < d_\nu(x)$ if and only if $x \in \mathcal{H}(\theta)$. Therefore ν corresponds to $\mathcal{H}(\theta)$ in the bijection of Theorem 4.1 (note that $\mathcal{H}(\theta) = \mathcal{H}^*(\theta)$ since θ is Dyck). This, by Theorem 4.1, implies that

$$|\nu| = \sum_{x \in \mathcal{H}(\theta)} |\theta(x)|,$$

and the result follows. ⋄

It would be interesting to find a generalization of Proposition 5.5 that would include all cbs's of a given size.

Proposition 5.5 makes it easy to compute the analogue of the generating function $\mathcal{F}(t,q,x)$ for self-dual connected Dyck skew shapes. Namely, let

$$\mathcal{G}(t,q,x) \stackrel{\text{def}}{=} \sum_{\nu} t^{dp(\nu)} q^{|\nu|} x^{w(\nu)}$$

where ν runs over all the self-dual connected Dyck skew shapes. Then it follows from the proof of Proposition 5.5 that $dp(\nu) = |\mathcal{H}(\nu)| = w(\nu)$ and $|\nu| = w(\nu)^2 - 2A(\theta)$ for all such ν, where θ is the outer border strip of ν. By Proposition 3.3 we therefore conclude that

$$\mathcal{G}(t,q,x) = \sum_{n \geq 1} q^{n^2} (tx)^n \, C_{n-1}\left(\frac{1}{q^2}\right)$$

where $C_n(q) \stackrel{\rm def}{=} \sum_\theta q^{A(\theta)}$ and θ runs over all the Dyck cbs's such that $w(\theta) = n+1$ (see, e.g., [15, Ex. 6.34, p. 235]). It then follows from a well known result (see [15, Ex. 6.34]) that

$$\mathcal{G}(t,q,x) = txq + \mathcal{G}(t,q,x)\,\mathcal{G}(t,q,xq^2).$$

This functional equation gives an interpretation of two terms of the functional equation (14) and can also be solved explicitly using the methods in [12, §4].

6 Applications

In this section we explain the connection between Dyck partitions and the Kazhdan-Lusztig polynomials of S_n, and we use the results of §4 to obtain some new identities for these polynomials. We refer the reader to [8] or [2] for any undefined notation and terminology concerning Coxeter groups.

For $u, v \in S_n$ we denote by $P_{u,v}(q) \in \mathbf{Z}[q]$ the corresponding Kazhdan-Lusztig polynomial. These polynomials play a fundamental role in representation theory, and in the algebraic geometry and topology of Schubert varieties (see, e.g., [9], [10]). We refer the reader to, e.g., [8] or [2] for the definition of these polynomials and further references about their applications.

For $i \in [n-1]$ let

$$W^{(i)} \stackrel{\rm def}{=} \{\sigma \in S_n : \ \ \sigma^{-1}(1) < \ldots < \sigma^{-1}(i),\ \sigma^{-1}(i+1) < \ldots < \sigma^{-1}(n)\}. \tag{15}$$

It is well known (see, e.g., [2]) and not hard to see, that the map

$$\sigma \mapsto (\sigma^{-1}(i) - i, \ldots, \sigma^{-1}(1) - 1) \tag{16}$$

is a poset isomorphism between $W^{(i)}$ (partially ordered by Bruhat order) and $\{\mu \in \mathcal{P} : \ \mu \subseteq (n-i)^i\}$ (partially ordered by containment of diagrams).

For $\sigma, \tau \in W^{(i)}$ define

$$P^{(i)}_{\sigma,\tau}(q) = \sum_{u \in W_{(i)}} (-1)^{inv(u)}\ P_{u\sigma,\tau}(q) \tag{17}$$

where $W_{(i)}$ is the Young subgroup of S_n generated by $\{(1,2), \ldots, (i-1,i), (i+1, i+2), \ldots, (n-1,n)\}$. These polynomials were first introduced and studied by Deodhar in [7].

The main connection between Dyck partitions and the Kazhdan-Lusztig polynomials of S_n is the following result, which is proved in [5, Theorem 5.1].

Theorem 6.1 *Let $i \in [n-1]$, and $\sigma, \tau \in W^{(i)}$. Then*

$$P^{(i)}_{\sigma,\tau}(q) = \begin{cases} q^{\frac{1}{2}(|\lambda\setminus\mu| - dp(\lambda\setminus\mu))}, & \text{if } \lambda \setminus \mu \text{ is Dyck,} \\ 0, & \text{otherwise,} \end{cases}$$

where μ and λ are the partitions corresponding to σ and τ, respectively, under the bijection (16).

Given this result, it is not surprising that combinatorial properties of Dyck partitions translate into properties of the Kazhdan-Lusztig polynomials.

