Matrix Decompositions Using Displacement Rank and Classes of Commutative Matrix Algebras

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ABSTRACT

Using the notion of displacement rank, we look for a unifying approach to representations of a matrix A as sums of products of matrices belonging to commutative matrix algebras. These representations are then considered in case A is the inverse of a Toeplitz or a Toeplitz plus Hankel matrix. Some well-known decomposition formulas for A (Gohberg-Semencul or Kailath et al., Gader, Bini-Pan, and Gohberg-Olshevsky) turn out to be special cases of the above representations. New formulas for A in terms of algebras of symmetric matrices are studied, and their computational aspects are discussed.

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1. INTRODUCTION

It is well known that the notion of displacement rank underlies many algorithms for solving Toeplitz systems of equations and that the same notion can be used to extend algorithms for Toeplitz matrices to other classes of matrices A [4, 5, 7, 10, 13–22, 24]. The main idea consists in looking for an

operator $\mathfrak{C}$ which transforms A into a low rank matrix $\mathfrak{C}(A)$ such that one could easily recover A from its image $\mathfrak{C}(A)$. A consequent expression for A is then obtained, which depends on the rank of $\mathfrak{C}(A)$ and is formulated in terms of (possibly) a few simple structured matrices. The classical Gohberg-Semencul [17] or Kailath et al. [22] formulas, the circulant type formulas of Gader [14] (see also Bini [7]), the ε-circulant type formulas of Gohberg and Olshevsky [16], and other known formulas involving a special algebra τ of matrices [7, 10] are all examples of the above technique. These formulas for A are then useful for solving computational problems—for example, a linear system—by means of any of a number of fast transforms (typically the FFT).

In the present paper we look for a unifying approach by exploiting a class of commutative matrix algebras (Section 2) containing, as particular instances, all algebras considered in the literature (τ, circulant, ε-circulant, Toeplitz triangular), with the sole exception of the group algebras different from circulant matrices used in [14]. This class of algebras is constructed with a technique which is similar, in spirit, to that used by Bapat and Sunder in their paper on hypergroups of matrices [6]. By this general approach we are able to formulate a decomposition theorem (Theorem 3.1) whose corollaries give the well-known splits for A based on the previously mentioned algebras.

New decomposition formulas for A are then obtained involving whole classes of algebras instead of singular algebras of matrices (Section 3: in particular Theorems 3.2 and 3.3). In Section 4 are listed, as particular instances, some interesting formulas for T^{-1} and $(T + H)^{-1}$ where T is a Toeplitz and $T + H$ is a Toeplitz plus Hankel matrix [18–20]. Especially in the case of $(T + H)^{-1}$, some of these formulas appear to be particularly simple and effective, as they involve only a few products of elements of the same algebra τ. Some computational aspects of these formulas are then investigated in Section 5.

All previous results are obtained using, as $\mathfrak{C}(A)$, the commutator $\mathfrak{C}_X(A) = AX - XA$ for different choices of X, depending on the matrix algebra involved. In fact, $\mathfrak{C}_X(A)$ turns out to be the most natural operator, as the matrix algebras considered throughout the paper are commutative.

For the sake of completeness we state Propositions 4.3 and 4.4, in Section 4, which show that for some convenient choices of X the images $\mathfrak{C}_X(T^{-1})$ or $\mathfrak{C}_X((T + H)^{-1})$ can always be expressed in terms of a number of columns or rows and columns of, respectively, T^{-1} and $(T + H)^{-1}$.

2.   A CLASS OF ALGEBRAS OF MATRICES

In this section we shall introduce a class of algebras of $n \times n$ matrices over $\mathbb{C}$, using a constructive criterion similar, in some ways, to that proposed by Bapat and Sunder in their paper on hypergroups of matrices [6]. This class

of algebras will be exploited to write an arbitrary square matrix over a ring as sums of matrix products, in the spirit of the literature on rank displacement operators [4, 5, 7, 10, 13–22, 24]. A general approach will follow from which one is able to regain, as special cases, the classical Gohberg-Semencul formulas [17] (or Kailath et al. [22]), the variants proposed by Gader in [14] and by Bini and Pan in [10] (see also Bini [7]), and the Gohberg-Olshevsky formulas [16].

Consider the lower Hessenberg matrix

$$
X = \begin{pmatrix}
r_{11} & b_1 & 0 & \cdots & 0 \\
r_{21} & r_{22} & b_2 & & \vdots \\
\vdots & & \ddots & \ddots & 0 \\
\vdots & & & \ddots & b_{n-1} \\
r_{n1} & \cdot & \cdot & \cdots & r_{nn}
\end{pmatrix},
\tag{2.1}
$$

and define $A_k = p_{k-1}(X)$, where $p_k(\lambda)$ is the characteristic polynomial of the top left $k \times k$ submatrix of X for $k = 1, \ldots, n$, and $p_0(\lambda) = 1$. We have

$$
p_0(\lambda) = 1,
$$

$$
p_1(\lambda) = \lambda - r_{11},
$$

$$
p_j(\lambda) = (\lambda - r_{jj})p_{j-1}(\lambda) - \sum_{m=1}^{j-1} r_{jm}\left(\prod_{i=m}^{j-1} b_i\right) p_{m-1}(\lambda), \qquad j = 2, \ldots, n;
$$

$$
A_1 = I,
$$

$$
A_2 = X - r_{11}I,
$$

$$
A_{j+1} = (X - r_{jj}I)A_j - \sum_{m=1}^{j-1} r_{jm}\left(\prod_{i=m}^{j-1} b_i\right) A_m, \qquad j = 2, \ldots, n.
\tag{2.2}
$$

Observe that, by Cayley-Hamilton theorem, $A_{n+1} = p_n(X) = 0$.

Let H_X be the space of matrices defined by

$$
H_X = \left\{ \sum_{k=1}^{n} a_k A_k \right\}
\tag{2.3}
$$

where the a_k's are complex parameters. The most important properties of H_X are listed in Proposition 2.1 below.

Recall that a square matrix is *nonderogatory* when there is only one eigenvector associated with each distinct eigenvalue λ. Moreover, it is known

that a matrix is nonderogatory if and only if its minimum polynomial is equal to its characteristic polynomial [28, pp. 14–16, 41].

Let J denote the matrix $[J]_{ij} = \delta_{i,\,n+1-j}$, $1 \leqslant i, j \leqslant n$. A square matrix A is *persymmetric* if $A^T = JAJ$ and *centrosymmetric* if $A = JAJ$.

PROPOSITION 2.1.

(i) H_X is commutative and closed under matrix multiplication. All matrices of the space H_X are symmetric (persymmetric, centrosymmetric) iff X is symmetric (persymmetric, centrosymmetric). Moreover · dim H_X = degree of the minimum polynomial of X.

(ii) If $b_i \neq 0 \; \forall i$, then X is nonderogatory and dim $H_X = n$.

(iii) Let $\mathbf{v}$ be an eigenvector of X, that is, $X\mathbf{v} = \lambda\mathbf{v}$, $\mathbf{v} \neq 0$, and $\lambda \in \mathbb{C}$. Then $(\sum_{k=1}^{n} a_k A_k)\mathbf{v} = \sum_{k=1}^{n} a_k p_{k-1}(\lambda)\mathbf{v}$, which implies that all matrices of H_X are simultaneously diagonalizable iff X is diagonalizable.

(iv) Assume $b_i \neq 0$ for all i. Then the first element v_1 of every eigenvector $\mathbf{v}$ of H_X is nonzero, and we can assume $v_1 = 1$.

Proof. Properties (i) and (iii) are trivial. Property (ii) follows from the nonsingularity of the top right $(n-1) \times (n-1)$ submatrix of $\lambda I - X$, which implies the uniqueness (up to a multiplicative factor) of the solution of the system $(\lambda I - X)\mathbf{v} = 0$, for every eigenvalue λ.

Regarding (iv), let the first element of $\mathbf{v}$ be zero. Then one easily calculates, by the structure of X, $v_i = 0$ for successive values of $i = 2, \ldots, n$.   ■

We will refer to the space H_X in (2.3) as a *Hessenberg algebra* (HA).

PROPOSITION 2.2.   The matrix A_t, $t = 1, \ldots, n$, has the form

$$A_t = \begin{pmatrix} 0 & \cdots & 0 & \prod_{i=1}^{t-1} b_i & 0 & \cdots & & 0 \\ \blacksquare & \cdots & \blacksquare & \blacksquare & \prod_{i=2}^{t} b_i & \ddots & & \vdots \\ \cdot & & & \blacksquare & & \ddots & & 0 \\ \cdot & & & & & \ddots & \prod_{i=n-t+1}^{n-1} b_i & \\ \cdot & & & & & & \blacksquare & \vdots \\ \cdot & & & & & & & \\ \blacksquare & \cdot & & \cdot & & \cdot & \cdot & \blacksquare \end{pmatrix} \qquad (2.4)$$

where the nonzero element in the first row is in position $(1, t)$.

Proof. The proof is by induction. The cases $t = 1, 2$ are trivial. Suppose (2.4) is verified for $t = 1, \ldots, j$. Then, using the commutativity of the A_k's and the formulas (2.2), for $t < j + 1$

$$\left[A_{j+1} \right] 1t = e_1^T A_j X e_t - \sum_{m=1}^{j} r_{jm} \left(\prod_{i=m}^{j-1} b_i \right) \left[A_m \right]_{1t}$$

$$= \left(\prod_{i=t}^{j-1} b_i \right) e_j^T \left(b_{t-1} e_{t-1} + \sum_{i=t}^{n} r_{it} e_i \right) - r_{jt} \left(\prod_{i=t}^{j-1} b_i \right) \left(\prod_{i=1}^{t-1} b_i \right) = 0.$$

Moreover, for $k = 1, \ldots, n - j$ and $t \geq k + j$,

$$\left[A_{j+1} \right]_{kt} = e_k^T X A_j e_t - \sum_{m=1}^{j} r_{jm} \left(\prod_{i=m}^{j-1} b_i \right) \left[A_m \right]_{kt} = e_k^T X A_j e_t$$

$$= \left(\sum_{i=1}^{k} r_{ki} e_i^T + b_k e_{k+1}^T \right) \left[\left(\prod_{i=t-j+1}^{t-1} b_i \right) e_{t-j+1} + \sum_{i=t-j+2}^{n} \left[A_j \right]_{it} e_i \right]$$

$$= \begin{cases} 0, & t > k + j, \\ \prod_{i=k}^{k+j-1} b_i, & t = k + j. \end{cases} \qquad \blacksquare$$

PROPOSITION 2.2 bis. *Let the matrix X in (2.1) be a tridiagonal matrix. Then the matrix A_t, $t = 1, \ldots, n$, has the form*

where $c_i = r_{i+1\,i}$, $i = 1, \ldots, n - 1$, $q_{tk} = \prod_{i=k}^{t-1} b_i \prod_{i=t-k+1}^{t-1} c_i$, $k = 1, \ldots, t$, and the nonzero element in the first row is in position $(1, t)$.

Proof. Left to the reader. ∎

As a consequence of Proposition 2.2 bis we have the following

COROLLARY 2.1. *Let the matrix X in (2.1) be centrosymmetric. Then*

$$A_n = \left(\prod_{i=1}^{n-1} b_i \right) J.$$

Assume $b_i \neq 0$ for all i, and define $X_i = q_{i1}^{-1} A_i$ $(q_{i1} = \prod_{j=1}^{i-1} b_j)$, $i = 1, \ldots, n$. Then we have $\mathbf{e}_1^T X_i = \mathbf{e}_i^T$, $i = 1, \ldots, n$, and H_X in (2.3) can be defined as the space spanned by the X_i's, that is, the space whose generic element is $H_X(\mathbf{a}) = \sum_{k=1}^n a_k X_k$. Observe that $\mathbf{e}_1^T H_X(\mathbf{a}) = \mathbf{a}^T$.

REMARK. If $\mathbf{v}$ is an eigenvector of X, that is, $X\mathbf{v} = \lambda\mathbf{v}$ for $\lambda \in \mathbb{C}$, $\mathbf{v} \neq 0$, then $X_j\mathbf{v} = q_{j1}^{-1} A_j\mathbf{v} = q_{j1}^{-1} p_{j-1}(X)\mathbf{v} = q_{j1}^{-1} p_{j-1}(\lambda)\mathbf{v}$. By left multiplying by $\mathbf{e}_1^T$ we have $q_{j1}^{-1} p_{j-1}(\lambda)v_1 = v_j$ and assuming $v_1 = 1$ [see Proposition 2.1(iv)],

$$X_j\mathbf{v} = v_j\mathbf{v}, \qquad j = 1, \ldots, n$$

—in other words, $\mathbf{v}$ is the eigenvector of X_j corresponding to an eigenvalue that is equal to the jth component of $\mathbf{v}$. A similar property holds for matrices forming hypergroups in the sense of Bapat and Sunder [6, 27].

The following proposition gives some information on the multiplication table of the X_i's.

PROPOSITION 2.3. *The following equality holds:*

$$X_i X_j = \sum_{k=1}^n [X_j]_{ik} X_k = \sum_{k=1}^n [X_i]_{jk} X_k = X_j X_i, \qquad (2.5)$$

and, as a direct consequence of (2.5), $\mathbf{e}_i^T X_j = \mathbf{e}_j^T X_i$.

If X is symmetric, then

$$X_i X_j = \sum_{k=1}^n [X_k]_{ij} X_k, \qquad 1 \leqslant i, j \leqslant n.$$

Proof. H_X is closed under multiplication. Thus $X_i X_j \in H_X$ and $\exists \{a_k\}_{k=1,\dots,n}$ such that $X_i X_j = \sum_{k=1}^{n} a_k X_k$. We have also $e_1^T X_i X_j = [a_1 \ a_2 \ \cdots \ a_n]$ and then $X_i X_j = \sum_{k=1}^{n} e_1^T X_i X_j e_k X_k = \sum_{k=1}^{n} [X_j]_{ik} X_k = X_j X_i = \sum_{k=1}^{n} [X_i]_{jk} X_k$.

If $X = X^T$, then $X_i = X_i^T$ and the equality $e_k^T X_j = e_j^T X_k$ implies $X_j e_k = X_k e_j$. Thus $X_i X_j = \sum_{k=1}^{n} e_i^T X_k e_j X_k = \sum_{k=1}^{n} [X_k]_{ij} X_k$. ∎

Observe that, in the symmetric case, the previous proposition says that the multiplication table of the X_i's has the same structure as that of the matrices $H_X(\mathbf{a}) = \sum_{k=1}^{n} a_k X_k$.

We shall state, in the following propositions, the relationship between HAs and group algebras and/or hypergroups in the sense of [6, 27].

Recall the definition of a *group algebra of matrices* (see [14]). Let $G = \{1, 2, \dots, n\}$ be a finite group of order n, with 1 denoting the identity element. A group matrix for G over $\mathbb{C}$ is an $n \times n$ matrix $A = (a_{ij})$, $a_{ij} \in \mathbb{C}$, $i, j \in G$, with the property that $a_{i,j} = a_{ki,kj}$ for every $k \in G$. The space of group matrices for G over $\mathbb{C}$ is an algebra of dimension n, which is called the group algebra for G over $\mathbb{C}$.

Let $\mathbb{C}[G]$ denote the group algebra for G over $\mathbb{C}$. Observe that there always exist $n \times n$ matrices $J_k \in \mathbb{C}[G]$ such that $e_1^T J_k = e_k^T$, $k = 1, \dots, n$. They are permutation matrices, and they span $\mathbb{C}[G]$. The set $\{J_1 = I, J_2, \dots, J_n\}$ is the right regular representation of G in $GL(n, \mathbb{C})$.

Recall that an $n \times n$ matrix is circulant if each row is derived from the row above by shifting right cyclically. The space of $n \times n$ circulant matrices is the group algebra for the cyclic group of order n [11].

PROPOSITION 2.4. *The space of $n \times n$ circulant matrices is the only group algebra which is also a Hessenberg algebra.*

Proof. Let H_X [defined in (2.3)] $\equiv \mathbb{C}[G]$ for a group G of order n. Observe that $b_i = 0$, for an index i, implies $e_1^T A_k e_n = 0$, $k = 1, \dots, n$ (see Proposition 2.2). This means that H_X cannot contain a matrix J_n whose first row is e_n^T, and thus it cannot be a group algebra. Let $b_i \neq 0$ for all i. The matrices X_i are well defined, and they are permutation matrices because H_X is a group algebra. As $e_i^T X_2 e_{i+1} = b_i/b_1 \neq 0$, $i = 1, \dots, n-1$, we have in particular that X_2 is the circulant matrix whose first row is e_2^T. H_X is also the space of all polynomials in X_2, and thus it is necessarily the space of circulant matrices. ∎

Recall the definition of a *hypergroup of matrices* given in [6]. The collection $\{A_1, A_2, \dots, A_n\}$ of n $n \times n$ matrices is a hypergroup of matrices if the following conditions are satisfied:

(a) $[A_k]_{ij} \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$ for all i, j, and k, and $A_1 = I$, the $n \times n$ identity matrix;

(b) the collection $\{A_1, A_2, \ldots, A_n\}$ is linearly independent and self-adjoint (in the sense of being closed under taking transposes); and

(c) $A_i A_j = \sum_{k=1}^n [A_j]_{ik} A_k$ for all i, j.

The A_k's span an n-dimensional algebra (not necessarily commutative), and we shall refer to this algebra as the algebra spanned by the hypergroup $\{A_1, A_2, \ldots, A_n\}$.

