Analisi Matematica I
Limiti di successioni (svolgimenti)

Svolgimento esercizio 1

(1) Infatti, per ogni $\varepsilon > 0$ si ha $\left|\frac{2n+3}{n+1} - 2\right| < \varepsilon \iff \frac{1}{n+1} < \varepsilon \iff n > \frac{1}{\varepsilon} - 1$, e quindi basta prendere $n_\varepsilon := \left[\frac{1}{\varepsilon}\right] - 1$.

(2) Infatti, per ogni $\varepsilon > 0$ si ha $\left|\frac{n^2+3n}{n^2+1} - 1\right| < \varepsilon \iff \frac{3n-1}{n^2+1} < \varepsilon \iff \varepsilon n^2 - 3n + \varepsilon - 1 > 0$, ed è sufficiente prendere $n > \frac{3+\sqrt{9+4\varepsilon-4\varepsilon^2}}{2\varepsilon}$, e quindi basta prendere $n_\varepsilon := \left[\frac{3+\sqrt{9+4\varepsilon-4\varepsilon^2}}{2\varepsilon}\right]$. Più semplicemente, poiché $\frac{3n-1}{n^2+1} < \frac{3n}{n^2} = \frac{3}{n}$, è sufficiente prendere $n > \frac{3}{\varepsilon}$, e quindi $n_\varepsilon := \left[\frac{3}{\varepsilon}\right]$.

(3) Infatti, per ogni $\varepsilon > 0$ si ha $\left|\sqrt{\frac{2n+3}{2n+1}} - 1\right| < \varepsilon \iff \frac{\frac{2n+3}{2n+1}-1}{\sqrt{\frac{2n+3}{2n+1}}+1} < \varepsilon \iff \frac{\frac{2}{2n+1}}{\sqrt{\frac{2n+3}{2n+1}}+1} < \varepsilon$. Poiché $\frac{\frac{2}{2n+1}}{\sqrt{\frac{2n+3}{2n+1}}+1} < \frac{2}{2n+1} < \frac{1}{n}$, è sufficiente prendere $n > \frac{1}{\varepsilon}$, e quindi $n_\varepsilon := \left[\frac{1}{\varepsilon}\right]$.

(4) Infatti, per ogni $\varepsilon > 0$ si ha $\left|\frac{4\sqrt{n}-3}{\sqrt{n}+1} - 4\right| < \varepsilon \iff \frac{7}{\sqrt{n}+1} < \varepsilon \iff n > \left(\frac{7}{\varepsilon} - 1\right)^2$, e quindi basta prendere $n_\varepsilon := \left[\left(\frac{7}{\varepsilon} - 1\right)^2\right]$, ma basterebbe anche $n_\varepsilon := \left[\frac{49}{\varepsilon^2}\right]$.

(5) Infatti, per ogni $\varepsilon > 0$ si ha $\left|\sqrt{n+1} - \sqrt{n}\right| < \varepsilon \iff \frac{1}{\sqrt{n+1}+\sqrt{n}} < \varepsilon$. Poiché $\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{2}{\sqrt{n}}$, è sufficiente prendere $n > \frac{4}{\varepsilon^2}$, e quindi $n_\varepsilon := \left[\frac{4}{\varepsilon^2}\right]$.

(6) Infatti, per ogni $M > 0$ si ha $\sqrt{n^2+n+1} > M \iff n^2 + n - (M^2 - 1) > 0$, ed è sufficiente prendere $n > \frac{-1+\sqrt{4M^2-3}}{2}$, e quindi $n_M := \left[\frac{\sqrt{4M^2-3}-1}{2}\right]$. Più semplicemente, poiché $\sqrt{n^2+n+1} > \sqrt{n^2} = n$, è sufficiente prendere $n > M$, e quindi $n_M := \left[M\right]$.

(7) Infatti, per ogni $M > 0$ si ha $\frac{n^2+1}{n+1} > M \iff \frac{n^2-Mn-(M-1)}{n+1} > 0 \iff n^2 - Mn - (M-1) > 0$, ed è sufficiente prendere $n > \frac{M+\sqrt{M^2+4M-4}}{2}$, e quindi $n_M := \left[\frac{M+\sqrt{M^2+4M-4}}{2}\right]$. Più semplicemente, poiché $\frac{n^2+1}{n+1} > \frac{n^2}{2n} = \frac{n}{2}$, è sufficiente prendere $n > 2M$, e quindi $n_M := \left[2M\right]$.

(8) Infatti, per ogni $M > 0$ si ha $\log_2 \frac{1}{n} < -M \iff \log_2 n > M \iff n > 2^M$, e quindi basta prendere $n_M := \left[2^M\right]$.

(9) Infatti, per ogni $M > 0$ si ha $\sqrt{n} - n < -M \iff n - \sqrt{n} - M > 0$, e quindi è sufficiente prendere $n > \left(\frac{1+\sqrt{1+4M}}{2}\right)^2$, e quindi $n_M := \left[\left(\frac{1+\sqrt{1+4M}}{2}\right)^2\right]$.

□

Svolgimento esercizio 2

(1) Si ha $\dfrac{n^3 + 2n^2 - \sqrt{n}}{n^2 + 3n - 1} = \dfrac{n^3(1 + o(1))}{n^2(1 + o(1))} = n(1 + o(1)) \to +\infty$.