Corollary 6.2 *Let $\tau \in W^{(i)}$. Then*

$$\sum_{\sigma \in W^{(i)}} P^{(i)}_{\sigma,\tau}(q) = \prod_{x \in \mathcal{H}^*(\lambda)} (1 + q^{\frac{|\theta(x)|-1}{2}})$$

where λ is the partition corresponding to τ under the bijection (16), and θ is the outer border strip of λ.

Proof. From Theorem 6.1 we have that

$$\sum_{\sigma \in W^{(i)}} P^{(i)}_{\sigma,\tau}(q) = \sum_{\{\mu \subseteq \lambda:\ \lambda\setminus\mu \text{ is Dyck}\}} q^{\frac{1}{2}(|\lambda\setminus\mu| - dp(\lambda\setminus\mu))},$$

and the result follows from Corollary 4.2. $\diamond$

Note that Corollary 6.2 implies, for $q = 1$, Corollary 6.6 of [5].

Given $v \in S_n$ and $i \in [n-1]$ it is well known (see, e.g., [8, §5.12] or [2, §2.4]) and easy to see, that there exist unique permutations $u \in W_{(i)}$ and $\sigma \in W^{(i)}$ such that $v = u\sigma$. We then define

$$\mathrm{inv}_i(v) \stackrel{\text{def}}{=} \mathrm{inv}(u).$$

Equivalently,

$$\begin{aligned} \mathrm{inv}_i(v) &= |\{1 \le a < b \le n:\ i \ge v(a) > v(b)\}| \\ &+ |\{1 \le a < b \le n:\ v(a) > v(b) > i\}| \\ &= \mathrm{inv}(v) - |\{1 \le a < b \le n:\ \ v(a) > i \ge v(b)\}|. \end{aligned}$$

For example, if $v = 81356742 \in S_8$ and $i = 4$ then $u = 13428567$, $\sigma = 51267834$, and $\mathrm{inv}_4(v) = 5$. We can then deduce from Corollary 6.2 the following remarkable identity for the Kazhdan-Lusztig polynomials of S_n, which is also the main result of this section.

Corollary 6.3 *Let $\tau \in S_n$, and $i \in [n-1]$. Then*

$$\sum_{\sigma \in S_n} (-1)^{inv_i(\sigma)} P_{\sigma,\tau}(q) = \begin{cases} \prod_{x \in \mathcal{H}^*(\lambda)} (1 + q^{\frac{|\theta(x)|-1}{2}}), & \text{if } \tau \in W^{(i)}, \\ 0, & \text{otherwise,} \end{cases}$$

where λ and θ have the same meaning as in Corollary 6.2.

Proof. We have that

$$\begin{aligned} \sum_{v \in S_n} (-1)^{inv_i(v)} \, P_{v,\tau}(q) &= \sum_{u \in W_{(i)}} \sum_{\sigma \in W^{(i)}} (-1)^{inv_i(u\sigma)} \, P_{u\sigma,\tau}(q) \\ &= \sum_{\sigma \in W^{(i)}} \sum_{u \in W_{(i)}} (-1)^{inv(u)} \, P_{u\sigma,\tau}(q) \\ &= \sum_{\sigma \in W^{(i)}} P^{(i)}_{\sigma,\tau}(q). \end{aligned}$$

If $\tau \in W^{(i)}$, then the result follows immediately from Corollary 6.2. If $\tau \notin W^{(i)}$ then there exists $j \in [n-1] \setminus \{i\}$ such that $\tau^{-1}(j) > \tau^{-1}(j+1)$. Hence

$$\begin{aligned} P^{(i)}_{\sigma,\tau}(q) &= \sum_{u \in W_{(i)}} (-1)^{inv(u)} P_{u\sigma,\tau}(q) \\ &= \sum_{\{u \in W_{(i)}:\, u^{-1}(j) > u^{-1}(j+1)\}} (-1)^{inv(u)} (P_{u\sigma,\tau}(q) - P_{(j,j+1)u\sigma,\tau}(q)) \\ &= 0 \end{aligned}$$

by a well known property of Kazhdan-Lusztig polynomials (see, e.g., [8, Corollary 7.14]) and the result again follows. ⋄

7 Open Problems

In this section we discuss some open problems and directions for further research arising from the present work, including possible analogues of Dyck partitions for other Coxeter groups.

From an enumerative point of view the main problem is certainly the following one.

Problem 7.1 *Give a formula for $\mathcal{F}(t, q, x)$, or even just for $\mathcal{F}(1, q, 1)$.*

The first few terms of $\mathcal{F}(1,q,1)$ are

$$\begin{aligned}\mathcal{F}(1,q,1) &= q+q^3+q^4+2q^5+3q^6+6q^7+11q^8+21q^9+39q^{10}+77q^{11}+150q^{12} \\ &+ 298q^{13}+589q^{14}+1187q^{15}+\ldots\end{aligned}$$

Recall that the coefficient of q^n in $\mathcal{F}(1,q,1)$ equals the number of connected Dyck skew shapes of size n. It would be interesting to know even the asymptotic behaviour of this coefficient. This, in turn, might shed light also on the following problems (we refer the reader to §§6.1 and 6.4 of [15] for the definitions, and further properties of, algebraic and D-finite formal power series).