In [6] is studied the subclass of the HA obtained for

$$X = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ 1 & a_2 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & a_n \end{pmatrix}, \tag{2.6}$$

where the a_i's are integers. Some conditions over the a_i's are given to make $\{A_1, A_2, \ldots, A_n\}$ a hypergroup of matrices:

$$0 = a_1 \leqslant \cdots \leqslant a_n \tag{2.7a}$$

or

$$0 = a_1 = a_n \leqslant a_2 = a_{n-1} \leqslant a_3 = a_{n-2} \leqslant \cdots . \tag{2.7b}$$

THEOREM 2.1. *In the class of HAs with $b_i = 1$, for all i, the only algebras spanned by a hypergroup are those obtained for*

(i) $X = $ *the circulant matrix whose first row is e_2^T, and*

(ii) $X = $ *the matrix (2.6) with the a_i's such that $[A_k]_{ij} \in \mathbb{Z}^+$ for all i, j, and k (for example, with the a_i's satisfying one of the conditions (2.7)).*

Proof.   By self-adjointness we have, in particular, $A_2 = A_i^T$ for some i. It cannot be that $i = 1$. If $i = 2$, then we have case (ii): trivially, the A_k's satisfy all the conditions defining a hypergroup except for the one which requires $[A_k]_{ij} \in \mathbb{Z}^+$ for all i, j, and k. Assume $i > 2$. Proposition 2.2, for $b_j = 1$,

$j = 1, \ldots, n$, and the condition $A_2 = A_i^T$ imply

$$
A_2 = \begin{pmatrix}
0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & r_{22} - r_{11} & 1 & \cdot & & & & \cdot \\
\vdots & r_{32} & & & \cdot & \cdot & & \cdot \\
0 & \vdots & & & \cdot & \cdot & & \cdot \\
1 & r_{i2} & & & & & & \cdot \\
0 & 1 & & \ddots & & & 0 \\
\vdots & \cdot & \ddots & \ddots & & & \cdot & 1 \\
0 & \cdots & 0 & 1 & r_{n\,n-i+2} & \cdots & r_{n\,n-1} & r_{nn} - r_{11}
\end{pmatrix}.
$$

Exploiting the equalities $e_i^T A_2 = e_2^T A_i = e_2^T A_2^T$ and $r_{i2} = r_{22} - r_{11}$, we have

$$
[A_i]_{32} = [A_3]_{i2} = [(A_2 - (r_{22} - r_{11})I)A_2]_{i2}
$$

$$
= \begin{cases}
[1 \; r_{22} - r_{11} \; r_{32} \; \cdots \; r_{i-1\,2} \; 0 \; 1 \; 0 \; \cdots \; 0] \\
[1 \; r_{22} - r_{11} \; \cdots \; r_{i-1\,2} \; r_{i2} \; 1 \; 0 \; \cdots \; 0]^T \\
[1 \; r_{22} - r_{11} \; r_{32} \; \cdots \; r_{n-1\,2} \; 0] \\
[1 \; r_{22} - r_{11} \; \cdots \; r_{n-1\,2} \; r_{n2}]^T
\end{cases}
$$

$$
= \begin{cases}
2 + (r_{22} - r_{11})^2 + r_{32}^2 + \cdots + r_{i-1\,2}^2, & i < n, \\
1 + (r_{22} - r_{11})^2 + r_{32}^2 + \cdots + r_{n-1\,2}^2, & i = n.
\end{cases}
$$

As $[A_i]_{32} = [A_2]_{23} = 1$, the only possibility is

$$
A_n^T = A_2 = \begin{pmatrix}
0 & 1 & 0 & \cdot & \cdot & & \cdot & 0 \\
0 & 0 & 1 & 0 & & & & \cdot \\
\cdot & 0 & r_{33} - r_{11} & 1 & \ddots & & & \cdot \\
\cdot & \vdots & \vdots & \ddots & \ddots & \ddots & & \cdot \\
\cdot & \vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\
\cdot & \vdots & \vdots & & & \ddots & \ddots & \\
0 & 0 & r_{n-1\,3} & \cdots & & r_{n-1\,n-1} - r_{11} & 1 \\
1 & 0 & 0 & \cdots & & 0 & 0
\end{pmatrix}.
$$

Moreover $A_2 A_n = \sum_{k=1}^{n} [A_2]_{nk} A_k = I$, that is, A_2 must be an orthonormal matrix. Then we have case (i), because A_2 must be the circulant matrix whose first row is $\mathbf{e}_2^T$.   ∎

Observe that this whole section could be rewritten with simple changes when X is an upper, instead of lower, Hessenberg matrix.

The following algebras are among the most important instances of HAs.

(1) Upper triangular Toeplitz matrices:

$$X = Z^T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \cdot & \cdot & \ddots & \vdots \\ \cdot & & \cdot & \ddots & 0 \\ & & & \cdot & \\ \cdot & & & & 1 \\ 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

(2) Circulant matrices [11]:

$$X = P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \cdot & \cdot & \cdot & \vdots \\ \vdots & & \cdot & \ddots & 0 \\ 0 & & & \cdot & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Eigenvalues of P:

$$\lambda_j = \omega^{j-1}, \, j = 1, \ldots, n \qquad (\omega = e^{i2\pi/n}).$$

Eigenvectors of H_p:

$$v_i^{(j)} = \omega^{(i-1)(j-1)}, \qquad i, j = 1, \ldots, n.$$

(3) ε-Circulant matrices [11]:

$$X = P_\varepsilon = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \cdot & \cdot & \ddots & \vdots \\ \vdots & & \cdot & \ddots & 0 \\ 0 & & & \cdot & 1 \\ \varepsilon & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Eigenvalues of P_ε:

$$\lambda_j = \left(\sqrt[n]{\varepsilon}\right)\omega^{j-1}, \qquad j = 1, \ldots, n.$$

Eigenvectors of H_{P_ε}:

$$v_i^{(j)} = \left(\sqrt[n]{\varepsilon}\right)^{i-1}\omega^{(i-1)(j-1)}, \qquad i, j = 1, \ldots, n.$$

Observe that $P_1 = P$ and $P_0 = Z^T$.

(4) Algebra τ [7, 8, 31]:

$$X = T_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Eigenvalues of T_2:

$$\lambda_j = 2\cos\frac{j\pi}{n+1}, \qquad j = 1, \ldots, n.$$

Eigenvectors of H_{T_2}:

$$v_i^{(j)} = \sqrt{\frac{2}{n+1}}\,[S_n]_{ij} = \sqrt{\frac{2}{n+1}}\,\sin\frac{ij\pi}{n+1}, \qquad i, j = 1, \ldots, n.$$

(5) Algebra Γ:

$$X = K_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

Eigenvalues of K_2:

$$\lambda_j = 2\cos\frac{2j-1}{2n+1}\pi, \qquad j = 1,\ldots, n.$$

Eigenvectors of H_{K_2}:

$$v_i^{(j)} = \frac{2}{\sqrt{2n+1}}\sin\frac{i(2j-1)}{2n+1}\pi, \qquad i,j = 1,\ldots, n.$$

(6) Algebra F^a (n odd):

$$X = F_2^a = \begin{pmatrix}
0 & 1 & 0 & \cdot & \cdot & \cdot & & & 0 \\
1 & \cdot & & \cdot & \cdot & & & & \\
0 & \cdot & & 0 & 1 & \cdot & & & \\
\cdot & & 1 & a & 1 & & \cdot & \\
\cdot & & & 1 & 0 & \cdot & & 0 \\
\cdot & & & & \cdot & \cdot & & 1 \\
0 & & \cdot & \cdot & \cdot & 0 & 1 & 0
\end{pmatrix}.$$

$\overset{\frac{n+1}{2}}{}$

It is known [28] that the eigenvalues of F_2^a are the eigenvalues of the following two matrices [whose orders are $(n-1)/2$ and $(n+1)/2$, respectively]:

$$\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & \cdot & \cdot & \cdot & \vdots \\
0 & \cdot & \cdot & \cdot & 0 \\
\vdots & \cdot & \cdot & \cdot & 1 \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}, \qquad
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & \cdot & \cdot & \cdot & \vdots \\
0 & \cdot & \cdot & 1 & 0 \\
\vdots & \cdot & 1 & 0 & 1 \\
0 & \cdots & 0 & 2 & a
\end{pmatrix}.$$

Moreover, the eigenvectors of F_2^a are obtained from the eigenvectors of the above two matrices by extending them antisymmetrically $(v_i^{(j)} = -v_{n+1-i}^{(j)})$ and symmetrically $(v_i^{(j)} = v_{n+1-i}^{(j)})$, respectively [28]. As the eigenvalues of the second matrix seem to be unknown in their explicit form (for $a \neq 0$), we only have

Known eigenvalues of F_2^a:

$$\lambda_j = 2\cos\frac{2j\pi}{n+1}, \qquad j = 1,\ldots, \frac{n-1}{2}.$$

Known eigenvectors of $H_{F_2^a}$:

$$v_i^{(j)} = \frac{2}{\sqrt{n+1}} \sin \frac{2ij}{n+1} \pi, \qquad i, j = 1, \ldots, \frac{n-1}{2}.$$

Observe that the algebras H_X at points 1, 2, 3, 4, and 5 satisfy a "cross-sum" condition

$$[H_X]_{i-1\,j} + [H_X]_{i+1\,j} = [H_X]_{i\,j-1} + [H_X]_{i\,j+1}$$

with different "border" conditions (see [7, 8, 11, 31]).

The algebras τ, Γ, and F^1 are considered in [6] as the most important specializations of a notion of hypergroup of matrices in the sense of Bapat and Sunder [6, 27].

The matrix algebra τ, introduced for the first time in [8], has a number of interesting applications in numerical linear algebra. Matrices of τ can be exploited to define spectral and computational properties of band symmetric Toeplitz matrices [8]. The evaluation of the multiplicative complexity of a set of bilinear forms defined by general symmetric Toeplitz matrices and band Toeplitz matrices is also related to the properties of the class τ [31, 9].

3. DISPLACEMENT OPERATORS RELATED
TO HESSENBERG ALGEBRAS

In this section we wish to show how an arbitrary square matrix A over a ring R (with identity) can be written as a sum of products of matrices which are elements of a Hessenberg algebra or are defined in terms of Hessenberg algebras. Because of the definition of A over R, the main results of this section will hold in case a_{ij} are matrices.

Our approach generalizes, in several respects, some results by Gader [14], which are inspired by previous results [22] dealing with displacement operators $\mathfrak{S}(A) = A - ZAZ^T$, where Z is the lower shift matrix [whose (i, j) element is 1 if $i - j = 1$ and 0 elsewhere]. Gader has shown in [14] how an arbitrary square matrix over R can be expressed as a sum of products of group matrices and matrices "close" to group matrices. In particular, for the group algebra of circulant matrices, the "shift" operator $A - ZAZ^T$ is re-

placed by the operator $A - P^T A P$, where P is the circulant matrix whose first row is $\mathbf{e}_2^T$.

Now, the use of the commutative HAs naturally suggests the introduction of displacement operators which are commutators of the form $\mathfrak{C}_X(A) = AX - XA$. Operators of this kind have been used, in particular cases, by Bini-Pan [10] (see also Pan [24], Bini [7], and Heinig and Rost [18]). A consequent expression for A is then obtained in terms of classes H_X where X is persymmetric or has Toeplitz form (see Theorem 3.1 below). As particular instances (related to particular choices of the HA) we retrieve the Gohberg-Semencul [17] (or Kailath et al. [22]) formulas, Gader's variant exploiting circulant matrices [14], the Bini-Pan formulas involving both τ and Toeplitz triangular matrices [10, 7], and the Gohberg-Olshevsky formulas exploiting ε-circulant matrices [16].

Two new decomposition formulas involving all possible symmetric HAs are then introduced in Theorems 3.2 and 3.3. The special case where the HA is the algebra τ will be considered in detail in Corollaries 3.2 and 3.3. These results will lead (see Section 4 below) to new formulas for the inverses of Toeplitz matrices T (or matrices "close" to Toeplitz matrices), which could be used to solve linear systems $T\mathbf{x} = \mathbf{f}$ by sine transforms (which can be implemented at the same cost as the FFT). New formulas will be also introduced for Toeplitz plus Hankel matrices, which are particularly simple and computationally economical.

Let $\mathrm{M}_n(R)$ be the space of $n \times n$ matrices over a ring R with identity, and let $A \in \mathrm{M}_n(R)$. Let X be a general matrix in $\mathrm{M}_n(R)$, and assume once and for all

$$[X]_{ij} r = r[X]_{ij}, \tag{3.1}$$

for any $r \in R$, $i, j = 1, \ldots, n$.

Set $\mathfrak{C}_X(A) = AX - XA$.

LEMMA 3.1. *We have*

$$\sum_{i,j=1}^{n} [\mathfrak{C}_X(A)]_{ij} [p(X^T)]_{ij} = \sum_{i,j=1}^{n} [p(X^T)]_{ij} [\mathfrak{C}_X(A)]_{ij} = 0,$$

where $p(X^T)$ is any polynomial in X^T with coefficients in R.

Proof.

$$\sum_{i,j=1}^{n} [AX - XA]_{ij} [p(X^T)]_{ij}$$

$$= \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} [A]_{ik}[X]_{kj} \right) [p(X^T)]_{ij}$$

$$- \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} [X]_{ik}[A]_{kj} \right) [p(X^T)]_{ij}$$

$$= \sum_{i,k=1}^{n} [A]_{ik} \sum_{j=1}^{n} [p(X^T)]_{ij}[X^T]_{jk}$$

$$- \sum_{k,j=1}^{n} [A]_{kj} \sum_{i=1}^{n} [X^T]_{ki}[p(X^T)]_{ij}$$

$$= \sum_{i,k=1}^{n} [A]_{ik} [p(X^T)X^T]_{ik} - \sum_{k,j=1}^{n} [A]_{kj} [X^T p(X^T)]_{kj} = 0. \quad \blacksquare$$

From the above lemma we easily obtain some orthogonality relations depending on a possible split of $\mathfrak{C}_X(A)$ as a sum of α rank one matrices. Analogous relations are obtained in [14] in a more specific context, using the operator $A - P^T A P$.

PROPOSITION 3.1. *Let* $\mathbf{x}_m = [x_1^{(m)} \ x_2^{(m)} \ \cdots \ x_n^{(m)}]^T$ *and* $\mathbf{y}_m = [y_1^{(m)} \ y_2^{(m)} \ \cdots \ y_n^{(m)}]^T$, $m = 1, \ldots, \alpha$, *be vectors of* R^n *such that* $\mathfrak{C}_X(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T$. *Then*

$$\sum_{m=1}^{\alpha} [p(X)\mathbf{x}_m]^T \mathbf{y}_m = 0,$$

where $p(X)$ *is any polynomial in* X *whose coefficients are in* R *and commute with all elements of* R.

Proof.

$$\sum_{m=1}^{\alpha} (p(X)\mathbf{x}_m)^T \mathbf{y}_m = \sum_{m=1}^{\alpha} \mathbf{x}_m^T p(X^T)\mathbf{y}_m = \sum_{m=1}^{\alpha} \sum_{i,j=1}^{n} x_i^{(m)}[p(X^T)]_{ij} y_j^{(m)}$$

$$= \sum_{m=1}^{\alpha} \sum_{i,j=1}^{n} x_i^{(m)} y_j^{(m)}[p(X^T)]_{ij}$$

$$= \sum_{i,j=1}^{n} [\mathbb{C}_X(A)]_{ij}[p(X^T)]_{ij} = 0. \qquad \blacksquare$$

Observe that the above orthogonality relations hold for a general matrix X obeying the equality (3.1).

Now we wish to prove that for some special structure conditions for X one obtains some different splits of a matrix $A \in M_n(R)$ depending on H_X. For this task it is necessary to reconsider some points of the previous section when $\mathbb{C}$ is replaced by R.

Assume X to be the lower Hessenberg matrix in (2.1), $X \in M_n(R)$, and assume once and for all that each b_i has an inverse in R. Define the matrices $X_k = (b_1 \cdots b_{k-1})^{-1} p_{k-1}(X)$, $k = 1, \ldots, n$, so that, by Proposition 2.2 [which still holds under the assumption (3.1)], $\mathbf{e}_1^T X_k = \mathbf{e}_k^T$, $k = 1, \ldots, n$, and consider the space spanned by $\{X_k\}$ in $M_n(R)$, that is, the space whose generic element is

$$H_X(\mathbf{z}) = \sum_{k=1}^{n} z_k X_k, \qquad z_k \in R, \quad k = 1, \ldots, n.$$

H_X is an n-dimensional space which is closed under matrix multiplication, as $H_X = \{\sum_{k=0}^{n-1} a_k X^k, a_k \in R\}$ and $p_n(X) = 0$. The space H_X is not commutative, because the equality $H_X(\mathbf{z}) H_X(\bar{\mathbf{z}}) = H_X(\bar{\mathbf{z}}) H_X(\mathbf{z})$ is not satisfied when the z_k's do not commute with the $\bar{z}_k$'s. However, we have $H_X(\mathbf{z}) p(X) = p(X) H_X(\mathbf{z})$ for all $\mathbf{z} \in R^n$ and for all polynomials $p(\lambda)$ whose coefficients are in R and commute with all elements of R. In particular, the previous equality holds when $p(X) = X_k$. Finally, observe that Proposition 2.3 still holds for $\{X_i\}$. In particular we have $\mathbf{e}_1^T H_X(\mathbf{z}) = \mathbf{z}^T$ and $\mathbf{e}_i^T X_j = \mathbf{e}_j^T X_i$.

In the proofs of the next three theorems (3.1–3.3) we shall use the following fact:

$$\operatorname{Ker} \mathfrak{C}_X = H_X. \tag{3.2}$$

It is known [23, p. 78] that if A and B are commuting matrices in $M_n(\mathbb{C})$ and A is nonderogatory, then B is a polynomial in A. Thus, if $R = \mathbb{C}$ and X is nonderogatory, then (3.2) holds, and consequently Theorems 3.1, 3.2, and 3.3 will hold (observe that in this case there is no restriction on the choice of the $[X]_{ij}$'s).

In the general case, that is, when a (F_2^a) R is a ring with identity, the assertion (3.2) is true when $X = P_\varepsilon, T_2, K_2, F_2$, as one can directly verify. Some of these special cases ($X = P_\varepsilon$, $X = T_2$) will be examined in Corollaries 3.1–3.3. One easily realizes (for example, considering the case $X = P_\varepsilon$) that assuming (3.2) forces (3.1) to be true. If $\operatorname{Ker} \mathfrak{C}_X \supset H_X$ properly ($\leftrightarrow$ dim Ker $\mathfrak{C}_X > n$), then the last addenda of all formulas stated in Theorems 3.1, 3.2, and 3.3 would be replaced by a matrix of Ker $\mathfrak{C}_X$ with first or last row (first or last column) the corresponding one of A.

Let $\hat{x}$ denote the vector $J\mathbf{x}$, where J is the *reflection* matrix (the permutation matrix whose ith row is $\mathbf{e}_{n+1-i}^T$). Hereafter the same symbol J will denote, depending on the context, the same reflection matrix for different values of n.

Let $X \in M_n(R)$ be the matrix in (2.1). Let (3.1) be satisfied, and assume that each b_i has inverse in R. X can be written as sum of two matrices:

$$X = \begin{pmatrix} r_{11} & b_1 & & \\ \vdots & & \ddots & \\ \vdots & & & b_{n-1} \\ r_{n1} & \cdots & \cdots & r_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} r_{11} & b_1 & & \\ \vdots & & \ddots & \\ \vdots & & & b_{n-1} \\ \beta & \cdots & \cdots & r_{nn} \end{pmatrix} + (r_{n1} - \beta)\mathbf{e}_n\mathbf{e}_1^T = X' + (r_{n1} - \beta)\mathbf{e}_n\mathbf{e}_1^T,$$

where $\beta \in R$.