(2) Si ha $\dfrac{n^3 + 2n^2 - \sqrt{n}}{n^2 + 3n - 1}\left(\dfrac{1}{n} - \dfrac{2}{n^2}\right) = \dfrac{n^3(1 + o(1))}{n^2(1 + o(1))}\dfrac{1}{n}(1 + o(1)) = 1 + o(1) \to 1$.

(3) Si ha $\dfrac{n^2 + 3n^{3/2} + \sqrt{n+3} + 1}{5n^3 + \sqrt[3]{n+7}} = \dfrac{n^2(1 + o(1))}{5n^3(1 + o(1))} = \dfrac{1}{5n}(1 + o(1)) \to 0$.

(4) Si ha $\dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^2+\sqrt[3]{n+7}} = \dfrac{n^2(1+o(1))}{5n^2(1+o(1))} = \dfrac{1}{5}(1+o(1)) \to \dfrac{1}{5}$.

(5) Si ha $\dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^{1/3}+\sqrt[3]{n+7}} = \dfrac{n^2(1+o(1))}{6n^{1/3}(1+o(1))} = \dfrac{n^{5/6}}{6}(1+o(1)) \to +\infty$.

(6) Si ha $\big(\dfrac{2n^2+3+\sqrt{n}}{n^2+1}-2\big)(n^{4/3}+2n+1) = \dfrac{\sqrt{n}+1}{n^2+1}\,(n^{4/3}+2n+1) = \dfrac{\sqrt{n}(1+o(1))}{n^2(1+o(1))}\,n^{4/3}(1+o(1)) = \dfrac{1}{n^{1/6}}(1+o(1)) \to 0$.

(7) Si ha $\big(\dfrac{2n^2+3+\sqrt{n}}{n^2+1}-2\big)(n^{3/2}+7n^2+\pi) = \dfrac{\sqrt{n}+1}{n^2+1}\,(n^{3/2}+7n^2+\pi) = \dfrac{\sqrt{n}(1+o(1))}{n^2(1+o(1))}\,7n^2(1+o(1)) = 7\sqrt{n}(1+o(1)) \to +\infty$.

(8) Si ha $\sqrt{\dfrac{2n^2+3+\sqrt{n}}{n^2+1}-2}\;(7n+2) = \sqrt{\dfrac{\sqrt{n}+1}{n^2+1}}\;(7n+2) = \sqrt{\dfrac{\sqrt{n}(1+o(1))}{n^2(1+o(1))}}\;7n(1+o(1)) = 7\sqrt[4]{n}(1+o(1)) \to +\infty$.

(9) Si ha $\big(\dfrac{2n^2+3+\sqrt{n}}{n^2+1}-2\big)\sqrt{7n+2} = \dfrac{\sqrt{n}+1}{n^2+1}\sqrt{7n+2} = \dfrac{\sqrt{n}(1+o(1))}{n^2(1+o(1))}\sqrt{7n}(1+o(1)) = \dfrac{\sqrt{7}}{n}(1+o(1)) \to 0$.

(10) Si ha $\sqrt{n+1}-\sqrt{n} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{2\sqrt{n}(1+o(1))} \to 0$.

(11) Si ha $\sqrt{n+1}-n = -n(1+o(1)) \to -\infty$.

(12) Si ha $\sqrt{n^2+1}-n = \dfrac{1}{\sqrt{n^2+1}+n} = \dfrac{1}{2n(1+o(1))} \to 0$.

(13) Si ha $\sqrt{n^2+n+1}-\sqrt{n} = n(1+o(1)) \to +\infty$.

(14) Si ha $\sqrt{n^2+n+1}-n = \dfrac{n+1}{\sqrt{n^2+n+1}+n} = \dfrac{n(1+o(1))}{2n(1+o(1))} = \dfrac{1}{2}(1+o(1)) \to \dfrac{1}{2}$.

(15) Si ha $\sqrt{n^2+\sqrt{n+1}}-n = \dfrac{\sqrt{n+1}}{\sqrt{n^2+n+1}+n} = \dfrac{\sqrt{n}(1+o(1))}{2n(1+o(1))} = \dfrac{1}{2\sqrt{n}(1+o(1))} \to 0$.

(16) Si ha $\big(\sqrt{n^2+\sqrt{n+1}}-n\big)^2 = \Big(\dfrac{\sqrt{n+1}}{\sqrt{n^2+n+1}+n}\Big)^2 = \Big(\dfrac{\sqrt{n}(1+o(1))}{2n(1+o(1))}\Big)^2 = \Big(\dfrac{1}{2\sqrt{n}(1+o(1))}\Big)^2 = \dfrac{1}{4n(1+o(1))} \to 0$.

(17) Si ha $\big(\sqrt{n^2+\sqrt{n+1}}-n\big)^2\sqrt{n+1} = \Big(\dfrac{\sqrt{n+1}}{\sqrt{n^2+n+1}+n}\Big)^2\sqrt{n+1} = \Big(\dfrac{\sqrt{n}(1+o(1))}{2n(1+o(1))}\Big)^2\sqrt{n}(1+o(1)) = \Big(\dfrac{1}{2\sqrt{n}(1+o(1))}\Big)^2\sqrt{n}(1+o(1)) = \dfrac{\sqrt{n}(1+o(1))}{4n(1+o(1))} = \dfrac{1}{4\sqrt{n}(1+o(1))} \to 0$.

(18) Si ha $\sqrt{n^2+n^3+1}-n = \dfrac{n^3+1}{\sqrt{n^2+n^3+1}+n} = \dfrac{n^3(1+o(1))}{n^{3/2}(1+o(1))} = n^{3/2}(1+o(1)) \to +\infty$.