Problem 7.2 *Is $\mathcal{F}(1,q,1)$ algebraic?*

Problem 7.3 *Is $\mathcal{F}(1,q,1)$ D-finite?*

Regarding the polynomials $D_n(t,q)$ (i.e., the coefficient of x^n in $\mathcal{F}(t,q,x)$) it is clear from their definition that there are polynomials $\tilde{D}_{n,k}(t) \in \mathbf{N}[t]$ such that

$$[q^k](D_n(t,q)) = \begin{cases} t\,\tilde{D}_{n,k}(t^2), & \text{if } k \equiv 1 \pmod 2, \\ \tilde{D}_{n,k}(t^2), & \text{if } k \equiv 0 \pmod 2, \end{cases}$$

and computer evidence strongly suggests that the following holds (we refer the reader to [14] or [4], for further information about unimodality and log-concavity).

Conjecture 7.4 *The polynomials $\tilde{D}_{n,k}(t)$ have only real zeros, for all $n,k \in \mathbf{P}$. In particular, they are log-concave and unimodal.*

This conjecture has been verified for $n \leq 12$.

Because of the close connection between Dyck partitions and the Kazhdan-Lusztig polynomials of S_n, and since Kazhdan-Lusztig polynomials are defined for any Coxeter group, it is natural to wonder if it is possible to define "Dyck partitions" for other Coxeter groups, and if these would turn out to be related to the corresponding Kazhdan-Lusztig polynomials. While this problem is probably too general, we feel that there are at least two special cases of it for which the answer should be yes.

Let S_n^B be the group of all signed permutations of size n. So S_n^B is the group of all permutations σ of $[\pm n]$ ($\stackrel{\text{def}}{=} [-n,n] \setminus \{0\}$) such that $\sigma(-a) = -\sigma(a)$ for all $a \in [n]$. It is well known (see, e.g., [2, §8.1]) that S_n^B is a Coxeter group. Now let

$$(S_n^B)^{(0)} \stackrel{\text{def}}{=} \{\sigma \in S_n^B : \ \sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(n)\}. \tag{18}$$

Given $\sigma \in (S_n^B)^{(0)}$ there is a unique k such that $\sigma^{-1}(k) < 0 < \sigma^{-1}(k+1)$. It is then known (see, e.g., [2]) that the map

$$\sigma \mapsto (-\sigma^{-1}(1), -\sigma^{-1}(2), \ldots, -\sigma^{-1}(k))$$

is a poset isomorphism between $(S_n^B)^{(0)}$ (partially ordered by Bruhat order) and the set of all *shifted* partitions (partially ordered by containment of their shifted diagrams) contained in the shifted partition $(n, n-1, \ldots, 2, 1)$. One can define analogues of the polynomials in (17) for every pair of elements of $(S_n^B)^{(0)}$ (see [7]). The general problem that we formulated above, therefore, becomes in this case the following.

Problem 7.5 *Is there an analogue of Dyck partitions for shifted partitions?*

Furthermore, one may ask if such "Dyck shifted partitions" are related to the corresponding polynomials.

Another special case of the general problem which we feel is particularly promising is that of affine Weyl groups of type A. Let $\tilde{S}_n$ be the group of all affine permutations of period n. So $\tilde{S}_n$ is the group of all bijections $\sigma : \mathbf{Z} \to \mathbf{Z}$ such that $\sigma(a+n) = \sigma(a) + n$ for all $a \in \mathbf{Z}$, and $\sum_{i=1}^{n} \sigma(i) = \binom{n+1}{2}$. It is well known (see, e.g., [2, §8.3]) that $\tilde{S}_n$ is a Coxeter group of type $\tilde{A}_{n-1}$. Let

$$(\tilde{S}_n)^{(0)} \stackrel{\text{def}}{=} \{\sigma \in \tilde{S}_n : \ \sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(n)\}. \tag{19}$$

Again, one can associate a polynomial, as in (17), to any pair of elements of $(\tilde{S}_n)^{(0)}$. Furthermore, it is known (see [1] and [11]) that there is a poset isomorphism between $(\tilde{S}_n)^{(0)}$, partially ordered by Bruhat order, and the set of all partitions which are n-cores, partially ordered by containment of their diagrams (recall that a partition λ is an *n-core* if there is no partition $\mu \subseteq \lambda$ such that $\lambda \setminus \mu$ is a connected border strip of size n). Therefore the general problem becomes in this case the following.

Problem 7.6 *Is there an analogue of Dyck partitions for n-cores?*

Furthermore, one may again ask if such "Dyck n-cores" are related to the corresponding polynomials.

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