THEOREM 3.1.

(i) Let X be persymmetric, $\beta r = r\beta \ \forall r \in R$, and $\beta \neq r_{n1}$. Then the equalities $\mathfrak{C}_X(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T$ and $\mathfrak{C}_{X^T}(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T$ imply, respectively,

$$(r_{n1} - \beta) A = - \sum_{m=1}^{\alpha} H_X(\hat{\mathbf{x}}_m) H_{X'}(\mathbf{y}_m) + (r_{n1} - \beta) H_X(\widehat{A\mathbf{e}_n}) \qquad \square$$

$$(3.3)$$

$$= \sum_{m=1}^{\alpha} H_{X'}(\hat{\mathbf{x}}_m) H_X(\mathbf{y}_m) + (r_{n1} - \beta) H_X(A^T\mathbf{e}_1) \qquad \square$$

$$(3.4)$$

and

$$(r_{n1} - \beta) A = \sum_{m=1}^{\alpha} H_{X'}(\mathbf{x}_m)^T H_X(\hat{\mathbf{y}}_m)^T + (r_{n1} - \beta) H_X(JA^T\mathbf{e}_n)^T \qquad \square$$

$$(3.5)$$

$$= - \sum_{m=1}^{\alpha} H_X(\mathbf{x}_m)^T H_{X'}(\hat{\mathbf{y}}_m)^T + (r_{n1} - \beta) H_X(A\mathbf{e}_1)^T. \qquad \square$$

$$(3.6)$$

(ii) Let X have Toeplitz structure, and let $b = b_i$, $i = 1, \ldots, n$. Then the equalities $\mathfrak{C}_X(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T$ and $\mathfrak{C}_{X^T}(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T$ imply, respectively,

$$bA = \sum_{m=1}^{\alpha} H_X(\hat{\mathbf{x}}_m) L(Z\hat{\mathbf{y}}_m) + bH_X(\widehat{A\mathbf{e}_n}) \qquad \square \qquad (3.7)$$

$$= - \sum_{m=1}^{\alpha} L(Z\mathbf{x}_m) H_X(\mathbf{y}_m) + bH_X(A^T\mathbf{e}_1) \qquad \square \qquad (3.8)$$

and

$$bA = - \sum_{m=1}^{\alpha} U(Z\hat{\mathbf{x}}_m) H_X(\hat{\mathbf{y}}_m)^T + bH_X(JA^T\mathbf{e}_n)^T \qquad \square \qquad (3.9)$$

$$= \sum_{m=1}^{\alpha} H_X(\mathbf{x}_m)^T U(Z\mathbf{y}_m) + bH_X(A\mathbf{e}_1)^T, \qquad \square \qquad (3.10)$$

where $L(\mathbf{z})$ $[U(\mathbf{z})]$ denotes $H_{Z^T}(\mathbf{z})^T$ $[H_{Z^T}(\mathbf{z})]$, that is, the lower [upper] triangular Toeplitz matrix with first column [row] $\mathbf{z}$ $[\mathbf{z}^T]$.

Proof. Assume $\mathfrak{C}_X(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T$. The main steps of the proofs of cases (i) and (ii) are equal. Thus we consider them in parallel, keeping in mind that X is persymmetric in both cases and that X' is persymmetric too. In the following we use the commutativity of X $[X']$ with $H_X(\mathbf{z})$ $[H_{X'}(\mathbf{z})]$.

(i) $\square$:

$$
\mathfrak{C}_X\!\left(- \sum_{m=1}^{\alpha} H_X(\hat{\mathbf{x}}_m) H_{X'}(\mathbf{y}_m) \right)
$$

$$
= - \sum_{m=1}^{\alpha} H_X(\hat{\mathbf{x}}_m)\left[H_{X'}(\mathbf{y}_m) X - X H_{X'}(\mathbf{y}_m) \right]
$$

$$
= -(r_{n1} - \beta) \sum_{m=1}^{\alpha} H_X(\hat{\mathbf{x}}_m)\left[H_{X'}(\mathbf{y}_m)\mathbf{e}_n \mathbf{e}_1^T - \mathbf{e}_n \mathbf{e}_1^T H_{X'}(\mathbf{y}_m) \right]
$$

$$
= (r_{n1} - \beta) \sum_{m=1}^{\alpha} H_X(\hat{\mathbf{x}}_m)\left(-\hat{\mathbf{y}}_m \mathbf{e}_1^T + \mathbf{e}_n \mathbf{y}_m^T\right) = (r_{n1} - \beta) \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T.
$$

(ii) $\square$:

$$
\mathfrak{C}_X\!\left(\sum_{m=1}^{\alpha} H_X(\hat{\mathbf{x}}_m) L(Z\hat{\mathbf{y}}_m) \right)
$$

$$
= \sum_{m=1}^{\alpha} H_X(\hat{\mathbf{x}}_m)\left[L(Z\hat{\mathbf{y}}_m) X - X L(Z\hat{\mathbf{y}}_m) \right]
$$

$$
= b \sum_{m=1}^{\alpha} H_X(\hat{\mathbf{x}}_m)\left[L(Z\hat{\mathbf{y}}_m)Z^T - Z^T L(Z\hat{\mathbf{y}}_m) \right]
$$

$$
= b \sum_{m=1}^{\alpha} H_X(\hat{\mathbf{x}}_m)\left(-\hat{\mathbf{y}}_m \mathbf{e}_1^T + \mathbf{e}_n \mathbf{y}_m^T\right) = b \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T.
$$

The last equality, in both cases (i) and (ii), follows from the following relation, which holds, for $i = 1, \dots, n$, by Propositions 2.3 and 3.1:

$$
\mathbf{e}_i^T \sum_{m=1}^{\alpha} H_X(\hat{\mathbf{x}}_m)\hat{\mathbf{y}}_m = \mathbf{e}_i^T \sum_{m=1}^{\alpha} \sum_{k=1}^{n} x_{n+1-k}^{(m)} X_k \hat{\mathbf{y}}_m
$$

$$
= \sum_{m=1}^{\alpha} \hat{\mathbf{x}}_m^T X_i \hat{\mathbf{y}}_m = \sum_{m=1}^{\alpha} \mathbf{x}_m^T X_i^T \mathbf{y}_m = 0.
$$

Thus, in case (i) [(ii)], the argument of $\mathfrak{C}_X$, whose last column is null, is obtained by subtracting from $(r_{n1} - \beta)A$ [from bA] a matrix $H_X(\mathbf{z})$, where $\mathbf{z}$ is defined by the equality $H_X(\mathbf{z})\mathbf{e}_n = (r_{n1} - \beta)A\mathbf{e}_n$ [$H_X(\mathbf{z})\mathbf{e}_n = bA\mathbf{e}_n$], i.e. $\mathbf{z} = (r_{n1} - \beta)JA\mathbf{e}_n$ [$\mathbf{z} = bJA\mathbf{e}_n$].

(i) $\square$:

$$\mathfrak{C}_X\left(\sum_{m=1}^{\alpha} H_{X'}(\hat{\mathbf{x}}_m) H_X(\mathbf{y}_m) \right)$$

$$= \sum_{m=1}^{\alpha} \left[H_{X'}(\hat{\mathbf{x}}_m) X - X H_{X'}(\hat{\mathbf{x}}_m) \right] H_X(\mathbf{y}_m)$$

$$= (r_{n1} - \beta) \sum_{m=1}^{\alpha} \left[H_{X'}(\hat{\mathbf{x}}_m)\mathbf{e}_n\mathbf{e}_1^T - \mathbf{e}_n\mathbf{e}_1^T H_{X'}(\hat{\mathbf{x}}_m) \right] H_X(\mathbf{y}_m)$$

$$= (r_{n1} - \beta) \sum_{m=1}^{\alpha} \left(\mathbf{x}_m\mathbf{e}_1^T - \mathbf{e}_n\hat{\mathbf{x}}_m^T \right) H_X(\mathbf{y}_m) = (r_{n1} - \beta) \sum_{m=1}^{\alpha} \mathbf{x}_m\mathbf{y}_m^T.$$

(ii) $\square$:

$$\mathfrak{C}_X\left(- \sum_{m=1}^{\alpha} L(Z\mathbf{x}_m) H_X(\mathbf{y}_m) \right)$$

$$= - \sum_{m=1}^{\alpha} \left[L(Z\mathbf{x}_m) X - X L(Z\mathbf{x}_m) \right] H_X(\mathbf{y}_m)$$

$$= -b \sum_{m=1}^{\alpha} \left[L(Z\mathbf{x}_m) Z^T - Z^T L(Z\mathbf{x}_m) \right] H_X(\mathbf{y}_m)$$

$$= b \sum_{m=1}^{\alpha} \left(\mathbf{x}_m\mathbf{e}_1^T - \mathbf{e}_n\hat{\mathbf{x}}_m^T \right) H_X(\mathbf{y}_m) = b \sum_{m=1}^{\alpha} \mathbf{x}_m\mathbf{y}_m^T.$$

The last equality, in both cases (i) and (ii), follows from the following relation, which holds, for $i = 1, \ldots, n$, by Propositions 2.3 and 3.1:

$$\sum_{m=1}^{\alpha} \hat{\mathbf{x}}_m^T H_X(\mathbf{y}_m)\mathbf{e}_{n+1-i} = \sum_{m=1}^{\alpha} \mathbf{x}_m^T \sum_{k=1}^{n} y_k^{(m)} J X_k \mathbf{e}_{n+1-i}$$

$$= \sum_{m=1}^{\alpha} \mathbf{x}_m^T \sum_{k=1}^{n} y_k^{(m)} X_k^T \mathbf{e}_i$$

$$= \sum_{m=1}^{\alpha} \mathbf{x}_m^T X_i^T \mathbf{y}_m = 0.$$

Thus, in case (i) [(ii)], the argument of $\mathfrak{C}_X$ is obtained by subtracting from $(r_{n1} - \beta)A$ $[bA]$ a matrix $H_X(\mathbf{z})$, and since its first row is null, $\mathbf{z} = (r_{n1} - \beta)A^T\mathbf{e}_1$ $[\mathbf{z} = bA^T\mathbf{e}_1]$.

Now assume $\mathfrak{C}_{X^T}(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m\mathbf{y}_m^T$. Then (i) [(ii)] ☐ and (i) [(ii)] ☐ follow, respectively, from (i) [(ii)] ☐ and (i) [(ii)] ☐ and from the equality $\mathfrak{C}_X(A^T) = -\mathfrak{C}_{X^T}(A)^T$. ∎

Now, by exploiting Theorem 3.1(ii) for $X = Z^T$, $X = P$, and $X = T_2$, respectively, and Theorem 3.1(i) for $X = P_\varepsilon = P_\beta + (\varepsilon - \beta)\mathbf{e}_n\mathbf{e}_1^T$, we regain some known results, which are listed in the following corollary.

Let $C(\mathbf{z})$, $\tau(\mathbf{z})$, and $C_\varepsilon(\mathbf{z})$ denote, respectively, the circulant matrix, the τ matrix, and the ε-circulant matrix whose first row is $\mathbf{z}^T$.

COROLLARY 3.1.

(i) *The Gohberg-Semencul* [17] *or Kailath et al.* [22] *formulas hold:*

$$\mathfrak{C}_{Z^T}(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m\mathbf{y}_m^T \quad \Rightarrow \quad A = \sum_{m=1}^{\alpha} U(\hat{\mathbf{x}}_m)L(Z\hat{\mathbf{y}}_m) + U(\widehat{A\mathbf{e}_n}) \quad (3.11)$$

$$= -\sum_{m=1}^{\alpha} L(Z\mathbf{x}_m)U(\mathbf{y}_m) + U(A^T\mathbf{e}_1); \quad (3.12)$$

$$\mathfrak{C}_Z(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m\mathbf{y}_m^T \quad \Rightarrow \quad A = -\sum_{m=1}^{\alpha} U(Z\hat{\mathbf{x}}_m)L(\hat{\mathbf{y}}_m) + L(JA^T\mathbf{e}_n)$$

$$(3.13)$$

$$= \sum_{m=1}^{\alpha} L(\mathbf{x}_m)U(Z\mathbf{y}_m) + L(A\mathbf{e}_1). \quad (3.14)$$

(ii) *The Gader* [14] *formulas hold:*

$$\mathfrak{C}_P(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m\mathbf{y}_m^T \quad \Rightarrow \quad A = \sum_{m=1}^{\alpha} C(\hat{\mathbf{x}}_m)L(Z\hat{\mathbf{y}}_m) + C(\widehat{A\mathbf{e}_n}) \quad (3.15)$$

$$= -\sum_{m=1}^{\alpha} L(Z\mathbf{x}_m)C(\mathbf{y}_m) + C(A^T\mathbf{e}_1); \quad (3.16)$$

$$\mathfrak{C}_{P^T}(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m\mathbf{y}_m^T \quad \Rightarrow \quad A = -\sum_{m=1}^{\alpha} U(Z\hat{\mathbf{x}}_m)C(\hat{\mathbf{y}}_m)^T + C(JA^T\mathbf{e}_n)^T$$

$$(3.17)$$

$$= \sum_{m=1}^{\alpha} C(\mathbf{x}_m)^T U(Z\mathbf{y}_m) + C(A\mathbf{e}_1)^T. \quad (3.18)$$

(iii) *The Bini-Pan* [10] *formulas hold*:

$$\mathfrak{S}_{T_2}(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T \quad \Rightarrow \quad A = \sum_{m=1}^{\alpha} \tau(\hat{\mathbf{x}}_m) L(Z\hat{\mathbf{y}}_m) + \tau\left(\widehat{A\mathbf{e}_n}\right) \quad (3.19)$$

$$= -\sum_{m=1}^{\alpha} L(Z\mathbf{x}_m)\tau(\mathbf{y}_m) + \tau\left(A^T\mathbf{e}_1\right) \quad (3.20)$$

$$= -\sum_{m=1}^{\alpha} U(Z\hat{\mathbf{x}}_m)\tau(\hat{\mathbf{y}}_m) + \tau\left(JA^T\mathbf{e}_n\right) \quad (3.21)$$

$$= \sum_{m=1}^{\alpha} \tau(\mathbf{x}_m)U(Z\mathbf{y}_m) + \tau(A\mathbf{e}_1). \quad (3.22)$$

(iv) *The Gohberg-Olshevsky* [16] *formulas hold*:

$$\mathfrak{S}_{P_\varepsilon}(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T \quad \Rightarrow \quad (\varepsilon - \beta)A$$

$$= -\sum_{m=1}^{\alpha} C_\varepsilon(\hat{\mathbf{x}}_m)C_\beta(\mathbf{y}_m) + (\varepsilon - \beta)C_\varepsilon\left(\widehat{A\mathbf{e}_n}\right) \quad (3.23)$$

$$= \sum_{m=1}^{\alpha} C_\beta(\hat{\mathbf{x}}_m)C_\varepsilon(\mathbf{y}_m) + (\varepsilon - \beta)C_\varepsilon\left(A^T\mathbf{e}_1\right); \quad (3.24)$$

$$\mathfrak{S}_{P_\varepsilon^T}(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T \quad \Rightarrow \quad (\varepsilon - \beta)A$$

$$= \sum_{m=1}^{\alpha} C_\beta(\mathbf{x}_m)^T C_\varepsilon(\hat{\mathbf{y}}_m)^T + (\varepsilon - \beta)C_\varepsilon\left(JA^T\mathbf{e}_n\right)^T \quad (3.25)$$

$$= -\sum_{m=1}^{\alpha} C_\varepsilon(\mathbf{x}_m)^T C_\beta(\hat{\mathbf{y}}_m)^T + (\varepsilon - \beta)C_\varepsilon(A\mathbf{e}_1)^T. \quad (3.26)$$

Define the following matrices (I_k is the identity matrix of dimension $k \times k$):

$$\Omega_1 = [0 \mid I_{n-1}], \qquad \text{dimension } (n-1) \times n,$$

$$\Omega_2 = [0 \mid I_{n-2} \mid 0], \qquad \text{dimension } (n-2) \times n,$$

$$\Omega_3 = [I_{n-1} \mid 0], \qquad \text{dimension } (n-1) \times n.$$

The matrices Ω_1, Ω_2, and Ω_3, applied to $\mathbf{z} = [z_1 \cdots z_n]^T$, give, respectively, $[z_2 \cdots z_n]^T$, $[z_2 \cdots z_{n-1}]^T$, and $[z_1 \cdots z_{n-1}]^T$.

In the next two theorems assume that X is symmetric. More explicitly, set

$$
X = \begin{pmatrix}
a_1 & b_1 & 0 & \cdots & & 0 \\
b_1 & a_2 & b_2 & \ddots & & \vdots \\
0 & b_2 & \ddots & \ddots & & 0 \\
\vdots & \ddots & \ddots & \ddots & & b_{n-1} \\
0 & \cdots & & 0 & b_{n-1} & a_n
\end{pmatrix}.
\tag{3.27}
$$

Assume that each b_i has inverse in R.

THEOREM 3.2. If $\mathfrak{C}_X(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T$ (X defined by (3.27)), then

$$
b_1 A = \sum_{m=1}^{\alpha} \left(\begin{array}{c|c} x_1^{(m)} & \mathbf{x}_m^T \Omega_1^T \\ \hline 0 & H_{X'}(\Omega_1 \mathbf{x}_m) \end{array} \right) H_X(\mathbf{y}_m) + b_1 H_X(A^T \mathbf{e}_1) \qquad \square
\tag{3.28}
$$

$$
= - \sum_{m=1}^{\alpha} H_X(\mathbf{x}_m) \left(\begin{array}{c|c} y_1^{(m)} & 0 \\ \hline \Omega_1 \mathbf{y}_m & H_{X'}(\Omega_1 \mathbf{y}_m) \end{array} \right) + b_1 H_X(A \mathbf{e}_1), \qquad \square
\tag{3.29}
$$

$$
b_{n-1} A = \sum_{m=1}^{\alpha} \left(\begin{array}{c|c} H_{X''}(X_{n-1}''^{-1} \Omega_3 \mathbf{x}_m) & 0 \\ \hline \mathbf{x}_m^T \Omega_3^T & x_n^{(m)} \end{array} \right) H_X(X_n^{-1} \mathbf{y}_m)
$$

$$
+ b_{n-1} H_X(X_n^{-1} A^T \mathbf{e}_n) \qquad \square
\tag{3.30}
$$

$$
= - \sum_{m=1}^{\alpha} H_X(X_n^{-1} \mathbf{x}_m) \left(\begin{array}{c|c} H_{X''}(X_{n-1}''^{-1} \Omega_3 \mathbf{y}_m) & \Omega_3 \mathbf{y}_m \\ \hline 0 & y_n^{(m)} \end{array} \right)
$$

$$
+ b_{n-1} H_X(X_n^{-1} A \mathbf{e}_n). \qquad \square
\tag{3.31}
$$

where X' and X'' are, respectively, the $(n-1) \times (n-1)$ lower right and upper left submatrices of X.