(19) Si ha $\sqrt{n^2+n^{1/3}+1}-n=\dfrac{n^{1/3}+1}{\sqrt{n^2+n^{1/3}+1}+n}=\dfrac{n^{1/3}(1+o(1))}{n^{3/2}(1+o(1))}=\dfrac{1}{n^{7/6}}(1+o(1))\to 0.$

(20) Si ha $\sqrt{n^2+n^3+1}-n^{3/2}=\dfrac{n^2+1}{\sqrt{n^2+n^3+1}+n^{3/2}}=\dfrac{n^2(1+o(1))}{2n^{3/2}(1+o(1))}=\dfrac{n^{1/2}}{2}(1+o(1))\to+\infty.$

(21) Si ha $\sqrt{n^3+n+1}-n^{3/2}=\dfrac{n+1}{\sqrt{n^3+n+1}+n^{3/2}}=\dfrac{n(1+o(1))}{2n^{3/2}(1+o(1))}=\dfrac{1}{2n^{1/2}}(1+o(1))\to 0.$

(22) Usando il risultato dell'esercizio (21), si ha $\dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^{1/3}+\sqrt[3]{n+7}}(\sqrt{n^3+n+1}-n^{3/2})=$ $\dfrac{n^2(1+o(1))}{6n^{1/3}(1+o(1))}\dfrac{1}{2n^{1/2}(1+o(1))}=\dfrac{n^{7/6}}{12}(1+o(1))\to+\infty.$

(23) Usando il risultato dell'esercizio (21), si ha $\dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^{1/3}+\sqrt[3]{n+7}}+\sqrt{n^3+n+1}-n^{3/2}=$ $\dfrac{n^2(1+o(1))}{6n^{1/3}(1+o(1))}+\dfrac{1}{2n^{1/2}(1+o(1))}=\dfrac{n^{5/3}}{6}(1+o(1))\to+\infty.$

(24) Usando il risultato dell'esercizio (21), si ha $\Big(\dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^{1/3}+\sqrt[3]{n+7}}\Big)^2+\sqrt{n^3+n+1}-n^{3/2}=$ $\Big(\dfrac{n^2(1+o(1))}{6n^{1/3}(1+o(1))}\Big)^2+\dfrac{1}{2n^{1/2}(1+o(1))}=\dfrac{n^{10/3}}{36}(1+o(1))\to+\infty.$

(25) Si ha $\sqrt[3]{n^3+2n^2}-n=\dfrac{2n^2}{(n^3+2n^2)^{2/3}+n(n^3+2n^2)^{1/3}+n^2}=\dfrac{2n^2}{3n^2(1+o(1))}\to\dfrac{2}{3}.$

(26) Si ha $\sqrt[3]{n^6-n^4+1}-n^2=\dfrac{-n^4+1}{(n^6-n^4+1)^{2/3}+n^2(n^6-n^4+1)^{1/3}+n^4}=\dfrac{-n^4(1+o(1))}{3n^4(1+o(1))}\to-\dfrac{1}{3}.$ □

Svolgimento esercizio 3

(1) Si ha $\dfrac{\sqrt{n+\sqrt{n+3}}-n5^{-\sqrt{n}}+3}{\big(n^5+3\,\mathrm{arctg}(n!)+7\big)^{2/7}}=\dfrac{\sqrt{n(1+o(1))}+3+o(1)}{\big(n^5(1+o(1))\big)^{2/7}}=\dfrac{\sqrt{n}(1+o(1))}{n^{10/7}(1+o(1))}\to 0.$

(2) Si ha $\dfrac{\sqrt{n+\sqrt{20^n+1}}+5\cdot 2^n\sqrt{n}+2}{3^n+8\cdot 5^{-n^2+n}+1}=\dfrac{(\sqrt[4]{20})^n(1+o(1))+5\sqrt{n}\cdot 2^n(1+o(1))}{3^n(1+o(1))}=\dfrac{(\sqrt[4]{20})^n(1+o(1))}{3^n(1+o(1))}\to$ 0, perché $\dfrac{\sqrt{n}\cdot 2^n}{(\sqrt[4]{20})^n}=\dfrac{\sqrt{n}}{(\sqrt[4]{\frac{5}{4}})^n}\to 0.$

(3) Si ha $\dfrac{n!}{(n+1)!-(n-1)!}=\dfrac{(n-1)!n}{(n-1)!n(n+1)-(n-1)!}=\dfrac{n}{n^2+n-1}=\dfrac{n}{n^2(1+o(1))}=\dfrac{1}{n}(1+$ $o(1))\to 0.$

(4) Si ha $\dfrac{n^{n-3}(n+3)!+n^{n-2}(n+2)!}{n!\cdot n^n}=\dfrac{(n+3)!}{n!n^3}+\dfrac{(n+2)!}{n!n^2}=\dfrac{(n+3)(n+2)(n+1)}{n^3}+\dfrac{(n+2)(n+1)}{n^2}=$ $2(1+o(1))\to 2.$

(5) Si ha $\dfrac{n!(2n+3\cos n)-(n+1)!}{n!(2n-\log_3 n)+2^{\log_3(n!)}} = \dfrac{n!(2n+3\cos n-(n+1))}{n!\cdot 2n(1+o(1))+(n!)^{\log_3 2}} = \dfrac{n!\cdot n(1+o(1))}{n!\cdot 2n(1+o(1))} \to \dfrac{1}{2}$.

(6) Si ha $\dfrac{2n!+(2n)!}{n^n+3n!} = \dfrac{(2n)!(1+o(1))}{n^n(1+o(1))} = \dfrac{2n}{n}\dfrac{2n-1}{n}\cdots\dfrac{n+1}{n}n!(1+o(1)) \geq n! \to +\infty$.

(7) Si ha $\dfrac{(\sqrt{n+1}+\sqrt{n})n!+3n^{51}+5^{n+1}}{(n-1)!\left(4n+n^{1/3}+\sin(n^5+3)\right)^{3/2}} = \dfrac{2\sqrt{n}\cdot n!(1+o(1))}{(n-1)!(4n)^{3/2}(1+o(1))} = \dfrac{2n\sqrt{n}(1+o(1))}{8n^{3/2}(1+o(1))} \to \dfrac{1}{4}$.

(8) Si ha $\dfrac{(\sqrt{n+1}-\sqrt{n})n!+3n^{51}+(n+1)5^{n+1}}{(n-1)!\sqrt{4n+2n^2+n^{3/2}\sin(n^5+3)}} = \dfrac{\frac{1}{2\sqrt{n}}n!(1+o(1))}{(n-1)!\cdot n\sqrt{2}(1+o(1))} = \dfrac{1}{2\sqrt{2n}}(1+o(1)) \to$ 0.

(9) Si ha $\dfrac{n^n+3^n}{2^{n\log_2 n}} = \dfrac{n^n(1+o(1))}{n^n} \to 1$.

(10) Si ha $\dfrac{n^n+n!}{2^{n^2}} = \dfrac{2^{n\log_2 n}(1+o(1))}{2^{n^2}} \to 0$.

(11) Si ha $\dfrac{n!(n^7+5n^2+1)2^n}{6^{n^2}} = \dfrac{n!\cdot n^7(1+o(1))}{6^{n^2-n\log_6 2}} = \dfrac{n!\cdot n^7(1+o(1))}{6^{n^2}(1+o(1))} \to 0$.

(12) Si ha $\dfrac{5^{n\log_5 n}-5^n}{n^{\log_5 n}+n^{n+\log_5 n}} = \dfrac{n^n(1+o(1))}{n^{n+\log_5 n}(1+o(1))} = \dfrac{1}{5^{(\log_5 n)^2}}(1+o(1)) \to 0$.

(13) Si ha $\dfrac{n!\cdot 3^{(n+1)!}+5^{(n+1)!}}{\left((n+1)!\right)^2} = \dfrac{5^{(n+1)!}(1+o(1))}{((n+1)!)^2} \to +\infty$, perché $\dfrac{n!\cdot 3^{(n+1)!}}{5^{(n+1)!}} = \dfrac{1}{n+1}\dfrac{(n+1)!}{(\frac{5}{3})^{(n+1)!}} \to 0$.

(14) Si ha $\dfrac{n!\cdot 7^{n!}-5^{(n+1)!}}{\left((n+1)!\right)^2+32^{n^2}+1} = -\dfrac{5^{(n+1)!}(1+o(1))}{32^{n^2}(1+o(1))} \to -\infty$, perché $\dfrac{5^{(n+1)!}}{n!\cdot 7^{n!}} = 5^{(n+1)!-\log_5(n!)-n!\log_7 5} =$ $5^{(n+1)!(1+o(1))} \to +\infty$, e $0 \leq \dfrac{\left((n+1)!\right)^2}{32^{n^2}} \leq (n+1)^2\dfrac{n^{2n}}{32^{n^2}} = \dfrac{n^2(1+o(1))}{32^{n^2-2n\log_{32} n}} \to 0$.

(15) Si ha $\dfrac{n!\cdot 7^{n(n+1)}+4^{(n+1)!}}{(n+1)^n} = \dfrac{4^{(n+1)!}(1+o(1))}{(n+1)^n} \to +\infty$.

(16) Si ha $\dfrac{(n!)^n\cdot 7^{n(n+1)}+4^{(n+1)!}}{\left((n-1)!\right)^n} = \dfrac{4^{(n+1)!}(1+o(1))}{4^{n\log_4(n-1)!}} \to +\infty$.

(17) Si ha $\dfrac{(n!\cdot 7^n)^{n+1}-4^{(n+1)!}+2n^n}{n^{(n-1)!}} = \dfrac{-4^{(n+1)!}(1+o(1))}{4^{(n-1)!\log_4 n}} \to -\infty$.

(18) Si ha $\dfrac{(n-3)!\,n^n-(n+1)!\,n^{n-4}}{2(n-4)!\,(n^n-n!\log_4 n)} \overset{(a)}{=} \dfrac{(n-3)n^4-(n+1)n(n-1)(n-2)(n-3)}{2n^4(1+o(1))} =$
$= \dfrac{2n^4(1+o(1))}{2n^4(1+o(1))} \to 1$, dove in (a) si è diviso numeratore e denominatore per $(n-4)!\,n^{n-4}$.

□

Svolgimento esercizio 4

(1) Si ha $a_n = \log_{12} n + \sqrt[12]{n} = \sqrt[12]{n}(1+o(1))$, e l'ordine di infinito di $\{a_n\}_{n\in\mathbb{N}}$ è $\frac{1}{12}$.

(2) Si ha $a_n = n^{150} + \left(\frac{3}{2}\right)^n = \left(\frac{3}{2}\right)^n (1 + o(1))$, e l'ordine di infinito di $\{a_n\}_{n\in\mathbb{N}}$ non esiste.