Proof. $\square$:

$$\mathfrak{C}_X\left(\sum_{m=1}^{\alpha} \left(\begin{array}{c|c} x_1^{(m)} & \mathbf{x}_m^T \Omega_1^T \\ \hline 0 & H_{X'}(\Omega_1 \mathbf{x}_m) \end{array} \right) H_X(\mathbf{y}_m) \right)$$

$$= \sum_{m=1}^{\alpha} \left[\left(\begin{array}{c|c} x_1^{(m)} & \mathbf{x}_m^T \Omega_1^T \\ \hline 0 & H_{X'}(\Omega_1 \mathbf{x}_m) \end{array} \right) X - X \left(\begin{array}{c|c} x_1^{(m)} & \mathbf{x}_m^T \Omega_1^T \\ \hline 0 & H_{X'}(\Omega_1 \mathbf{x}_m) \end{array} \right) \right] H_X(\mathbf{y}_m)$$

$$= \sum_{m=1}^{\alpha} \left(\begin{array}{c} \mathbf{x}_m^T X - (a_1 + b_1)\mathbf{x}_m^T + b_1 x_1^{(m)} \mathbf{e}_1^T \\ \hline \left(b_1 \Omega_1 \mathbf{x}_m \mid H_{X'}(\Omega_1 \mathbf{x}_m) X' \right) - \left(\begin{array}{c} b_1 \mathbf{x}_m^T \\ 0 \end{array} \right) - \\ \left(0 \mid X' H_{X'}(\Omega_1 \mathbf{x}_m) \right) \end{array} \right) H_X(\mathbf{y}_m)$$

$$= \sum_{m=1}^{\alpha} \left(\begin{array}{c} \mathbf{x}_m^T X - (a_1 + b_1)\mathbf{x}_m^T + b_1 x_1^{(m)} \mathbf{e}_1^T \\ b_1 x_2^{(m)} \mathbf{e}_1^T - b_1 \mathbf{x}_m^T \\ b_1 x_3^{(m)} \mathbf{e}_1^T \\ \vdots \\ b_1 x_n^{(m)} \mathbf{e}_1^T \end{array} \right) H_X(\mathbf{y}_m) = b_1 \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T .$$

The last equality follows from the following relations, which hold, for $i = 1, \ldots, n$, by Propositions 2.3 and 3.1:

$$\left(\sum_{m=1}^{\alpha} \mathbf{x}_m^T X H_X(\mathbf{y}_m) \right) \mathbf{e}_i = \left(\sum_{m=1}^{\alpha} \mathbf{x}_m^T X \sum_{k=1}^{n} y_k^{(m)} X_k \right) \mathbf{e}_i = \sum_{m=1}^{\alpha} \mathbf{x}_m^T X X_i \mathbf{y}_m = 0,$$

$$\left(\sum_{m=1}^{\alpha} \mathbf{x}_m^T H_X(\mathbf{y}_m) \right) \mathbf{e}_i = \left(\sum_{m=1}^{\alpha} \mathbf{x}_m^T \sum_{k=1}^{n} y_k^{(m)} X_k \right) \mathbf{e}_i = \sum_{m=1}^{\alpha} \mathbf{x}_m^T X_i \mathbf{y}_m = 0.$$

Thus the argument of $\mathfrak{C}_X$ is obtained by subtracting from $b_1 A$ a matrix $H_X(\mathbf{z})$, and its first row is null, that is, $\mathbf{z} = b_1 A^T \mathbf{e}_1$.

$\boxed{}$: By Proposition 2.3 we have $X_i X_2 = \sum_{k=1}^{n} [X_2]_{ik} X_k$. Exploiting this equality for $i = n, n - 1, \ldots, 2$, we obtain $X_n Q_{n+1-i} = X_{i-1}$, $i = n, n - 1, \ldots, 2$, where the Q_i's are the polynomials in X defined as follows:

$$Q_0 = I, \qquad Q_1 = b_{n-1}^{-1} b_1 [X_2 - b_1^{-1}(a_n - a_1)I],$$

$$Q_{i+1} = b_{n-i-1}^{-1} b_1 \{ Q_i [X_2 - b_1^{-1}(a_{n-i} - a_1)I] - b_1^{-1} b_{n-i} Q_{i-1} \},$$

$$i = 1, \ldots, n - 2.$$

In particular $X_n Q_{n-1} = X_1 = I$ and $X_n^{-1} = Q_{n-1}$, i.e., X_n has an inverse in $M_n(R)$, and X_n^{-1} is a polynomial in X whose coefficients commute with all elements of R. Thus we can say that $\forall \mathbf{y} \in R^n$ there always exists $\mathbf{z}$ such that $\mathbf{e}_n^T H_X(\mathbf{z}) = \mathbf{y}^T$ and we have $\mathbf{z} = X_n^{-1}\mathbf{y}$. The same assertions hold for X_{n-1}'' and $H_{X''}$. Thus we have

$$\mathfrak{C}_X \left(\sum_{m=1}^{\alpha} \left(\begin{array}{c|c} H_{X''}(X_{n-1}''^{-1} \Omega_3 \mathbf{x}_m) & 0 \\ \hline \mathbf{x}_m^T \Omega_3^T & x_n^{(m)} \end{array} \right) H_X(X_n^{-1} \mathbf{y}_m) \right)$$

$$= \sum_{m=1}^{\alpha} \left[\left(\begin{array}{c|c} H_{X''}(X_{n-1}''^{-1} \Omega_3 \mathbf{x}_m) & 0 \\ \hline \mathbf{x}_m^T \Omega_3^T & x_n^{(m)} \end{array} \right) X \right.$$

$$\left. - X \left(\begin{array}{c|c} H_{X''}(X_{n-1}''^{-1} \Omega_3 \mathbf{x}_m) & 0 \\ \hline \mathbf{x}_m^T \Omega_3^T & x_n^{(m)} \end{array} \right) \right] H_X(X_n^{-1} \mathbf{y}_m)$$

$$= \sum_{m=1}^{\alpha} \left(\begin{array}{c} \left(H_{X''}(\cdots) X'' \mid b_{n-1} \Omega_3 \mathbf{x}_m \right) \\ -\begin{pmatrix} 0 \\ b_{n-1} \mathbf{x}_m^T \end{pmatrix} - \left(X'' H_{X''}(\cdots) \mid 0 \right) \\ \hline \mathbf{x}_m^T X - (b_{n-1} + a_n)\mathbf{x}_m^T + b_{n-1} x_n^{(m)} \mathbf{e}_n^T \end{array} \right) H_X(X_n^{-1} \mathbf{y}_m)$$

$$= \sum_{m=1}^{\alpha} \{ b_{n-1}(\mathbf{x}_m \mathbf{e}_n^T - \mathbf{e}_{n-1} \mathbf{x}_m^T) + \mathbf{e}_n \mathbf{x}_m^T [X - (b_{n-1} + a_n)I] \}$$

$$\times H_X(X_n^{-1} \mathbf{y}_m)$$

$$= b_{n-1} \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T.$$

The last equality follows from the following relations, which hold, for $i = 1, \ldots, n$, by Propositions 2.3 and 3.1:

$$\sum_{m=1}^{\alpha} \mathbf{x}_m^T \sum_{k=1}^{n} \left(X_n^{-1} \mathbf{y}_m \right)_k X_k \mathbf{e}_i = \sum_{m=1}^{\alpha} \mathbf{x}_m^T X_i X_n^{-1} \mathbf{y}_m = 0,$$

$$\sum_{m=1}^{\alpha} \mathbf{x}_m^T X \sum_{k=1}^{n} \left(X_n^{-1} \mathbf{y}_m \right)_k X_k \mathbf{e}_i = \sum_{m=1}^{\alpha} \mathbf{x}_m^T X X_i X_n^{-1} \mathbf{y}_m = 0.$$

Thus the argument of $\mathfrak{C}_X$ is obtained by subtracting from $b_{n-1} A$ a matrix $H_X(\mathbf{z})$, and its last row is null, that is, $\mathbf{z} = b_{n-1} X_n^{-1} A^T \mathbf{e}_n$.
⬜ and ⬜ follow, respectively, from ⬜ and ⬜ and the equality $\mathfrak{C}_X(A^T) = -\mathfrak{C}_X(A)^T$. ∎
In the following theorem we assume that the matrix X is centrosymmetric.

THEOREM 3.3.   *Let X defined in (3.27) be centrosymmetric. If $\mathfrak{C}_X(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T$, then*

$$b_1(A + JAJ) = \sum_{m=1}^{\alpha} \begin{pmatrix} x_1^{(m)} & \mathbf{x}_m^T \Omega_2^T & x_n^{(m)} \\ \hline 0 & H_{X'}(\Omega_2 \mathbf{x}_m) & 0 \\ \hline x_n^{(m)} & \mathbf{x}_m^T \Omega_2^T J & x_1^{(m)} \end{pmatrix} H_X(\mathbf{y}_m)$$

$$+ b_1 H_X\left((A + JAJ)^T \mathbf{e}_1 \right) \quad \square \qquad (3.32)$$

$$= -\sum_{m=1}^{\alpha} H_X(\mathbf{x}_m) \begin{pmatrix} y_1^{(m)} & 0 & y_n^{(m)} \\ \hline \Omega_2 \mathbf{y}_m & H_{X'}(\Omega_2 \mathbf{y}_m) & J\Omega_2 \mathbf{y}_m \\ \hline y_n^{(m)} & 0 & y_1^{(m)} \end{pmatrix}$$

$$+ b_1 H_X((A + JAJ)\mathbf{e}_1), \quad \square \qquad (3.33)$$

where X' is the $(n - 2) \times (n - 2)$ principal submatrix obtained from X by deleting the first and last rows and columns.

Proof. $\boxed{}$:

$$\mathfrak{C}_X\left(\sum_{m=1}^{\alpha} \left(\begin{array}{c|c|c} x_1^{(m)} & \mathbf{x}_m^T \Omega_2^T & x_n^{(m)} \\ \hline 0 & H_{X'}(\Omega_2 \mathbf{x}_m) & 0 \\ \hline x_n^{(m)} & \mathbf{x}_m^T \Omega_2^T J & x_1^{(m)} \end{array}\right) H_X(\mathbf{y}_m) \right)$$

$$= \sum_{m=1}^{\alpha} \left[\left(\begin{array}{c|c|c} x_1^{(m)} & \mathbf{x}_m^T \Omega_2^T & x_n^{(m)} \\ \hline 0 & H_{X'}(\Omega_2 \mathbf{x}_m) & 0 \\ \hline x_n^{(m)} & \mathbf{x}_m^T \Omega_2^T J & x_1^{(m)} \end{array}\right) X \right.$$

$$\left. - X \left(\begin{array}{c|c|c} x_1^{(m)} & \mathbf{x}_m^T \Omega_2^T & x_n^{(m)} \\ \hline 0 & H_{X'}(\Omega_2 \mathbf{x}_m) & 0 \\ \hline x_n^{(m)} & \mathbf{x}_m^T \Omega_2^T J & x_1^{(m)} \end{array}\right) \right] H_X(\mathbf{y}_m)$$

$$= \sum_{m=1}^{\alpha} \left(\begin{array}{c} \mathbf{x}_m^T X - (a_1 + b_1)\mathbf{x}_m^T + b_1 x_1^{(m)}\mathbf{e}_1^T + b_1 x_n^{(m)}\mathbf{e}_n^T \\ \hline \left(b_1 \Omega_2 \mathbf{x}_m \mid H_{X'} X' \mid b_{n-1}\widehat{\Omega_2 \mathbf{x}_m} \right) - \begin{pmatrix} b_1 \mathbf{x}_m^T \\ 0 \end{pmatrix} \\ - \left(0 \mid X' H_{X'} \mid 0 \right) - \begin{pmatrix} 0 \\ b_{n-1}\hat{\mathbf{x}}_m^T \end{pmatrix} \\ \hline \hat{\mathbf{x}}_m^T X - (b_{n-1} + a_n)\hat{\mathbf{x}}_m^T + b_{n-1} x_n^{(m)}\mathbf{e}_1^T + b_{n-1} x_1^{(m)}\mathbf{e}_n^T \end{array} \right) H_X(\mathbf{y}_m)$$

$$= \sum_{m=1}^{\alpha} \left(\begin{array}{c} \mathbf{x}_m^T X - (a_1 + b_1)\mathbf{x}_m^T + b_1\left(x_1^{(m)}\mathbf{e}_1^T + x_n^{(m)}\mathbf{e}_n^T \right) \\ b_1 x_2^{(m)}\mathbf{e}_1^T + b_{n-1} x_{n-1}^{(m)}\mathbf{e}_n^T - b_1 \mathbf{x}_m^T \\ b_1 x_3^{(m)}\mathbf{e}_1^T + b_{n-1} x_{n-2}^{(m)}\mathbf{e}_n^T \\ \vdots \\ b_1 x_{n-2}^{(m)}\mathbf{e}_1^T + b_{n-1} x_3^{(m)}\mathbf{e}_n^T \\ b_1 x_{n-1}^{(m)}\mathbf{e}_1^T + b_{n-1} x_2^{(m)}\mathbf{e}_n^T - b_{n-1}\hat{\mathbf{x}}_m^T \\ \hat{\mathbf{x}}_m^T X - (a_n + b_{n-1})\hat{\mathbf{x}}_m^T + b_{n-1}\left(x_n^{(m)}\mathbf{e}_1^T + x_1^{(m)}\mathbf{e}_n^T \right) \end{array} \right) H_X(\mathbf{y}_m)$$

$$= b_1 \sum_{m=1}^{\alpha} \left(\mathbf{x}_m \mathbf{y}_m^T + \hat{\mathbf{x}}_m \hat{\mathbf{y}}_m^T \right)$$

$$= b_1 \mathfrak{C}_X(A + JAJ).$$

The last equality but one follows from the relations obtained in the proof of Theorem 3.2 and from the following other relations, which hold, for $i = 1, \ldots, n$, by Proposition 2.3, Corollary 2.1, and Proposition 3.1:

$$\left(\sum_{m=1}^{\alpha} \hat{\mathbf{x}}_m^T H_X(\mathbf{y}_m) \right) \mathbf{e}_i = \left(\sum_{m=1}^{\alpha} \hat{\mathbf{x}}_m^T \sum_{k=1}^{n} y_k^{(m)} X_k \right) \mathbf{e}_i = \sum_{m=1}^{\alpha} \mathbf{x}_m^T J X_i \mathbf{y}_m = 0,$$

$$\left(\sum_{m=1}^{\alpha} \hat{\mathbf{x}}_m^T X H_X(\mathbf{y}_m) \right) \mathbf{e}_i = \left(\sum_{m=1}^{\alpha} \hat{\mathbf{x}}_m^T X \sum_{k=1}^{n} y_k^{(m)} X_k \right) \mathbf{e}_i = \sum_{m=1}^{\alpha} \mathbf{x}_m^T J X X_i \mathbf{y}_m = 0.$$

Thus the argument of $\mathfrak{C}_X$ is obtained by subtracting from $b_1(A + JAJ)$ a matrix $H_X(\mathbf{z})$, and its first row is null, that is, $\mathbf{z} = b_1(A + JAJ)^T \mathbf{e}_1$.
$\square$ follows from $\square$ and the equality $\mathfrak{C}_X(A^T) = -\mathfrak{C}_X(A)^T$. ∎

Theorems 3.2 and 3.3 allow us, in principle, to split A in terms of symmetric HAs in many different ways (i.e. for all possible choices of H_X). The above theorems and all consequent formulas for A are new. These formulas differ from those of Theorem 3.1 in that they are constructed, in several important cases, in terms of the *same* HA. The most interesting cases —from a computational point of view—are those in which the eigenvalues and the eigenvectors of H_X are known in explicit form. These are the cases of the algebras τ and Γ defined in Section 2. In the following only the case of the algebra τ is considered in detail, as the related formulas for A a Toeplitz or Toeplitz plus Hankel matrix (Section 4) are the most simple and convenient. Then consider Theorems 3.2 and 3.3 for $X = T_2$; set

$$\tau^1(\mathbf{z}) = \left(\begin{array}{c|c} z_1 & \mathbf{z}^T \Omega_1^T \\ \hline 0 & \tau(\Omega_1 \mathbf{z}) \end{array} \right),$$

$$\tau^2(\mathbf{z}) = \left(\begin{array}{c|c|c} z_1 & \mathbf{z}^T \Omega_2^T & z_n \\ \hline 0 & \tau(\Omega_2 \mathbf{z}) & 0 \\ \hline z_n & \mathbf{z}^T \Omega_2^T J & z_1 \end{array} \right),$$

$$\tau^3(\mathbf{z}) = \left(\begin{array}{c|c} \tau(J\Omega_3 \mathbf{z}) & 0 \\ \hline \mathbf{z}^T \Omega_3^T & z_n \end{array} \right).$$

COROLLARY 3.2. *If* $\mathfrak{S}_{T_2}(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T$, *then*

$$A = \sum_{m=1}^{\alpha} \tau^1(\mathbf{x}_m)\tau(\mathbf{y}_m) + \tau(A^T \mathbf{e}_1) \tag{3.34}$$

$$= -\sum_{m=1}^{\alpha} \tau(\mathbf{x}_m)\tau^1(\mathbf{y}_m)^T + \tau(A\mathbf{e}_1) \tag{3.35}$$

$$= \sum_{m=1}^{\alpha} \tau^3(\mathbf{x}_m)\tau(\hat{\mathbf{y}}_m) + \tau(JA^T \mathbf{e}_n) \tag{3.36}$$

$$= -\sum_{m=1}^{\alpha} \tau(\hat{\mathbf{x}}_m)\tau^3(\mathbf{y}_m)^T + \tau(JA\mathbf{e}_n). \tag{3.37}$$

COROLLARY 3.3. *If* $\mathfrak{S}_{T_2}(A) = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T$, *then*

$$A + JAJ = \sum_{m=1}^{\alpha} \tau^2(\mathbf{x}_m)\tau(\mathbf{y}_m) + \tau\big((A + JAJ)^T \mathbf{e}_1\big) \tag{3.38}$$

$$= -\sum_{m=1}^{\alpha} \tau(\mathbf{x}_m)\tau^2(\mathbf{y}_m)^T + \tau\big((A + JAJ)\mathbf{e}_1\big). \tag{3.39}$$

4. APPLICATIONS TO TOEPLITZ AND TOEPLITZ PLUS HANKEL MATRICES

Let $T = (t_{i-j})$ and $H = (h_{i+j-2})$, $i, j = 1, \ldots, n$, denote, respectively, a Toeplitz matrix and a Hankel matrix of dimension $n \times n$ with elements in $\mathbb{C}$.

Let $\mathbf{s}_{\cdot i}$ and $\mathbf{s}_{i \cdot}$ denote, respectively, the ith column and the ith row of $S = T^{-1}$, that is $T\mathbf{s}_{\cdot i} = \mathbf{e}_i$ and $\mathbf{e}_i^T = \mathbf{s}_{i \cdot}^T T$. Let $\mathbf{w}_{\cdot i}$ and $\mathbf{w}_{i \cdot}$ denote, respectively, the ith column and the ith row of $W = (T + H)^{-1}$, that is, $(T + H)\mathbf{w}_{\cdot i} = \mathbf{e}_i$ and $\mathbf{e}_i^T = \mathbf{w}_{i \cdot}^T (T + H)$. Throughout this section T and $T + H$ are assumed to be nonsingular matrices. Set

$$\mathbf{a}(t_{-n}) = \begin{bmatrix} t_{-n} & t_{-n+1} & \cdots & t_{-1} \end{bmatrix}^T,$$

$$\mathbf{b}(t_n) = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}^T,$$

$$\mathbf{c}(h_{-1}) = \begin{bmatrix} h_{-1} & h_0 & \cdots & h_{n-2} \end{bmatrix}^T,$$

$$\mathbf{d}(h_{2n-1}) = \begin{bmatrix} h_n & h_{n+1} & \cdots & h_{2n-1} \end{bmatrix}^T.$$

PROPOSITION 4.1.   *For every t_n and t_{-n} in $\mathbb{C}$ we have*

(i) $\mathfrak{C}_Z(T^{-1}) = T^{-1}Z - ZT^{-1} = \boldsymbol{\gamma}\hat{\mathbf{s}}_{\cdot 1}^T - \mathbf{s}_{\cdot 1}\hat{\boldsymbol{\gamma}}^T,$

(ii) $\mathfrak{C}_{Z^T}(T^{-1}) = T^{-1}Z^T - Z^TT^{-1} = \boldsymbol{\delta}\hat{\mathbf{s}}_{\cdot n}^T - \mathbf{s}_{\cdot n}\hat{\boldsymbol{\delta}}^T,$

(iii) $\mathfrak{C}_{e_i e_j^T}(T^{-1}) = T^{-1}\mathbf{e}_i\mathbf{e}_j^T - \mathbf{e}_i\mathbf{e}_j^TT^{-1} = \mathbf{s}_{\cdot i}\mathbf{e}_j^T - \mathbf{e}_i\hat{\mathbf{s}}_{\cdot n+1-j}^T,$

where $\boldsymbol{\gamma} \equiv \boldsymbol{\gamma}(t_{-n})$ and $\boldsymbol{\delta} \equiv \boldsymbol{\delta}(t_n)$ are defined by

$$T\boldsymbol{\gamma}(t_{-n}) = \mathbf{a}(t_{-n}),$$

$$(4.1)$$

$$T\boldsymbol{\delta}(t_n) = \mathbf{b}(t_n).$$

Proof.   The first two equalities are well known (see for example [18, p. 16]). The third equality can be easily calculated.   ∎

PROPOSITION 4.2 [19].   *We have*

$$\mathfrak{C}_{T_2}\left((T + H)^{-1}\right) = \mathbf{x}_1\mathbf{w}_{1\cdot}^T + \mathbf{x}_2\mathbf{w}_{n\cdot}^T - \mathbf{w}_{\cdot 1}\mathbf{x}_3^T - \mathbf{w}_{\cdot n}\mathbf{x}_4^T,$$

where $\mathbf{x}_i$, $1 \leqslant i \leqslant 4$, are defined by

$$(T + H)\mathbf{x}_1 = \mathbf{b}(t_n) + \mathbf{c}(h_{-1}), \qquad (T + H)\mathbf{x}_2 = \mathbf{a}(t_{-n}) + \mathbf{d}(h_{2n-1}),$$

$$(T + H)^T\mathbf{x}_3 = \hat{\mathbf{a}}(t_{-n}) + \mathbf{c}(h_{-1}), \qquad (T + H)^T\mathbf{x}_4 = \hat{\mathbf{b}}(t_n) + \mathbf{d}(h_{2n-1}).$$

$$(4.2)$$

The following Proposition 4.3 lets us express the vectors $\boldsymbol{\gamma}(t_n)$ and $\boldsymbol{\delta}(t_n)$ defined in Proposition 4.1 in terms of columns $\mathbf{s}_{\cdot i}$ of T^{-1}. Proposition 4.4 has a similar meaning regarding Toeplitz plus Hankel matrices: the vectors $\mathbf{x}_i$ of Proposition 4.2 are expressed in terms of rows and columns of $W = (T + H)^{-1}$. Rost and Heinig consider the particular cases $s_{11} = s_{nn} \neq 0$ [18] and $w_{11}w_{nn} - w_{n1}w_{1n} \neq 0$ [20]. We extend their results to every nonsingular matrix T and $T + H$.

PROPOSITION 4.3.   *For $i = 1, \ldots, n$ we have*

$$s_{ni} = 0 \quad \Rightarrow \quad T(Z\mathbf{s}_{\cdot i} - \mathbf{s}_{\cdot i+1}) = \left[\hat{\mathbf{a}}(0)^T \mathbf{s}_{\cdot i}\right] \mathbf{e}_1$$

$$\Rightarrow \quad Z\mathbf{s}_{\cdot i} - \mathbf{s}_{\cdot i+1} = \left[\hat{\mathbf{a}}(0)^T \mathbf{s}_{\cdot i}\right] \mathbf{s}_{\cdot 1}; \tag{4.3}$$

$$s_{ni} \neq 0 \quad \Rightarrow \quad T\left(\frac{\mathbf{s}_{\cdot i+1} - Z\mathbf{s}_{\cdot i}}{s_{ni}}\right) = \mathbf{a}\left(-\frac{\hat{\mathbf{a}}(0)^T \mathbf{s}_{\cdot i}}{s_{ni}}\right)$$

$$\Rightarrow \quad \gamma\left(-\frac{\hat{\mathbf{a}}(0)^T \mathbf{s}_{\cdot i}}{s_{ni}}\right) = \frac{\mathbf{s}_{\cdot i+1} - Z\mathbf{s}_{\cdot i}}{s_{ni}}; \tag{4.4}$$

$$s_{1i} = 0 \quad \Rightarrow \quad T\left(Z^T\mathbf{s}_{\cdot i} - \mathbf{s}_{\cdot i-1}\right) = \left[\hat{\mathbf{b}}(0)^T \mathbf{s}_{\cdot i}\right] \mathbf{e}_n$$

$$\Rightarrow \quad Z^T\mathbf{s}_{\cdot i} - \mathbf{s}_{\cdot i-1} = \left[\hat{\mathbf{b}}(0)^T \mathbf{s}_{\cdot i}\right] \mathbf{s}_{\cdot n}; \tag{4.5}$$

$$s_{1i} \neq 0 \quad \Rightarrow \quad T\left(\frac{\mathbf{s}_{\cdot i-1} - Z^T\mathbf{s}_{\cdot i}}{s_{1i}}\right) = \mathbf{b}\left(-\frac{\hat{\mathbf{b}}(0)^T \mathbf{s}_{\cdot i}}{s_{1i}}\right)$$

$$\Rightarrow \quad \delta\left(-\frac{\hat{\mathbf{b}}(0)^T \mathbf{s}_{\cdot i}}{s_{1i}}\right) = \frac{\mathbf{s}_{\cdot i-1} - Z^T\mathbf{s}_{\cdot i}}{s_{1i}}. \tag{4.6}$$

Proof.   We have

$$TZ\mathbf{s}_{\cdot i} = \begin{pmatrix} t_{-1} & \cdots & \cdots & t_{-n+1} & 0 \\ t_0 & t_{-1} & \cdots & t_{-n+2} & 0 \\ \vdots & & & \vdots & \vdots \\ t_{n-2} & \cdots & \cdots & t_0 & 0 \end{pmatrix} \mathbf{s}_{\cdot i} = \begin{pmatrix} \hat{\mathbf{a}}(0)^T \mathbf{s}_{\cdot i} \\ (\mathbf{e}_1^T T - t_{-n+1}\mathbf{e}_n^T)\mathbf{s}_{\cdot i} \\ \vdots \\ (\mathbf{e}_{n-1}^T T - t_{-1}\mathbf{e}_n^T)\mathbf{s}_{\cdot i} \end{pmatrix}.$$

As $\mathbf{s}_{\cdot i} = T^{-1}\mathbf{e}_i$, we obtain

$$
TZ\mathbf{s}_{\cdot i} = \begin{pmatrix} \hat{\mathbf{a}}(0)^T\mathbf{s}_{\cdot i} \\ -t_{-n+1}s_{ni} \\ \vdots \\ -t_{-1}s_{ni} \end{pmatrix} + \mathbf{e}_{i+1} \quad \Rightarrow \quad T(Z\mathbf{s}_{\cdot i} - \mathbf{s}_{\cdot i+1})
$$

$$
= \begin{pmatrix} \hat{\mathbf{a}}(0)^T\mathbf{s}_{\cdot i} \\ -t_{-n+1}s_{ni} \\ \vdots \\ -t_{-1}s_{ni} \end{pmatrix} \quad (\mathbf{s}_{\cdot n+1} = 0).
$$

From the last equality one obtains (4.3) and (4.4) respectively for $s_{ni} = 0$ and $s_{ni} \neq 0$. As regards (4.5) and (4.6), we develop in a similar manner the product $TZ^T\mathbf{s}_{\cdot i}$ instead of $TZ\mathbf{s}_{\cdot i}$. ■

Set

$$
\mathbf{u}^i = T_2\mathbf{w}_{\cdot i} - \mathbf{w}_{\cdot i-1} - \mathbf{w}_{\cdot i+1}, \qquad \mathbf{v}^i = T_2\mathbf{w}_{i\cdot} - \mathbf{w}_{i-1\cdot} - \mathbf{w}_{i+1\cdot},
$$
$$
\Delta_1 = \Delta_1(i,j) = w_{1i}w_{nj} - w_{ni}w_{1j}, \quad \Delta_2 = \Delta_2(i,j) = w_{i1}w_{jn} - w_{in}w_{j1},
$$
$$
\mathbf{z}_1 = w_{ni}\mathbf{w}_{\cdot j} - w_{nj}\mathbf{w}_{\cdot i}, \qquad\qquad \mathbf{z}_3 = w_{in}\mathbf{w}_{j\cdot} - w_{jn}\mathbf{w}_{i\cdot},
$$
$$
\mathbf{z}_2 = w_{1j}\mathbf{w}_{\cdot i} - w_{1i}\mathbf{w}_{\cdot j}, \qquad\qquad \mathbf{z}_4 = w_{j1}\mathbf{w}_{i\cdot} - w_{i1}\mathbf{w}_{j\cdot}.
$$

PROPOSITION 4.4.   *For $i,j = 1, \ldots, n$ and $i \neq j$ we have:*

(a) $\Delta_1 = 0 \Rightarrow$

$$
-w_{nj}\mathbf{u}^i + w_{ni}\mathbf{u}^j = \left[\hat{\mathbf{a}}(-h_n)^T + \mathbf{c}(-t_1)^T\right]\mathbf{z}_1\mathbf{w}_{\cdot 1}
$$
$$
+ \left[\mathbf{d}(-t_{-1})^T + \hat{\mathbf{b}}(-h_{n-2})^T\right]\mathbf{z}_1\mathbf{w}_{\cdot n}, \tag{1}
$$

$$
w_{1j}\mathbf{u}^i - w_{1i}\mathbf{u}^j = \left[\hat{\mathbf{a}}(-h_n)^T + \mathbf{c}(-t_1)^T\right]\mathbf{z}_2\mathbf{w}_{\cdot 1}
$$
$$
+ \left[\mathbf{d}(-t_{-1})^T + \hat{\mathbf{b}}(-h_{n-2})^T\right]\mathbf{z}_2\mathbf{w}_{\cdot n}; \tag{2}
$$

(b) $\Delta_1 \neq 0 \implies$

$$(T + H)\left(\frac{-w_{nj}\mathbf{u}^i + w_{ni}\mathbf{u}^j}{\Delta_1}\right) = \mathbf{b}(t_n) + \mathbf{c}(h_{-1}) \tag{3}$$

for
$$h_{-1} = \frac{\left[\hat{\mathbf{a}}(-h_n)^T + \mathbf{c}(-t_1)^T\right]\mathbf{z}_1}{\Delta_1} - t_1,$$

$$t_n = \frac{\left[\mathbf{d}(-t_{-1})^T + \hat{\mathbf{b}}(-h_{n-2})^T\right]\mathbf{z}_1}{\Delta_1} - h_{n-2},$$

$$(T + H)\left(\frac{w_{1j}\mathbf{u}^i - w_{1i}\mathbf{u}^j}{\Delta_1}\right) = \mathbf{a}(t_{-n}) + \mathbf{d}(h_{2n-1}) \tag{4}$$

for

$$t_{-n} = \frac{\left[\hat{\mathbf{a}}(-h_n)^T + \mathbf{c}(-t_1)^T\right]\mathbf{z}_2}{\Delta_1} - h_n,$$

$$\mathring{h}_{2n-1} = \frac{\left[\mathbf{d}(-t_{-1})^T + \hat{\mathbf{b}}(-h_{n-2})^T\right]\mathbf{z}_2}{\Delta_1} - t_{-1}.$$

(c) $\Delta_2 = 0 \implies$

$$-w_{jn}\mathbf{v}^i + w_{in}\mathbf{v}^j = \left[\mathbf{b}(-h_n)^T + \mathbf{c}(-t_{-1})^T\right]\mathbf{z}_3\mathbf{w}_{1\cdot}$$

$$+ \left[\mathbf{d}(-t_1)^T + \mathbf{a}(-h_{n-2})^T\right]\mathbf{z}_3\mathbf{w}_{n\cdot}, \tag{5}$$

$$w_{j1}\mathbf{v}^i - w_{i1}\mathbf{v}^j = \left[\mathbf{b}(-h_n)^T + \mathbf{c}(-t_{-1})^T\right]\mathbf{z}_4\mathbf{w}_{1\cdot}$$

$$+ \left[\mathbf{d}(-t_1)^T + \mathbf{a}(-h_{n-2})^T\right]\mathbf{z}_4\mathbf{w}_{n\cdot}; \tag{6}$$

(d) $\Delta_2 \neq 0 \Rightarrow$

$$(T + H)^T \left(\frac{-w_{jn}\mathbf{v}^i + w_{in}\mathbf{v}^j}{\Delta_2} \right) = \hat{\mathbf{a}}(t_{-n}) + \mathbf{c}(h_{-1}) \qquad (7)$$

for

$$h_{-1} = \frac{\left[\mathbf{b}(-h_n)^T + \mathbf{c}(-t_{-1})^T \right] \mathbf{z}_3}{\Delta_2} - t_{-1},$$

$$t_{-n} = \frac{\left[\mathbf{d}(-t_1)^T + \mathbf{a}(-h_{n-2})^T \right] \mathbf{z}_3}{\Delta_2} - h_{n-2}, (7)$$

$$(T + H)^T \left(\frac{w_{j1}\mathbf{v}^i - w_{i1}\mathbf{v}^j}{\Delta_2} \right) = \hat{\mathbf{b}}(t_n) + \mathbf{d}(h_{2n-1}) \qquad (8)$$

for

$$t_n = \frac{\left[\mathbf{b}(-h_n)^T + \mathbf{c}(-t_{-1})^T \right] \mathbf{z}_4}{\Delta_2} - h_n,$$

$$h_{2n-1} = \frac{\left[\mathbf{d}(-t_1)^T + \mathbf{a}(-h_{n-2})^T \right] \mathbf{z}_4}{\Delta_2} - t_1.$$

Proof. Develop the product $(T + H)T_2\mathbf{w}_{\cdot i}$. To this end observe that the following equalities hold:

$$TZ + HZ^T = \begin{pmatrix} \hat{\mathbf{a}}(0)^T + \mathbf{c}(0)^T \\ \mathbf{e}_1^T(T + H) - h_0\mathbf{e}_1^T - t_{-n+1}\mathbf{e}_n^T \\ \vdots \\ \mathbf{e}_{n-1}^T(T + H) - h_{n-2}\mathbf{e}_1^T - t_{-1}\mathbf{e}_n^T \end{pmatrix},$$

$$TZ^T + HZ = \begin{pmatrix} \mathbf{e}_2^T(T + H) - t_1\mathbf{e}_1^T - h_n\mathbf{e}_n^T \\ \vdots \\ \mathbf{e}_n^T(T + H) - t_{n-1}\mathbf{e}_1^T - h_{2n-2}\mathbf{e}_n^T \\ \mathbf{d}(0)^T + \hat{\mathbf{b}}(0)^T \end{pmatrix},$$

and, as $\mathbf{w}_{\cdot i} = (T + H)^{-1}\mathbf{e}_i$, they imply

$$(T + H)T_2\mathbf{w}_{\cdot i} = \begin{pmatrix} \left[\hat{\mathbf{a}}(-h_n)^T + \mathbf{c}(-t_1)^T\right]\mathbf{w}_{\cdot i} \\ -w_{1i}(t_2 + h_0) - w_{ni}(t_{-n+1} + h_{n+1}) \\ \vdots \\ -w_{1i}(t_{n-1} + h_{n-3}) - w_{ni}(t_{-2} + h_{2n-2}) \\ \left[\mathbf{d}(-t_{-1})^T + \hat{\mathbf{b}}(-h_{n-2})^T\right]\mathbf{w}_{\cdot i} \end{pmatrix} + \mathbf{e}_{i-1} + \mathbf{e}_{i+1}$$

$$\Rightarrow \quad (T + H)(T_2\mathbf{w}_{\cdot i} - \mathbf{w}_{\cdot i-1} - \mathbf{w}_{\cdot i+1})$$

$$= \begin{pmatrix} \left[\hat{\mathbf{a}}(-h_n)^T + \mathbf{c}(-t_1)^T\right]\mathbf{w}_{\cdot i} \\ -w_{1i}(t_2 + h_0) - w_{ni}(t_{-n+1} + h_{n+1}) \\ \vdots \\ -w_{1i}(t_{n-1} + h_{n-3}) - w_{ni}(t_{-2} + h_{2n-2}) \\ \left[\mathbf{d}(-t_{-1})^T + \hat{\mathbf{b}}(-h_{n-2})^T\right]\mathbf{w}_{\cdot i} \end{pmatrix}. \quad (4.7)$$

Now write (4.7) with i replaced by j, and consider the linear combination of this new relation with (4.7):

$$(T + H)(\alpha \mathbf{u}^i + \beta \mathbf{u}^j)$$

$$= \begin{pmatrix} \left[\hat{\mathbf{a}}(-h_n)^T + \mathbf{c}(-t_1)^T\right](\alpha \mathbf{w}_{\cdot i} + \beta \mathbf{w}_{\cdot j}) \\ -(\alpha w_{1i} + \beta w_{1j})(t_2 + h_0) - (\alpha w_{ni} + \beta w_{nj})(t_{-n+1} + h_{n+1}) \\ \vdots \\ -(\alpha w_{1i} + \beta w_{1j})(t_{n-1} + h_{n-3}) - (\alpha w_{ni} + \beta w_{nj})(t_{-2} + h_{2n-2}) \\ \left[\mathbf{d}(-t_{-1})^T + \hat{\mathbf{b}}(-h_{n-2})^T\right](\alpha \mathbf{w}_{\cdot i} + \beta \mathbf{w}_{\cdot j}) \end{pmatrix}$$

Observe that the positions $\alpha = -w_{nj}$, $\beta = w_{ni}$ imply (1) and (3), and the positions $\alpha = w_{1j}$, $\beta = -w_{1i}$ imply (2) and (4).