(3) Si ha $a_n = n^{1/2}(1 + n^{1/4}) = n^{3/4}(1 + o(1))$, e l'ordine di infinito di $\{a_n\}_{n\in\mathbb{N}}$ è $\frac{3}{4}$.

(4) Si ha $a_n = \dfrac{1}{\sqrt{n+1} - \sqrt{n}} = \sqrt{n+1} + \sqrt{n} = 2\sqrt{n}(1 + o(1))$, e l'ordine di infinito di $\{a_n\}_{n\in\mathbb{N}}$ è $\frac{1}{2}$.

(5) Si ha $a_n = \dfrac{n!}{(n-1)!} - 3 = n - 3 = n(1 + o(1))$, e l'ordine di infinito di $\{a_n\}_{n\in\mathbb{N}}$ è 1.

(6) Si ha $a_n = \dfrac{(n+1)! - (n-1)!}{n!} = \dfrac{n(n+1) - 1}{n} = n(1 + o(1))$, e l'ordine di infinito di $\{a_n\}_{n\in\mathbb{N}}$ è 1.

(7) Si ha $a_n = \dfrac{5^{2n\log_5 n} - 3^{n\log_3(n^2)} + n^4}{7^{n^2\log_7(n^2)} - 4^{2n^2\log_4 n} + n^2} = \dfrac{n^{2n} - (n^2)^n + n^4}{(n^2)^{n^2} - n^{2n^2} + n^2} = n^2$. L'ordine di infinito di $\{a_n\}_{n\in\mathbb{N}}$ è 2.

(8) Si ha $a_n = \dfrac{\sqrt{n + \sqrt{n+3}} - n^3 4^{-\sqrt{n}} + 5}{\left(n^5 - 4\,\mathrm{arctg}(n^n) + 12\right)^{1/16}} = \dfrac{\sqrt{n(1+o(1))} + 5 + o(1)}{\left(n^5(1+o(1))\right)^{1/16}} = \dfrac{\sqrt{n}(1+o(1))}{n^{5/16}(1+o(1))} = n^{3/16}(1 + o(1))$, e l'ordine di infinito di $\{a_n\}_{n\in\mathbb{N}}$ è $\frac{3}{16}$.

(9) Si ha $a_n = \dfrac{2n!\,(n^n - n!\log_2 n)}{(n-3)!\,n^n - (n+1)!\,n^{n-4}} \overset{(a)}{=} \dfrac{2n^4 n(n-1)(n-2)(1+o(1))}{n^4 - (n+1)n(n-1)(n-2)} = \dfrac{2n^7(1+o(1))}{2n^3(1+o(1))} = n^4(1 + o(1))$, dove in (a) si è diviso numeratore e denominatore per $(n-3)!\,n^{n-4}$. L'ordine di infinito di $\{a_n\}_{n\in\mathbb{N}}$ è 4.

(10) Si ha $a_n = \dfrac{((n+1)!)^2\,5^{n\log_5 n} - (n!)^2\,3^{(n-1)\log_3 n}}{((n-1)!)^2\,4^{(n-2)\log_4 n} - ((n-2)!)^2\,6^{(n+1)\log_6 n}} = \dfrac{((n+1)!)^2 n^n - (n!)^2 n^{n-1}}{((n-1)!)^2 n^{n-2} - ((n-2)!)^2 n^{n+1}} \overset{(a)}{=} \dfrac{(n+1)^2 n^2 (n-1)^2 n^2 - n^2(n-1)^2 n}{(n-1)^2 + n^3} = \dfrac{n^8(1+o(1))}{n^3(1+o(1))} = n^5(1 + o(1))$, dove in (a) si è diviso numeratore e denominatore per $((n-2)!)^2\,n^{n-2}$. L'ordine di infinito di $\{a_n\}_{n\in\mathbb{N}}$ è 5.

(11) Si ha $a_n = \dfrac{1}{\sqrt{n}} - \left(\dfrac{2}{3}\right)^n = \dfrac{1}{\sqrt{n}}(1 + o(1))$, e l'ordine di infinitesimo di $\{a_n\}_{n\in\mathbb{N}}$ è $\frac{1}{2}$.

(12) Si ha $a_n = \dfrac{\sqrt{n} + n^{7/3}}{n^5 + n^3 + 8} = \dfrac{n^{7/3}(1+o(1))}{n^5(1+o(1))} = \dfrac{1}{n^{8/3}}(1 + o(1))$, e l'ordine di infinitesimo di $\{a_n\}_{n\in\mathbb{N}}$ è $\frac{8}{3}$.

(13) Si ha $a_n = 2 - \dfrac{2n^2}{n^2+n} = \dfrac{2n}{n^2+n} = \dfrac{2}{n}(1 + o(1))$, e l'ordine di infinitesimo di $\{a_n\}_{n\in\mathbb{N}}$ è 1.

(14) Si ha $a_n = \dfrac{n!}{(n+1)! - (n-1)!} = \dfrac{n}{n^2 + n - 1} = \dfrac{1}{n}(1 + o(1))$, e l'ordine di infinitesimo di $\{a_n\}_{n\in\mathbb{N}}$ è 1.

(15) Si ha $a_n = \dfrac{(n^2+1)(7^{n!} + 3^{n^2})}{(7^{n!} + 9^{n^2+n})(n^3 + \log_4 n)} = \dfrac{n^2 7^{n!}(1+o(1))}{n^3 7^{n!}(1+o(1))} = \dfrac{1}{n}(1 + o(1))$. L'ordine di infinitesimo di $\{a_n\}_{n\in\mathbb{N}}$ è 1.