As regards (5), (6), (7), and (8), develop in a similar manner $(T + H)^T T_2\mathbf{w}_i$. instead of $(T + H)T_2\mathbf{w}_{\cdot i}$ and proceed as above. ∎

Now the previous propositions can be exploited to write in explicit form the inverses of T and of $T + H$, taking into account the results of Section 3. All formulas considered for T^{-1} and $(T + H)^{-1}$ have a common reference to

the Hessenberg algebras introduced in Section 2. Besides all the formulas well known for T^{-1} (Gohberg-Semencul [17], Ammar-Gader [4, 5], and others), new formulas involving Hessenberg algebras are introduced (Theorems 4.2, 4.4, and 4.5). All possible formulas for the inverse of a Toeplitz matrix, obtained using the formulas of Section 3, are listed in the Appendix.

A first class of formulas for T^{-1}, where T is a *general* (nonsymmetric) $n \times n$ Toeplitz matrix with complex values, can be derived from Proposition 4.1 and Corollary 3.1(i), (ii), (iii), (iv). These formulas include well-known formulas for T^{-1} and all their possible variants. We obtain 16 different formulas for T^{-1}, but only four of them are here mentioned (the others are listed in the Appendix).

THEOREM 4.1.

$$T^{-1} = L(Z\mathbf{s}_{.n})U(\hat{\boldsymbol{\delta}}) - [L(Z\boldsymbol{\delta}) - I]U(\hat{\mathbf{s}}_{.n}), \qquad (4.8)$$

$$T^{-1} = [L(Z(\mathbf{e}_n - \boldsymbol{\delta})) + I]C(\hat{\mathbf{s}}_{.n}) - L(Z\mathbf{s}_{.n})C(\hat{\mathbf{e}}_n - \hat{\boldsymbol{\delta}}), \qquad (4.9)$$

$$T^{-1} = L(Z\mathbf{s}_{.1})\tau(\hat{\boldsymbol{\gamma}}) - L(Z\boldsymbol{\gamma})\tau(\hat{\mathbf{s}}_{.1}) + L(Z\mathbf{s}_{.n})\tau(\hat{\boldsymbol{\delta}})$$

$$- [L(Z\boldsymbol{\delta}) - I]\tau(\hat{\mathbf{s}}_{.n}), \qquad (4.10)$$

$$T^{-1} = (\varepsilon - \beta)^{-1}[C_\beta(\hat{\mathbf{s}}_{.n})C_\varepsilon(\varepsilon\hat{\mathbf{e}}_n - \hat{\boldsymbol{\delta}}) - C_\beta(\beta\hat{\mathbf{e}}_n - \hat{\boldsymbol{\delta}})C_\varepsilon(\hat{\mathbf{s}}_{.n})]. \qquad (4.11)$$

Proof. Use Corollary 3.1 [respectively (3.12), (3.16), (3.20), and (3.24)] and Proposition 4.1. ■

Another class of formulas for T^{-1}, where T is a general $n \times n$ Toeplitz matrix on $\mathbb{C}$, could be obtained from Proposition 4.1 and Theorem 3.2. This last theorem will be exploited only for $X = T_2$, that is, only its consequences in Corollary 3.2 will be considered in detail. The reasons for this special choice were explained in the previous section, and they are now related to the computational cost of solving a linear system $T\mathbf{x} = \mathbf{f}$ or $(T + H)\mathbf{x} = \mathbf{f}$.

THEOREM 4.2.

$$T^{-1} = \tau^1(\boldsymbol{\gamma})\tau(\hat{\mathbf{s}}_{\cdot 1}) - \tau^1(\mathbf{s}_{\cdot 1})\tau(\hat{\boldsymbol{\gamma}}) + \left[\tau^1(\boldsymbol{\delta}) + I\right]\tau(\hat{\mathbf{s}}_{\cdot n}) - \tau^1(\mathbf{s}_{\cdot n})\tau(\hat{\boldsymbol{\delta}}),$$

$$(4.12)$$

$$T^{-1} = \tau(\mathbf{s}_{\cdot 1})\left[\tau^1(\hat{\boldsymbol{\gamma}})^T + I\right] - \tau(\boldsymbol{\gamma})\tau^1(\hat{\mathbf{s}}_{\cdot 1})^T + \tau(\mathbf{s}_{\cdot n})\tau^1(\hat{\boldsymbol{\delta}})^T$$

$$- \tau(\boldsymbol{\delta})\tau^1(\hat{\mathbf{s}}_{\cdot n})^T, \tag{4.13}$$

$$T^{-1} = \left[\tau^3(\boldsymbol{\gamma}) + I\right]\tau(\mathbf{s}_{\cdot 1}) - \tau^3(\mathbf{s}_{\cdot 1})\tau(\boldsymbol{\gamma}) + \tau^3(\boldsymbol{\delta})\tau(\mathbf{s}_{\cdot n}) - \tau^3(\mathbf{s}_{\cdot n})\tau(\boldsymbol{\delta}),$$

$$(4.14)$$

$$T^{-1} = \tau(\hat{\mathbf{s}}_{\cdot 1})\tau^3(\hat{\boldsymbol{\gamma}})^T - \tau(\hat{\boldsymbol{\gamma}})\tau^3(\hat{\mathbf{s}}_{\cdot 1})^T + \tau(\hat{\mathbf{s}}_{\cdot n})\left[\tau^3(\hat{\boldsymbol{\delta}})^T + I\right]$$

$$- \tau(\hat{\boldsymbol{\delta}})\tau^3(\hat{\mathbf{s}}_{\cdot n})^T. \tag{4.15}$$

Proof. Use Corollary 3.2 and Proposition 4.1. ∎

Now let T be *symmetric*. This implies

$$\mathbf{s}_{\cdot 1} = \hat{\mathbf{s}}_{\cdot n} \quad \text{and} \quad \hat{\boldsymbol{\delta}} = \boldsymbol{\gamma} + (t_n - t_{-n})\mathbf{s}_{\cdot 1}. \tag{4.16}$$

As before, we can use Corollary 3.1 and Proposition 4.1 to write T^{-1} in explicit form. From Corollary 3.1(i), (ii), (iv), we obtain the same formulas as for the nonsymmetric case, that is, (4.8), (4.9), (4.11). From Corollary 3.1(iii), we obtain a simpler version of the formula (4.10):

THEOREM 4.3. *For T symmetric we have*

$$T^{-1} = \left[L(Z\mathbf{s}_{\cdot 1}) + L(Z\hat{\mathbf{s}}_{\cdot 1})J\right]\tau(\hat{\boldsymbol{\gamma}}) - \left[L(Z\hat{\boldsymbol{\gamma}}) + L(Z\boldsymbol{\gamma})J - I\right]\tau(\mathbf{s}_{\cdot 1}).$$

$$(4.17)$$

Proof. Use (4.16) in (4.10). ∎

A class of new formulas for T^{-1}, in the symmetric case, can be obtained directly from (4.12), (4.13), (4.14), and (4.15) using (4.16) [or, in other terms, by exploiting Corollary 3.2 and Proposition 4.1 where (4.16) is assumed].

THEOREM 4.4.   *For T symmetric we have*

$$T^{-1} = \left[\tau^1(\hat{\boldsymbol{\gamma}}) + \tau^1(\boldsymbol{\gamma})J + I\right]\tau(\mathbf{s}_{\cdot 1}) - \left[\tau^1(\mathbf{s}_{\cdot 1}) + \tau^1(\hat{\mathbf{s}}_{\cdot 1})J\right]\tau(\hat{\boldsymbol{\gamma}}),$$

$$(4.18)$$

$$T^{-1} = \tau(\mathbf{s}_{\cdot 1})\left[\tau^1(\hat{\boldsymbol{\gamma}})^T + J\tau^1(\boldsymbol{\gamma})^T + I\right] - \tau(\hat{\boldsymbol{\gamma}})\left[\tau^1(\mathbf{s}_{\cdot 1})^T + J\tau^1(\hat{\mathbf{s}}_{\cdot 1})^T\right],$$

$$(4.19)$$

$$T^{-1} = \left[\tau^3(\boldsymbol{\gamma}) + \tau^3(\hat{\boldsymbol{\gamma}})J + I\right]\tau(\mathbf{s}_{\cdot 1}) - \left[\tau^3(\hat{\mathbf{s}}_{\cdot 1}) + \tau^3(\mathbf{s}_{\cdot 1})J\right]\tau(\hat{\boldsymbol{\gamma}}),$$

$$(4.20)$$

$$T^{-1} = \tau(\mathbf{s}_{\cdot 1})\left[\tau^3(\boldsymbol{\gamma})^T + J\tau^3(\hat{\boldsymbol{\gamma}})^T + I\right] - \tau(\hat{\boldsymbol{\gamma}})\left[\tau^3(\hat{\mathbf{s}}_{\cdot 1})^T + J\tau^3(\mathbf{s}_{\cdot 1})^T\right].$$

$$(4.21)$$

∎

The last case is related to the application of Corollary 3.3 to a symmetric Toeplitz matrix T.

THEOREM 4.5.   *For T symmetric we have*

$$T^{-1} = \left[\tau^2(\hat{\boldsymbol{\gamma}}) + I\right]\tau(\mathbf{s}_{\cdot 1}) - \tau^2(\mathbf{s}_{\cdot 1})\tau(\hat{\boldsymbol{\gamma}}), \qquad (4.22)$$

$$T^{-1} = \tau(\mathbf{s}_{\cdot 1})\left[\tau^2(\hat{\boldsymbol{\gamma}})^T + I\right] - \tau(\hat{\boldsymbol{\gamma}})\tau^2(\mathbf{s}_{\cdot 1})^T. \qquad (4.23)$$

Proof.   Use Corollary 3.3 and Proposition 4.1 where (4.16) is assumed.
∎

We know that $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ can always be expressed in terms of some columns of T^{-1} (see Proposition 4.3). In particular, when $s_{11} = s_{nn} \neq 0$, all previous formulas expressed in terms of $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$ can be conveniently rewritten in terms of the first and the last columns (in the symmetric case only one of them) of T^{-1}, using the equalities [18]

$$\boldsymbol{\gamma} = -\frac{1}{s_{nn}}Z\mathbf{s}_{\cdot n} \quad \text{and} \quad \boldsymbol{\delta} = -\frac{1}{s_{11}}Z^T\mathbf{s}_{\cdot 1}. \qquad (4.24)$$

Here we only write some representative formulas in the symmetric case, which will be analyzed in detail—in the next section—to evaluate the computational cost of solving a system $T\mathbf{x} = \mathbf{f}$.

THEOREM 4.6.   *If T is symmetric and $s_{11} = s_{nn} \neq 0$, then*

$$T^{-1} = \frac{1}{s_{11}}\left[L(\mathbf{s}_{.1})U(\mathbf{s}_{.1}) - L(Z\hat{\mathbf{s}}_{.1})U(Z\hat{\mathbf{s}}_{.1}) \right] \qquad (4.25)$$

(*Gohberg-Semencul formula* [17, 5]),

$$T^{-1} = \frac{1}{s_{11}}\left[L(\mathbf{s}_{.1})C(\mathbf{s}_{.1}) - L(Z\hat{\mathbf{s}}_{.1})C\left(P^{T}\hat{\mathbf{s}}_{.1}\right) \right] \qquad (4.26)$$

(*Ammar-Gader formula* [5]),

$$T^{-1} = \frac{1}{s_{nn}}\left\{ \left[L(\mathbf{s}_{.1}) + L\left(Z^{2}\hat{\mathbf{s}}_{.1}\right)J \right]\tau(\mathbf{s}_{.1}) \right.$$

$$\left. - \left[L(Z\mathbf{s}_{.1}) + L\left(Z\hat{\mathbf{s}}_{.1}\right)J \right]\tau\left(Z^{T}\mathbf{s}_{.1}\right) \right\} \qquad (4.27)$$

(*Bini-Pan formula* [10]), *and*

$$T^{-1} = \frac{1}{s_{11}}(\varepsilon - \beta)^{-1}\left[C_{\beta}(\mathbf{s}_{.1})C_{\varepsilon}\left(\varepsilon s_{11}\mathbf{e}_1 + Z\hat{\mathbf{s}}_{.1}\right) \right.$$

$$\left. - C_{\beta}\left(\beta s_{11}\mathbf{e}_1 + Z\hat{\mathbf{s}}_{.1} \right)C_{\varepsilon}(\mathbf{s}_{.1}) \right] \qquad (4.28)$$

(*Gohberg-Olshevsky formula* [16]). *Moreover we have*

$$T^{-1} = \frac{1}{s_{nn}}\left\{ \left[\tau^{1}(\mathbf{s}_{.1}) + \tau^{1}(\hat{\mathbf{s}}_{.1})J \right]\tau\left(Z^{T}\mathbf{s}_{.1}\right) \right.$$

$$\left. - \left[\tau^{1}\left(Z^{T}\mathbf{s}_{.1}\right) + \tau^{1}(Z\hat{\mathbf{s}}_{.1})J - s_{nn}I \right]\tau(\mathbf{s}_{.1}) \right\}, \qquad (4.29)$$

$$T^{-1} = \frac{1}{s_{nn}}\left\{ \tau^{2}(\mathbf{s}_{.1})\tau\left(Z^{T}\mathbf{s}_{.1}\right) - \left[\tau^{2}\left(Z^{T}\mathbf{s}_{.1}\right) - s_{nn}I \right]\tau(\mathbf{s}_{.1}) \right\}. \quad (4.30)$$

   Proof.   Use the equalities (4.24) and, respectively, (4.8), (4.9), (4.17), (4.11), (4.18), and (4.22).   ∎

Now consider the case of a *general* Toeplitz plus Hankel matrix $T + H$. A first class of formulas for $W = (T + H)^{-1}$ is obtained in terms of τ and H_{Z^T}, in the spirit of [10, 7]. This class of formulas can be derived from Proposition 4.2 and Corollary 3.1(iii).

THEOREM 4.7.

$$W = \tau(\hat{\mathbf{x}}_1)L(Z\hat{\mathbf{w}}_{1.}) + \tau(\hat{\mathbf{x}}_2)L(Z\hat{\mathbf{w}}_{n.}) - \tau(\hat{\mathbf{w}}_{.1})L(Z\hat{\mathbf{x}}_3)$$
$$- \tau(\hat{\mathbf{w}}_{.n})[L(Z\hat{\mathbf{x}}_4) - I], \tag{4.31}$$

$$W = -[L(Z\mathbf{x}_1) - I]\tau(\mathbf{w}_{1.}) - L(Z\mathbf{x}_2)\tau(\mathbf{w}_{n.}) + L(Z\mathbf{w}_{.1})\tau(\mathbf{x}_3)$$
$$+ L(Z\mathbf{w}_{.n})\tau(\mathbf{x}_4), \tag{4.32}$$

$$W = -U(Z\hat{\mathbf{x}}_1)\tau(\hat{\mathbf{w}}_{1.}) - [U(Z\hat{\mathbf{x}}_2) - I]\tau(\hat{\mathbf{w}}_{n.}) + U(Z\hat{\mathbf{w}}_{.1})\tau(\hat{\mathbf{x}}_3)$$
$$+ U(Z\hat{\mathbf{w}}_{.n})\tau(\hat{\mathbf{x}}_4), \tag{4.33}$$

$$W = \tau(\mathbf{x}_1)U(Z\mathbf{w}_{1.}) + \tau(\mathbf{x}_2)U(Z\mathbf{w}_{n.}) - \tau(\mathbf{w}_{.1})[U(Z\mathbf{x}_3) - I]$$
$$- \tau(\mathbf{w}_{.n})U(Z\mathbf{x}_4). \tag{4.34}$$

Proof.   Use Corollary 3.1(iii) and Proposition 4.2.   ∎

A new class of formulas for $(T + H)^{-1}$ is also obtained from Proposition 4.2 and Corollary 3.2. For computational convenience, only the algebra τ is considered: the rank of $\mathfrak{C}_X((T + H)^{-1})$ is small for $X = T_2$, and efficient sine transforms are involved in solving $(T + H)\mathbf{x} = \mathbf{f}$. The main computational aspects are discussed in the next section.