(16) Si ha $a_n = \dfrac{n^{2(n+4)}\, n! - (n-1)!\, n^{2n+7}}{(n+1)!\, n^{2n+9} + (n^2)^n\, (n+2)!} \overset{(a)}{=} \dfrac{n^8\, n - n^7}{(n+1)n\, n^9 + (n+2)(n+1)n} = \dfrac{n^9(1+o(1))}{n^{11}(1+o(1))} = \dfrac{1}{n^2}(1+o(1))$, dove in (a) si è diviso numeratore e denominatore per $(n-1)!\, n^{2n}$. L'ordine di infinitesimo di $\{a_n\}_{n\in\mathbb{N}}$ è 2.

□

Svolgimento esercizio 5

(1) Si ha $\left(1+\dfrac{1}{2n}\right)^n = \sqrt{\left(1+\dfrac{1}{2n}\right)^{2n}} \to \sqrt{e}$, sfruttando la continuità di $x \to \sqrt{x}$.

(2) Si ha $\left(1+\dfrac{1}{2n+1}\right)^n = \sqrt{\dfrac{\left(1+\frac{1}{2n+1}\right)^{2n+1}}{1+\frac{1}{2n+1}}} \to \sqrt{e}$, sfruttando la continuità di $x \to \sqrt{x}$.

(3) Si ha $\left(1+\dfrac{1}{n}\right)^{2n} = \left(\left(1+\dfrac{1}{n}\right)^n\right)^2 \to e^2$.

(4) Si ha $\dfrac{(n+1)^n}{n^n+3} = \dfrac{(n+1)^n}{n^n(1+o(1))} = \left(\dfrac{n+1}{n}\right)^n(1+o(1)) = \left(1+\dfrac{1}{n}\right)^n(1+o(1)) \to e$.

(5) Si ha $\dfrac{(n+1)^n}{n^n+n^2} = \dfrac{(n+1)^n}{n^n(1+o(1))} = \left(1+\dfrac{1}{n}\right)^n(1+o(1)) \to e$.

(6) Si ha $\dfrac{(2n+1)^n}{(2n)^n+n^4} = \dfrac{(2n+1)^n}{(2n)^n(1+o(1))} = \left(\dfrac{2n+1}{2n}\right)^n(1+o(1)) = \left(1+\dfrac{1}{2n}\right)^n(1+o(1)) \to \sqrt{e}$, usando il risultato dell'esercizio (1).

(7) Si ha $\dfrac{(2n)^n+2^n}{(2n+1)^n} = \dfrac{(2n)^n(1+o(1))}{(2n+1)^n} \to \dfrac{1}{\sqrt{e}}$, usando il risultato dell'esercizio (6).

(8) Si ha $\dfrac{(2n+1)^n}{2n^n+1} = \dfrac{(2n+1)^n}{2n^n(1+o(1))} = \dfrac{1}{2}\left(\dfrac{2n+1}{n}\right)^n(1+o(1)) = \dfrac{1}{2}\left(2+\dfrac{1}{n}\right)^n(1+o(1)) \to +\infty$, perché $\left(2+\frac{1}{n}\right)^n \geq 2^n \to +\infty$.

(9) Si ha $(n+1)^n - n! = n^n\left(\dfrac{n+1}{n}\right)^n - n! = en^n(1+o(1)) - n! = en^n(1+o(1)) \to +\infty$.

(10) Si ha $(n+1)^{n+1} - n^{n+1} = n^{n+1}\left\{\left(\dfrac{n+1}{n}\right)^{n+1} - 1\right\} = n^{n+1}\big(e+o(1)-1\big) = (e-1)n^{n+1}(1+o(1)) \to +\infty$.

(11) Si ha $\dfrac{(n+1)^{n+1} - n^{n+1}}{(n-1)^{n+1} - n!} \overset{(a)}{=} \dfrac{(e-1)n^{n+1}(1+o(1))}{n^{n+1}(1-\frac{1}{n})^n(1-\frac{1}{n}) - n!} = \dfrac{(e-1)(1+o(1))}{e^{-1}(1+o(1))+o(1)} \to e(e-1)$, dove in (a) si è usato lo sviluppo asintotico dell'esercizio (10).

(12) Si ha $(n+1)^{n!} - n^{2n} = (n+1)^{n!}\left\{1 - \left(\dfrac{n^2}{(n+1)^{(n-1)!}}\right)^n\right\} = (n+1)^{n!}(1+o(1)) \to +\infty$, poiché $\frac{n^2}{(n+1)^{(n-1)!}} \to 0$.

(13) Si ha $\dfrac{(n+1)^n+n!}{n^n+5^n-n!}=\dfrac{(n+1)^n(1+o(1))}{n^n(1+o(1))}=\Big(\dfrac{n+1}{n}\Big)^n(1+o(1))\to e.$

(14) Si ha $\dfrac{(2n+1)^n+n!+1}{(2n+2)^n-n!+n^2}=\dfrac{(2n+1)^n(1+o(1))}{(2n+2)^n(1+o(1))}=\Big(\Big(\dfrac{2n+2}{2n+1}\Big)^n\Big)^{-1}(1+o(1))=$
$=\Big(\Big(1+\dfrac{1}{2n+1}\Big)^n\Big)^{-1}(1+o(1))\to\dfrac{1}{\sqrt{e}}$, usando il risultato dell'esercizio (2).