THEOREM 4.8.

$$W = [\tau^1(\mathbf{x}_1) + I]\tau(\mathbf{w}_{1.}) + \tau^1(\mathbf{x}_2)\tau(\mathbf{w}_{n.}) - \tau^1(\mathbf{w}_{.1})\tau(\mathbf{x}_3)$$
$$- \tau^1(\mathbf{w}_{.n})\tau(\mathbf{x}_4), \tag{4.35}$$

$$W = -\tau(\mathbf{x}_1)\tau^1(\mathbf{w}_{1.})^T - \tau(\mathbf{x}_2)\tau^1(\mathbf{w}_{n.})^T + \tau(\mathbf{w}_{.1})[\tau^1(\mathbf{x}_3)^T + I]$$
$$+ \tau(\mathbf{w}_{.n})\tau^1(\mathbf{x}_4)^T, \tag{4.36}$$

$$W = \tau^3(\mathbf{x}_1)\tau(\hat{\mathbf{w}}_{1.}) + [\tau^3(\mathbf{x}_2) + I]\tau(\hat{\mathbf{w}}_{n.}) - \tau^3(\mathbf{w}_{.1})\tau(\hat{\mathbf{x}}_3)$$
$$- \tau^3(\mathbf{w}_{.n})\tau(\hat{\mathbf{x}}_4), \tag{4.37}$$

$$W = -\tau(\hat{\mathbf{x}}_1)\tau^3(\mathbf{w}_{1.})^T - \tau(\hat{\mathbf{x}}_2)\tau^3(\mathbf{w}_{n.})^T + \tau(\hat{\mathbf{w}}_{.1})\tau^3(\mathbf{x}_3)^T$$
$$+ \tau(\hat{\mathbf{w}}_{.n})[\tau^3(\mathbf{x}_4)^T + I]. \tag{4.38}$$

Proof. Use Corollary 3.2 and Proposition 4.2. ∎

Let $T + H$ be *symmetric*, that is, $T = T^T$. This implies

$$\mathbf{w}_{i\cdot} = \mathbf{w}_{\cdot i}, \qquad \mathbf{x}_3 = \mathbf{x}_1 + (t_{-n} - t_n)\mathbf{w}_{\cdot n}, \qquad \mathbf{x}_4 = \mathbf{x}_2 + (t_n - t_{-n})\mathbf{w}_{\cdot 1}.$$

$$(4.39)$$

Thus, in the symmetric case, Proposition 4.2 becomes

$$\mathfrak{C}_{T_2}(W) = \mathbf{x}_1 \mathbf{w}_{\cdot 1}^T + \mathbf{x}_2 \mathbf{w}_{\cdot n}^T - \mathbf{w}_{\cdot 1}\big[\mathbf{x}_1^T + (t_{-n} - t_n)\mathbf{w}_{\cdot n}^T\big]$$
$$- \mathbf{w}_{\cdot n}\big[\mathbf{x}_2^T + (t_n - t_{-n})\mathbf{w}_{\cdot 1}^T\big].$$

This last formula can be simplified by expressing $\mathbf{x}_1$ and $\mathbf{x}_2$ as suggested by Proposition 4.4 and observing that $\Delta_1(i, j) = \Delta_2(i, j)$:

$$\mathfrak{C}_{T_2}\big((T + H)^{-1}\big) = \mathbf{x}_1 \mathbf{w}_{\cdot 1}^T + \mathbf{x}_2 \mathbf{w}_{\cdot n}^T - \mathbf{w}_{\cdot 1}\mathbf{x}_1^T - \mathbf{w}_{\cdot n}\mathbf{x}_2^T. \qquad (4.40)$$

In this last case the formulas obtained are those of Theorems 4.7 and 4.8, but with $\mathbf{w}_{1\cdot}$, $\mathbf{w}_{n\cdot}$, $\mathbf{x}_3$, and $\mathbf{x}_4$ replaced by $\mathbf{w}_{\cdot 1}$, $\mathbf{w}_{\cdot n}$, $\mathbf{x}_1$, and $\mathbf{x}_2$, respectively.

Let $T + H$ be *persymmetric*, that is, $JH = H^T J$. This implies

$$\mathbf{w}_{i\cdot} = \hat{\mathbf{w}}_{\cdot n + 1 - i}, \qquad \hat{\mathbf{x}}_3 = \mathbf{x}_2 + (h_{-1} - h_{2n-1})\mathbf{w}_{\cdot n},$$
$$\hat{\mathbf{x}}_4 = \mathbf{x}_1 + (h_{2n-1} - h_{-1})\mathbf{w}_{\cdot 1}. \qquad (4.41)$$

Thus, in the persymmetric case, Proposition 4.2 becomes

$$\mathfrak{C}_{T_2}(W) = \mathbf{x}_1 \hat{\mathbf{w}}_{\cdot n}^T + \mathbf{x}_2 \hat{\mathbf{w}}_{\cdot 1}^T - \mathbf{w}_{\cdot 1}\big[\hat{\mathbf{x}}_2^T + (h_{-1} - h_{2n-1})\hat{\mathbf{w}}_{\cdot n}^T\big]$$
$$- \mathbf{w}_{\cdot n}\big[\hat{\mathbf{x}}_1^T + (h_{2n-1} - h_{-1})\hat{\mathbf{w}}_{\cdot 1}^T\big].$$

This last formula can be simplified by expressing $\mathbf{x}_1$ and $\mathbf{x}_2$ as suggested by Proposition 4.4 and observing that $\Delta_1(i, j) = -\Delta_2(n + 1 - i, n + 1 - j)$:

$$\mathfrak{C}_{T_2}\big((T + H)^{-1}\big) = \mathbf{x}_1 \hat{\mathbf{w}}_{\cdot n}^T + \mathbf{x}_2 \hat{\mathbf{w}}_{\cdot 1}^T - \mathbf{w}_{\cdot 1}\hat{\mathbf{x}}_2^T - \mathbf{w}_{\cdot n}\hat{\mathbf{x}}_1^T. \qquad (4.42)$$

In this last case the formulas obtained are those of Theorems 4.7 and 4.8, with $\mathbf{w}_{1\cdot}$, $\mathbf{w}_{n\cdot}$, $\mathbf{x}_3$, and $\mathbf{x}_4$ replaced by $\hat{\mathbf{w}}_{\cdot n}$, $\hat{\mathbf{w}}_{\cdot 1}$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_1$, respectively.

Finally, let $T + H$ be *centrosymmetric*, that is, $T = T^T$ and $JH = H^T J$. This implies that (4.39) and (4.41) hold simultaneously, and in any case Proposition 4.2 becomes

$$\mathfrak{C}_{T_2}\big((T + H)^{-1}\big) = \mathbf{x}_1 \mathbf{w}_{\cdot 1}^T + \hat{\mathbf{x}}_1 \hat{\mathbf{w}}_{\cdot 1}^T - \mathbf{w}_{\cdot 1}\mathbf{x}_1^T - \hat{\mathbf{w}}_{\cdot 1}\hat{\mathbf{x}}_1^T. \qquad (4.43)$$

Now exploit Corollary 3.1(iii), Corollary 3.2, and Corollary 3.3—taking into account (4.43)—to obtain, respectively, the following Theorems 4.9, 4.10, 4.11.

THEOREM 4.9.   *For $T + H$ centrosymmetric we have*

$$W = \tau(\mathbf{x}_1)[L(Z\mathbf{w}_{.1}) + JL(Z\hat{\mathbf{w}}_{.1})] - \tau(\mathbf{w}_{.1})[L(Z\mathbf{x}_1) + JL(Z\hat{\mathbf{x}}_1) - I],$$
$$(4.44)$$

$$W = [L(Z\mathbf{w}_{.1}) + L(Z\hat{\mathbf{w}}_{.1})J]\tau(\mathbf{x}_1) - [L(Z\mathbf{x}_1) + L(Z\hat{\mathbf{x}}_1)J - I]\tau(\mathbf{w}_{.1}),$$
$$(4.45)$$

$$W = [U(Z\mathbf{w}_{.1}) + U(Z\hat{\mathbf{w}}_{.1})J]\tau(\mathbf{x}_1) - [U(Z\mathbf{x}_1) + U(Z\hat{\mathbf{x}}_1)J - I]\tau(\mathbf{w}_{.1}),$$
$$(4.46)$$

$$W = \tau(\mathbf{x}_1)[U(Z\mathbf{w}_{.1}) + JU(Z\hat{\mathbf{w}}_{.1})] - \tau(\mathbf{w}_{.1})[U(Z\mathbf{x}_1) + JU(Z\hat{\mathbf{x}}_1) - I].$$
$$(4.47)$$

∎

THEOREM 4.10.   *For $T + H$ centrosymmetric we have*

$$W = [\tau^1(\mathbf{x}_1) + \tau^1(\hat{\mathbf{x}}_1)J + I]\tau(\mathbf{w}_{.1}) - [\tau^1(\mathbf{w}_{.1}) + \tau^1(\hat{\mathbf{w}}_{.1})J]\tau(\mathbf{x}_1),$$
$$(4.48)$$

$$W = \tau(\mathbf{w}_{.1})[\tau^1(\mathbf{x}_1)^T + J\tau^1(\hat{\mathbf{x}}_1)^T + I] - \tau(\mathbf{x}_1)[\tau^1(\mathbf{w}_{.1})^T + J\tau^1(\hat{\mathbf{w}}_{.1})^T],$$
$$(4.49)$$

$$W = [\tau^3(\hat{\mathbf{x}}_1) + \tau^3(\mathbf{x}_1)J + I]\tau(\mathbf{w}_{.1}) - [\tau^3(\hat{\mathbf{w}}_{.1}) + \tau^3(\mathbf{w}_{.1})J]\tau(\mathbf{x}_1),$$
$$(4.50)$$

$$W = \tau(\mathbf{w}_{.1})[\tau^3(\hat{\mathbf{x}}_1)^T + J\tau^3(\mathbf{x}_1)^T + I] - \tau(\mathbf{x}_1)[\tau^3(\hat{\mathbf{w}}_{.1})^T + J\tau^3(\mathbf{w}_{.1})^T].$$
$$(4.51)$$

∎

THEOREM 4.11.   *For $T + H$ centrosymmetric we have*

$$W = \left[\tau^2(\mathbf{x}_1) + I\right]\tau(\mathbf{w}_{\cdot 1}) - \tau^2(\mathbf{w}_{\cdot 1})\tau(\mathbf{x}_1), \qquad (4.52)$$

$$W = \tau(\mathbf{w}_{\cdot 1})\left[\tau^2(\mathbf{x}_1)^T + I\right] - \tau(\mathbf{x}_1)\tau^2(\mathbf{w}_{\cdot 1})^T. \qquad (4.53)$$

■

REMARK.   We know that the $\mathbf{x}_i$'s can always be expressed in terms of some rows and some columns of $(T + H)^{-1}$ (see Proposition 4.4). In particular, if $\Delta = w_{11}w_{nn} - w_{n1}w_{1n} \neq 0$ [20], then

$$\mathbf{x}_1 = \Delta^{-1}\left[-w_{nn}(T_2\mathbf{w}_{\cdot 1} - \mathbf{w}_{\cdot 2}) + w_{n1}(T_2\mathbf{w}_{\cdot n} - \mathbf{w}_{\cdot n-1})\right],$$

$$\mathbf{x}_2 = \Delta^{-1}\left[w_{1n}(T_2\mathbf{w}_{\cdot 1} - \mathbf{w}_{\cdot 2}) - w_{11}(T_2\mathbf{w}_{\cdot n} - \mathbf{w}_{\cdot n-1})\right],$$

$$\mathbf{x}_3 = \Delta^{-1}\left[-w_{nn}(T_2\mathbf{w}_{1\cdot} - \mathbf{w}_{2\cdot}) + w_{1n}(T_2\mathbf{w}_{n\cdot} - \mathbf{w}_{n-1\cdot})\right],$$

$$\mathbf{x}_4 = \Delta^{-1}\left[w_{n1}(T_2\mathbf{w}_{1\cdot} - \mathbf{w}_{2\cdot}) - w_{11}(T_2\mathbf{w}_{n\cdot} - \mathbf{w}_{n-1\cdot})\right].$$

Recall that $\mathbf{x}_1$ can be expressed as follows:

$$\mathbf{x}_1 = \Delta_1^{-1}\left[-w_{nj}(T_2\mathbf{w}_{\cdot i} - \mathbf{w}_{\cdot i-1} - \mathbf{w}_{\cdot i+1}) + w_{ni}(T_2\mathbf{w}_{\cdot j} - \mathbf{w}_{\cdot j-1} - \mathbf{w}_{\cdot j+1})\right],$$

where i and j are such that $\Delta_1 = w_{1i}w_{nj} - w_{ni}w_{1j} \neq 0$ [see (3) in Proposition 4.4]. In particular, if $\Delta = w_{11}w_{nn} - w_{n1}w_{1n} \neq 0$ and $T + H$ is centrosymmetric, then we have

$$\mathbf{x}_1 = \Delta^{-1}\left[(w_{n1}J - w_{11}I)(T_2\mathbf{w}_{\cdot 1} - \mathbf{w}_{\cdot 2})\right].$$

Clearly Theorem 4.11 gives the most convenient representations for $W = (T + H)^{-1}$ in the centrosymmetric case. In fact (4.52) will be analyzed in detail, as regards its computational meaning, in the next section.

5.   COMPUTATIONAL ASPECTS

In this section the matrices T and $T + H$ and all the vectors have complex entries. The expression $T = T^T$ ($JH = H^TJ$) means that T (H) is a complex symmetric (persymmetric) matrix.

The formulas (4.25) and (4.26) are exploited in [21, 5] for solving the real Toeplitz linear system $T\mathbf{x} = \mathbf{f}$, where T is nonsingular, $T = T^T$, and $s_{11} \neq 0$ (for instance, T real positive definite), in a two-phase calculation. The first phase is the computation of $\mathbf{s}_{.1}$. The second phase exploits the structural properties of the matrices involved in (4.25) and (4.26) for computing $T^{-1}\mathbf{f}$ by means of a number of fast real discrete Fourier transforms (FFT).

In detail, if $\phi(n)$ is the amount of computation required by a real FFT of order n [it is known that $\phi(n) = O(n \log n)$ arithmetic operations], then the second phase of the algorithm described above requires at most $14\phi(n) + O(n)$ computations for the Gohberg-Semencul formula (4.25) and $9\phi(n) + O(n)$ computations for the Ammar-Gader formula (4.26). The efficiency of the Ammar-Gader formula is mainly due to the introduction of circulant matrices in the formula for T^{-1} and to the fact that circulant matrices are more directly related to FFT and numerical convolution with respect to triangular Toeplitz matrices. This is even more evident in the Gohberg-Olshevsky formula (4.11) (and in its variants listed in the Appendix), where only ε-circulant matrices are present (see also [4]).

If T and $\mathbf{f}$ have complex elements, then the number of transforms (14 and 9) does not change. However, in this case some (possibly all) of the transforms involved become complex Fourier transforms, and it is known that the cost of an order n complex Fourier transform is about twice that of an order n real Fourier transform [25].

Now we can exploit the formulas expressing the inverse of a Toeplitz plus Hankel matrix $T + H$ obtained in the previous section (Theorems 4.8, 4.10, 4.11) for solving the more general linear system $(T + H)\mathbf{x} = \mathbf{f}$. Notice that these formulas are structurally identical to the corresponding ones for T^{-1} (Theorems 4.2, 4.4, 4.5); thus the second phase does not change if the coefficient matrix is Toeplitz instead of Toeplitz plus Hankel ($H = 0$).

Here we examine in detail the second phase in the solution of $(T + H)\mathbf{x} = \mathbf{f}$ when $T + H$ is nonsingular, $T = T^T$, and $JH = H^TJ$. We'll show that, if $\sigma(n)$ is the amount of computation required by a real *sine transform* of order n, then the computation of $(T + H)^{-1}\mathbf{f}$, where $T + H$ and $\mathbf{f}$ have real elements, using the formula (4.52)

$$(T + H)^{-1} = \left[\tau^2(\mathbf{x}_1) + I\right]\tau(\mathbf{w}_{.1}) - \tau^2(\mathbf{w}_{.1})\tau(\mathbf{x}_1) \qquad (5.1)$$

[$\mathbf{x}_1$ defined in (4.2)], requires at most $5\sigma(n) + 5\sigma(n - 2) + O(n)$ computations.

To prove this assertion first recall that all matrices of τ of order n are simultaneously diagonalized by the sine matrix S_n defined in Section 2, i.e.

$$\tau(\mathbf{z}) = \frac{2}{n + 1} S_n D_n(\mathbf{z}) S_n,$$

where

$$D_n(\mathbf{z}) = \mathrm{diag}\left[\left(\sin\frac{i\pi}{n+1}\right)^{-1}\mathbf{e}_i^T S_n\mathbf{z}, i = 1,\ldots,n\right].$$

Call the matrix by vector product $S_n\mathbf{z}$, $\mathbf{z} \in \mathbb{C}^n$, the *sine transform* of $\mathbf{z}$—more specifically, an *order n sine transform*. In [25] it is proved that the linear transform $S_n'\mathbf{z}$, $\mathbf{z} \in \mathbb{R}^{n+1}$, where $[S_n']_{ij} = \sin[(i-1)(j-1)\pi/(n+1)]$, $i,j = 1,\ldots,n+1$, can be obtained by computing a real Fourier transform of order $n+1$. The matrix S_n here considered is—apart the multiplicative factor $[2/(n+1)]^{1/2}$— the $n \times n$ lower right submatrix of S_n'; so an order n sine transform $S_n\mathbf{z}$, $\mathbf{z} \in \mathbb{R}^n$, can be computed by a FFT with $O((n+1)\log(n+1))$ arithmetic operations.