(15) Si ha $\dfrac{(2n+1)^n+(2n)^n}{(2n+2)^n-(2n+1)^n}=\dfrac{1+\frac{(2n)^n}{(2n+1)^n}}{\frac{(2n+2)^n}{(2n+1)^n}+1}=\dfrac{1+\big((\frac{2n+1}{2n})^n\big)^{-1}}{(\frac{2n+2}{2n+1})^n+1}=\dfrac{1+\big((1+\frac{1}{2n})^n\big)^{-1}}{(1+\frac{1}{2n+1})^n+1}\to\dfrac{1+\frac{1}{\sqrt{e}}}{\sqrt{e}+1}=$
$\dfrac{1}{\sqrt{e}}$, usando i risultati degli esercizi (1) e (2).

(16) Si ha $\dfrac{n^{n-3}+(n-3)^n}{6n^n+7n^{n/2}}=\dfrac{n^{-3}+(1-\frac{3}{n})^n}{6+o(1)}\to\dfrac{1}{6e^3}.$

(17) Si ha $\dfrac{(3n)^n-n^{3n}}{(n-1)^{3n}+(n+3)!}\overset{(a)}{=}\dfrac{-n^{3n}(1+o(1))}{(n-1)^{3n}+o(1)}=-\Big(\dfrac{n}{n-1}\Big)^{3n}(1+o(1))\to-e^{-3}$, dove in (a) si sono usati i risultati $\dfrac{3^nn^n}{n^{3n}}=\dfrac{3^n}{n^{2n}}\to 0$, e $\dfrac{(n+3)!}{(n-1)^{3n}}=\dfrac{(n+3)(n+2)(n+1)n}{(n-1)^{2n}}\cdot\dfrac{(n-1)!}{(n-1)^n}\to 0.$

(18) Si ha $\big(\sqrt{1+e^{-n}}-1\big)\big(e^n-3^n+n^2\big)=\dfrac{e^{-n}}{\sqrt{1+e^{-n}}+1}\big(e^n-3^n+n^2\big)=\dfrac{e^{-n}}{2(1+o(1))}e^n(1+o(1))=$
$\dfrac{1}{2}(1+o(1))\to\dfrac{1}{2}.$

(19) Si ha $\dfrac{(e^n+1)(n+\log n)}{(e^n+2^n)(2+\log n^6)}=\dfrac{e^n(1+o(1))n(1+o(1))}{e^n(1+o(1))6\log n(1+o(1))}=\dfrac{n}{6\log n}(1+o(1))\to+\infty.$

(20) Si ha $\dfrac{(n^2+5n+7)e^{1/n}}{(n+1)(\sqrt{n+3}-\sqrt{n})}=\dfrac{n^2(1+o(1))}{n(1+o(1))}\dfrac{\sqrt{n+3}+\sqrt{n}}{3}=n(1+o(1))\cdot 2\sqrt{n}(1+o(1))=$
$2n^{3/2}(1+o(1))\to+\infty.$

(21) Si ha $\sqrt[n]{2n}=\sqrt[n]{2}\cdot\sqrt[n]{n}\to 1.$

(22) Si ha $\sqrt[n]{n^3}=\big(\sqrt[n]{n}\big)^3\to 1.$

(23) Intanto $\sqrt[2n+1]{-n}=-\sqrt[2n+1]{n}$. Poi $1\le\sqrt[2n+1]{n}\le\sqrt[2n+1]{2n+1}\to 1$. Per il teorema dei due carabinieri, si ha $\sqrt[2n+1]{n}\to 1$, e quindi $\sqrt[2n+1]{-n}\to-1$.

(24) Intanto $\sqrt[n]{n^2+3}=\sqrt[n]{n^2\Big(1+\frac{3}{n^2}\Big)}=\big(\sqrt[n]{n}\big)^2\sqrt[n]{1+\frac{3}{n^2}}$. Poi, definitivamente si ha $1\le\sqrt[n]{1+\frac{3}{n^2}}\le$ $\sqrt[n]{2}\to 1$. Per il teorema dei due carabinieri, si ha $\sqrt[n]{1+\dfrac{3}{n^2}}\to 1$, e quindi $\sqrt[n]{n^2+3}=\big(\sqrt[n]{n}\big)^2\sqrt[n]{1+\frac{3}{n^2}}\to$ 1.

(25) Intanto $\sqrt[n]{2^n+n^2}=\sqrt[n]{2^n\Big(1+\frac{n^2}{2^n}\Big)}=2\sqrt[n]{1+\frac{n^2}{2^n}}$. Poi, definitivamente si ha $1\le\sqrt[n]{1+\frac{n^2}{2^n}}\le$ $\sqrt[n]{2}\to 1$. Per il teorema dei due carabinieri, si ha $\sqrt[n]{1+\dfrac{n^2}{2^n}}\to 1$, e quindi $\sqrt[n]{2^n+n^2}=2\sqrt[n]{1+\frac{n^2}{2^n}}\to$ 2.

(26) Intanto $\sqrt{n+\sqrt{n}}-\sqrt{n} = \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} = \frac{\sqrt{n}}{2\sqrt{n}(1+o(1))} \to \frac{1}{2}$. Inoltre, $\sqrt[n]{2^{n+1}+n^2} = \sqrt[n]{2^{n+1}\Big(1+\frac{n^2}{2^{n+1}}\Big)} = 2\sqrt[n]{2}\sqrt[n]{1+\frac{n^2}{2^{n+1}}}$. Poi, definitivamente si ha $1 \le \sqrt[n]{1+\frac{n^2}{2^{n+1}}} \le \sqrt[n]{2} \to 1$. Per il teorema dei due carabinieri, si ha $\sqrt[n]{1+\dfrac{n^2}{2^{n+1}}} \to 1$, e quindi $\big(\sqrt{n+\sqrt{n}}-\sqrt{n}\big)\sqrt[n]{2^{n+1}+n^2} \to 1$.

(27) Si ha $\dfrac{(n^2+5n+7)e^{1/n}}{(n+1)(\sqrt{n+3}-\sqrt{n})} = \dfrac{n^2(1+o(1))}{n(1+o(1))}\dfrac{\sqrt{n+3}+\sqrt{n}}{3} = n(1+o(1))\cdot 2\sqrt{n}(1+o(1)) = 2n^{3/2}(1+o(1)) \to +\infty$.

(28) Si ha $n^2(2n+\sqrt{n})^{1/n} - \cos(n^3) = n^2\sqrt[n]{2n}\Big(1+\dfrac{1}{2\sqrt{n}}\Big)^{1/n} - \cos(n^3) = n^2(1+o(1)) - \cos(n^3) = n^2(1+o(1)) \to +\infty$, poiché $1 \le (1+\frac{1}{2\sqrt{n}})^{1/n} \le 2^{1/n} \to 1$.

(29) Si ha $\big(4n^n-(n+1)^n\big)^{1/n} = n\Big\{4-\Big(\dfrac{n+1}{n}\Big)^n\Big\}^{1/n} = n(4-e+o(1))^{1/n} = n(1+o(1)) \to +\infty$, poiché $1 \le (4-e+o(1))^{1/n} \le 4^{1/n} \to 1$.

(30) Si ha $\big((n+1)^{n+1}-n^{n+1}\big)^{1/n} = n\sqrt[n]{n}\Big\{\Big(\dfrac{n+1}{n}\Big)^{n+1}-1\Big\}^{1/n} = n(1+o(1))(e+o(1)-1)^{1/n} = n(1+o(1)) \to +\infty$, poiché $1 \le (e-1+o(1))^{1/n} \le e^{1/n} \to 1$.

□

Svolgimento esercizio 6

(1) Si ha $\dfrac{n!}{n^{n/2}} = \dfrac{n^n e^{-n}\sqrt{2\pi n}(1+o(1))}{n^{n/2}} = n^{n/2}e^{-n}\sqrt{2\pi n}(1+o(1)) = \sqrt{\dfrac{n^n}{e^{2n}}}\sqrt{2\pi n}(1+o(1)) \to +\infty$.

(2) Si ha $\sqrt[n]{n!} = \sqrt[n]{n^n e^{-n}\sqrt{2\pi n}(1+o(1))} = \dfrac{n}{e}\big(\sqrt[n]{2\pi n}\big)^{1/2}(1+o(1)) \to +\infty$, in quanto $\sqrt[n]{2\pi n} \to 1$, e $\sqrt[n]{\frac{1}{2}} \le \sqrt[n]{1+o(1)} \le \sqrt[n]{2}$, definitivamente, per cui $\sqrt[n]{1+o(1)} \to 1$.

(3) Si ha $\dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{n}\sqrt[n]{n^n e^{-n}\sqrt{2\pi n}(1+o(1))} = \dfrac{1}{e}\big(\sqrt[n]{2\pi n}\big)^{1/2}(1+o(1)) \to \dfrac{1}{e}$.

(4) Si ha $\dfrac{\sqrt[2n]{n!}}{n} = \dfrac{1}{n}\Big(\sqrt[n]{n^n e^{-n}\sqrt{2\pi n}(1+o(1))}\Big)^{1/2} = \dfrac{1}{n}\sqrt{\dfrac{n}{e}}\big(\sqrt[n]{2\pi n}\big)^{1/4}(1+o(1)) \to 0$.

(5) Si ha $\dfrac{1}{n}\sqrt[n]{\dfrac{(2n)!}{n!}} = \dfrac{1}{n}\sqrt[n]{\dfrac{(2n)^{2n}e^{-2n}\sqrt{2\pi\cdot 2n}(1+o(1))}{n^n e^{-n}\sqrt{2\pi n}(1+o(1))}} = \dfrac{1}{n}\dfrac{4n}{e}\sqrt[n]{\sqrt{2}}(1+o(1)) \to \dfrac{4}{e}$.

(6) $\sqrt[n]{\dbinom{2n}{n}} = \sqrt[n]{\dfrac{(2n)!}{n!\cdot n!}} = \sqrt[n]{\dfrac{(2n)^{2n}e^{-2n}\sqrt{2\pi\cdot 2n}(1+o(1))}{n^{2n}e^{-2n}2\pi n(1+o(1))}} = \dfrac{4}{(\sqrt[n]{\pi n})^{1/2}}(1+o(1)) \to 4$.

(7) $\sqrt[n]{\dbinom{4n}{2n}} = \sqrt[n]{\dfrac{(4n)!}{(2n)!\cdot(2n)!}} = \sqrt[n]{\dfrac{(4n)^{4n}e^{-4n}\sqrt{2\pi\cdot 4n}(1+o(1))}{(2n)^{4n}e^{-4n}2\pi\cdot 2n(1+o(1))}} = \dfrac{16}{(\sqrt[n]{2\pi n})^{1/2}}(1+o(1)) \to 16$.

□