Let

$$\mathbf{z}_1 = S_n\mathbf{f},$$

$$\mathbf{z}_2^A = D_n(\mathbf{w}_{\cdot 1})\mathbf{z}_1, \qquad \mathbf{z}_2^B = D_n(\mathbf{x}_1)\mathbf{z}_1,$$

$$\mathbf{z}_3^A = S_n\mathbf{z}_2^A, \qquad \mathbf{z}_3^B = S_n\mathbf{z}_2^B,$$

$$\mathbf{z}_4^A = S_{n-2}\Omega_2\mathbf{z}_3^A, \qquad \mathbf{z}_4^B = S_{n-2}\Omega_2\mathbf{z}_3^B,$$

$$\mathbf{z}_5^A = D_{n-2}(\Omega_2\mathbf{x}_1)\mathbf{z}_4^A, \qquad \mathbf{z}_5^B = D_{n-2}(\Omega_2\mathbf{w}_{\cdot 1})\mathbf{z}_4^B,$$

$$\mathbf{z}_6 = \frac{2}{n-1}S_{n-2}(\mathbf{z}_5^A - \mathbf{z}_5^B).$$

Then

$$(T+H)^{-1}\mathbf{f}$$

$$= \frac{2}{n+1}\left[\begin{pmatrix} \mathbf{e}_1^T\mathbf{z}_6 + \mathbf{e}_1^T\mathbf{x}_1\mathbf{e}_1^T\mathbf{z}_3^A + \mathbf{e}_n^T\mathbf{x}_1\mathbf{e}_n^T\mathbf{z}_3^A - w_{11}\mathbf{e}_1^T\mathbf{z}_3^B - w_{n1}\mathbf{e}_n^T\mathbf{z}_3^B \\ \mathbf{z}_6 \\ \mathbf{e}_{n-2}^T\mathbf{z}_6 + \mathbf{e}_n^T\mathbf{x}_1\mathbf{e}_1^T\mathbf{z}_3^A + \mathbf{e}_1^T\mathbf{x}_1\mathbf{e}_n^T\mathbf{z}_3^A - w_{n1}\mathbf{e}_1^T\mathbf{z}_3^B - w_{11}\mathbf{e}_n^T\mathbf{z}_3^B \end{pmatrix} + \mathbf{z}_3^A\right].$$

Recall that if $\Delta = w_{11}w_{nn} - w_{n1}w_{1n} = w_{11}^2 - w_{n1}^2 \neq 0$, then

$$x_1 = \Delta^{-1}(w_{n1}J - w_{11}I)(T_2\mathbf{w}_{\cdot 1} - \mathbf{w}_{\cdot 2}).$$

It is clear that if $T+H$ and $\mathbf{f}$ have complex elements, then the number of sine transforms (five of order n plus five of order $n-2$) does not change. However, some (possibly all) of the transforms become complex sine trans-

forms, and an order n complex sine transform can be trivially obtained by computing two order n real sine transforms.

The expression (5.1) appears to be particularly economical and convenient for representing the inverse of a Toeplitz plus Hankel (centrosymmetric) matrix, as it is constructed only in terms of two products of τ matrices of order n or $n - 2$.

The formulas for the inverse of a general $T + H$, which could be explicitly derived from Bini-Pan decompositions [see Corollary 3.1(iii)] [7, 10], seem to be the only formulas present in the actual literature. As these formulas consist of sums of matrix products involving both τ and triangular Toeplitz matrices, they are less efficient than (5.1) and all other decompositions involving only τ matrices listed in the previous section.

The formula for the elements of $(T + H)^{-1}$ introduced by Rost and Heinig in [19] is essentially recursive and doesn't show an explicit representation of $(T + H)^{-1}$ like those listed in the present paper.

Notice that the results obtained by Jain, Ammar, and Gader [21, 5] for the second phase in the resolution of a symmetric Toeplitz system hold in the important but restrictive case $s_{11} \neq 0$. Thus the formula (5.1) could be competitive also for $H = 0$.

On the other hand, the implementation of (5.1), as well as the implementation of all formulas involving only τ matrices, requires the calculation of different order sine transforms (n and $n - 2$ or n and $n - 1$), and the fast transforms are not usually assumed to be of arbitrary size.

Efficient algorithms for fast Fourier transforms (an order n sine transform can be computed through an order $n + 1$ FFT) were recently developed for dimensions that are products of powers of small prime integers [1, 2, 29, 30]. If m ($m - 2$ or $m - 1$) is a convenient dimension for computing an FFT, then $m - 2$ or $m - 1$ (m) can be also a convenient dimension only in a limited number of cases. This fact could be observed in detail by considering all possible solutions of the equation $x + 2 = y$ or $x + 1 = y$ in the set of integers of the form $p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$, where the p_i's are fixed primes [3, 12, 26].

Further developments with formulas involving fast transforms of the same dimension will be considered in a future paper.

APPENDIX

Formulas for the inverse of a Toeplitz matrix:

(a) $\mathbb{C}_{Z^T}(T^{-1}) = \delta \hat{\mathbf{s}}_{\cdot n}^T - \mathbf{s}_{\cdot n} \hat{\delta}^T \Rightarrow$

(1) $T^{-1} = U(\hat{\delta}) L(Z\mathbf{s}_{\cdot n}) - U(\hat{\mathbf{s}}_{\cdot n}) [L(Z\delta) - I]$

(2) $\qquad = L(Zs_{\cdot n})U(\hat{\boldsymbol{\delta}}) - [L(Z\boldsymbol{\delta}) - I]U(\hat{s}_{\cdot n}),$ (4.8)

(b) $\mathfrak{C}_Z(T^{-1}) = \boldsymbol{\gamma}\hat{s}_{\cdot 1}^T - s_{\cdot 1}\hat{\boldsymbol{\gamma}}^T \Rightarrow$

(3) $\qquad T^{-1} = U(Z\hat{s}_{\cdot 1})L(\boldsymbol{\gamma}) - [U(Z\hat{\boldsymbol{\gamma}}) - I]L(s_{\cdot 1})$

(4) $\qquad = L(\boldsymbol{\gamma})U(Z\hat{s}_{\cdot 1}) - L(s_{\cdot 1})[U(Z\hat{\boldsymbol{\gamma}}) - I],$

(c) $\mathfrak{C}_P(T^{-1}) = s_{\cdot n}(\hat{e}_n^T - \hat{\boldsymbol{\delta}}^T) - (e_n - \boldsymbol{\delta})\hat{s}_{\cdot n}^T \Rightarrow$

(5) $\qquad T^{-1} = C(\hat{s}_{\cdot n})\big[L(Z(e_n - \boldsymbol{\delta})) + I\big] - C(\hat{e}_n - \hat{\boldsymbol{\delta}})L(Zs_{\cdot n})$

(6) $\qquad = \big[L(Z(e_n - \boldsymbol{\delta})) + I\big]C(\hat{s}_{\cdot n}) - L(Zs_{\cdot n})C(\hat{e}_n - \hat{\boldsymbol{\delta}}),$

(4.9)

(d) $\mathfrak{C}_{PT}(T^{-1}) = s_{\cdot 1}(\hat{e}_1^T - \hat{\boldsymbol{\gamma}}^T) - (e_1 - \boldsymbol{\gamma})\hat{s}_{\cdot 1}^T \Rightarrow$

(7) $\qquad T^{-1} = \big[U(Z(\hat{e}_1 - \hat{\boldsymbol{\gamma}})) + I\big]C(s_{\cdot 1})^T - U(Z\hat{s}_{\cdot 1})C(e_1 - \boldsymbol{\gamma})^T$

(8) $\qquad = C(s_{\cdot 1})^T\big[U(Z(\hat{e}_1 - \hat{\boldsymbol{\gamma}})) + I\big] - C(e_1 - \boldsymbol{\gamma})^T U(Z\hat{s}_{\cdot 1}),$

(e) $\mathfrak{C}_{T_2}(T^{-1}) = \boldsymbol{\gamma}\hat{s}_{\cdot 1}^T - s_{\cdot 1}\hat{\boldsymbol{\gamma}}^T + \boldsymbol{\delta}\hat{s}_{\cdot n}^T - s_{\cdot n}\hat{\boldsymbol{\delta}}^T \Rightarrow$

(9) $\qquad T^{-1} = \tau(\hat{\boldsymbol{\gamma}})L(Zs_{\cdot 1}) - \tau(\hat{s}_{\cdot 1})L(Z\boldsymbol{\gamma}) + \tau(\hat{\boldsymbol{\delta}})L(Zs_{\cdot n})$

$\qquad\qquad - \tau(\hat{s}_{\cdot n})\big[L(Z\boldsymbol{\delta}) - I\big]$

(10) $\qquad = L(Zs_{\cdot 1})\tau(\hat{\boldsymbol{\gamma}}) - L(Z\boldsymbol{\gamma})\tau(\hat{s}_{\cdot 1})$

$\qquad\qquad + L(Zs_{\cdot n})\tau(\hat{\boldsymbol{\delta}}) - \big[L(Z\boldsymbol{\delta}) - I\big]\tau(\hat{s}_{\cdot n})$ (4.10)

(11) $\qquad = U(Z\hat{s}_{\cdot 1})\tau(\boldsymbol{\gamma}) - \big[U(Z\hat{\boldsymbol{\gamma}}) - I\big]\tau(s_{\cdot 1}) + U(Z\hat{s}_{\cdot n})\tau(\boldsymbol{\delta})$

$\qquad\qquad - U(Z\hat{\boldsymbol{\delta}})\tau(s_{\cdot n})$

$$(12) \qquad = \tau(\boldsymbol{\gamma})U(Z\hat{\mathbf{s}}_{\cdot 1}) - \tau(\mathbf{s}_{\cdot 1})[U(Z\hat{\boldsymbol{\gamma}}) - I] + \tau(\boldsymbol{\delta})U(Z\hat{\mathbf{s}}_{\cdot n})$$

$$- \tau(\mathbf{s}_{\cdot n})U(Z\hat{\boldsymbol{\delta}}),$$

(f) $\mathfrak{C}_{P_\varepsilon}(T^{-1}) = \mathbf{s}_{\cdot n}(\varepsilon \hat{\mathbf{e}}_n^T - \hat{\boldsymbol{\delta}}^T) - (\varepsilon \mathbf{e}_n - \boldsymbol{\delta})\hat{\mathbf{s}}_{\cdot n}^T \Rightarrow$

$$(13) \quad (\varepsilon - \beta)T^{-1} = C_\varepsilon(\varepsilon \hat{\mathbf{e}}_n - \hat{\boldsymbol{\delta}})C_\beta(\hat{\mathbf{s}}_{\cdot n}) - C_\varepsilon(\hat{\mathbf{s}}_{\cdot n})C_\beta(\beta \hat{\mathbf{e}}_n - \hat{\boldsymbol{\delta}})$$

$$(14) \qquad = C_\beta(\hat{\mathbf{s}}_{\cdot n})C_\varepsilon(\varepsilon \hat{\mathbf{e}}_n - \hat{\boldsymbol{\delta}}) - C_\beta(\beta \hat{\mathbf{e}}_n - \hat{\boldsymbol{\delta}})C_\varepsilon(\hat{\mathbf{s}}_{\cdot n}),$$

$$(4.11)$$

(g) $\mathfrak{C}_{P_\varepsilon^T}(T^{-1}) = \mathbf{s}_{\cdot 1}(\varepsilon \hat{\mathbf{e}}_1^T - \hat{\boldsymbol{\gamma}}^T) - (\varepsilon \mathbf{e}_1 - \boldsymbol{\gamma})\hat{\mathbf{s}}_{\cdot 1}^T \Rightarrow$

(15)

$$(\varepsilon - \beta)T^{-1} = C_\beta(\mathbf{s}_{\cdot 1})^T C_\varepsilon(\varepsilon \mathbf{e}_1 - \boldsymbol{\gamma})^T - C_\beta(\beta \mathbf{e}_1 - \boldsymbol{\gamma})^T C_\varepsilon(\mathbf{s}_{\cdot 1})^T$$

$$(16) \qquad = C_\varepsilon(\varepsilon \mathbf{e}_1 - \boldsymbol{\gamma})^T C_\beta(\mathbf{s}_{\cdot 1})^T - C_\varepsilon(\mathbf{s}_{\cdot 1})^T C_\beta(\beta \mathbf{e}_1 - \boldsymbol{\gamma})^T,$$

(h) $\mathfrak{C}_{T_2}(T^{-1}) = \boldsymbol{\gamma}\hat{\mathbf{s}}_{\cdot 1}^T - \mathbf{s}_{\cdot 1}\hat{\boldsymbol{\gamma}}^T + \boldsymbol{\delta}\hat{\mathbf{s}}_{\cdot n}^T - \mathbf{s}_{\cdot n}\hat{\boldsymbol{\delta}}^T \Rightarrow$

$$(17) \qquad T^{-1} = \tau^1(\boldsymbol{\gamma})\tau(\hat{\mathbf{s}}_{\cdot 1}) - \tau^1(\mathbf{s}_{\cdot 1})\tau(\hat{\boldsymbol{\gamma}}) + [\tau^1(\boldsymbol{\delta}) + I]\tau(\hat{\mathbf{s}}_{\cdot n})$$

$$- \tau^1(\mathbf{s}_{\cdot n})\tau(\hat{\boldsymbol{\delta}}) \qquad (4.12)$$

$$(18) \qquad = \tau(\mathbf{s}_{\cdot 1})[\tau^1(\hat{\boldsymbol{\gamma}})^T + I] - \tau(\boldsymbol{\gamma})\tau^1(\hat{\mathbf{s}}_{\cdot 1})^T + \tau(\mathbf{s}_{\cdot n})\tau^1(\hat{\boldsymbol{\delta}})^T$$

$$- \tau(\boldsymbol{\delta})\tau^1(\hat{\mathbf{s}}_{\cdot n})^T \qquad (4.13)$$

$$(19) \qquad = [\tau^3(\boldsymbol{\gamma}) + I]\tau(\mathbf{s}_{\cdot 1}) - \tau^3(\mathbf{s}_{\cdot 1})\tau(\boldsymbol{\gamma}) + \tau^3(\boldsymbol{\delta})\tau(\mathbf{s}_{\cdot n})$$

$$- \tau^3(\mathbf{s}_{\cdot n})\tau(\boldsymbol{\delta}) \qquad (4.14)$$

$$(20) \qquad = \tau(\hat{\mathbf{s}}_{\cdot 1})\tau^3(\hat{\boldsymbol{\gamma}})^T - \tau(\hat{\boldsymbol{\gamma}})\tau^3(\hat{\mathbf{s}}_{\cdot 1})^T + \tau(\hat{\mathbf{s}}_{\cdot n})[\tau^3(\hat{\boldsymbol{\delta}})^T + I]$$

$$- \tau(\hat{\boldsymbol{\delta}})\tau^3(\hat{\mathbf{s}}_{\cdot n})^T. \qquad (4.15)$$

Formulas for the inverse of a symmetric Toeplitz matrix $[\mathbf{s}_{\cdot 1} = \hat{\mathbf{s}}_{\cdot n},$
$\hat{\delta} = \gamma + (t_n - t_{-n})\mathbf{s}_{\cdot 1}]$:

(1)–(8) identical to those of the general case,

(a) $\mathfrak{C}_{T_2}(T^{-1}) = \gamma\hat{\mathbf{s}}_{\cdot 1}^T - \mathbf{s}_{\cdot 1}\hat{\gamma}^T + \hat{\gamma}\mathbf{s}_{\cdot 1}^T - \hat{\mathbf{s}}_{\cdot 1}\gamma^T \Rightarrow$

(9) $\qquad T^{-1} = \tau(\hat{\gamma})[L(Z\mathbf{s}_{\cdot 1}) + JL(Z\hat{\mathbf{s}}_{\cdot 1})]$

$\qquad\qquad - \tau(\mathbf{s}_{\cdot 1})[L(Z\hat{\gamma}) + JL(Z\gamma) - I]$

(10) $\qquad = [L(Z\mathbf{s}_{\cdot 1}) + L(Z\hat{\mathbf{s}}_{\cdot 1})J]\tau(\hat{\gamma})$

$\qquad\qquad - [L(Z\hat{\gamma}) + L(Z\gamma)J - I]\tau(\mathbf{s}_{\cdot 1})$ $\qquad$ (4.17)

(11) $\qquad = [U(Z\mathbf{s}_{\cdot 1}) + U(Z\hat{\mathbf{s}}_{\cdot 1})J]\tau(\hat{\gamma})$

$\qquad\qquad - [U(Z\hat{\gamma}) + U(Z\gamma)J - I]\tau(\mathbf{s}_{\cdot 1})$

(12) $\qquad = \tau(\hat{\gamma})[U(Z\mathbf{s}_{\cdot 1}) + JU(Z\hat{\mathbf{s}}_{\cdot 1})]$

$\qquad\qquad - \tau(\mathbf{s}_{\cdot 1})[U(Z\hat{\gamma}) + JU(Z\gamma) - I],$

(13)–(16) identical to those of the general case,

(b) $\mathfrak{C}_{T_2}(T^{-1}) = \gamma\hat{\mathbf{s}}_{\cdot 1}^T - \mathbf{s}_{\cdot 1}\hat{\gamma}^T + \hat{\gamma}\mathbf{s}_{\cdot 1}^T - \hat{\mathbf{s}}_{\cdot 1}\gamma^T \Rightarrow$

(17) $\qquad T^{-1} = [\tau^1(\hat{\gamma}) + \tau^1(\gamma)J + I]\tau(\mathbf{s}_{\cdot 1})$

$\qquad\qquad - [\tau^1(\mathbf{s}_{\cdot 1}) + \tau^1(\hat{\mathbf{s}}_{\cdot 1})J]\tau(\hat{\gamma})$ $\qquad$ (4.18)

(18) $\qquad = \tau(\mathbf{s}_{\cdot 1})[\tau^1(\hat{\gamma})^T + J\tau^1(\gamma)^T + I]$

$\qquad\qquad - \tau(\hat{\gamma})[\tau^1(\mathbf{s}_{\cdot 1})^T + J\tau^1(\hat{\mathbf{s}}_{\cdot 1})^T]$ $\qquad$ (4.19)

(19) $\qquad = [\tau^3(\gamma) + \tau^3(\hat{\gamma})J + I]\tau(\mathbf{s}_{\cdot 1})$

$\qquad\qquad - [\tau^3(\hat{\mathbf{s}}_{\cdot 1}) + \tau^3(\mathbf{s}_{\cdot 1})J]\tau(\hat{\gamma})$ $\qquad$ (4.20)

(20) $\qquad = \tau(\mathbf{s}_{\cdot 1})[\tau^3(\gamma)^T + J\tau^3(\hat{\gamma})^T + I]$

$\qquad\qquad - \tau(\hat{\gamma})[\tau^3(\hat{\mathbf{s}}_{\cdot 1})^T + J\tau^3(\mathbf{s}_{\cdot 1})^T],$ $\qquad$ (4.21)

(21) $\qquad T^{-1} = [\tau^2(\hat{\gamma}) + I]\tau(\mathbf{s}_{\cdot 1}) - \tau^2(\mathbf{s}_{\cdot 1})\tau(\hat{\gamma})$ $\qquad$ (4.22)

(22) $\qquad = \tau(\mathbf{s}_{\cdot 1})[\tau^2(\hat{\gamma})^T + I] - \tau(\hat{\gamma})\tau^2(\mathbf{s}_{\cdot 1})^T.$ $\qquad$ (4.23)

γ and δ are defined in (4.1). They can be always expressed in terms of some columns of T^{-1} (Proposition 4.3). For instance, if $s_{11} = s_{nn} \neq 0$, then

$$\gamma = -\frac{1}{s_{nn}} Z s_{\cdot n}, \qquad \delta = -\frac{1}{s_{11}} Z^T s_{\cdot 1}.$$

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