Linear Algebra and its Applications 432 (2010) 3210–3230

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

A complete solution of the normal Hankel problem

V.N. Chugunov *, Kh.D. Ikramov

Institute of Numerical Mathematics, Russian Academy of Sciences, ul.Gubkina 8, Moscow 119991, Russia
Faculty of Computational Mathematics and Cybernetics, Moscow State University, Leninskie Gory, Moscow 119992, Russia

ARTICLE INFO

Article history:
Received 15 June 2009
Accepted 12 January 2010
Available online 18 February 2010

Submitted by R. Horn

AMS classification:
15A21

Keywords:
Hankel matrix
Normal matrix
Toeplitz matrix
Backward identity
Circulant
Hankel circulant
Separable circulant

ABSTRACT

The normal Hankel problem is the one of characterizing the matrices that are normal and Hankel at the same time. We give a complete solution of this problem.

1. Introduction

The normal Toeplitz problem (NTP) is the one of characterizing the matrices that are normal and Toeplitz at the same time. This problem was posed and solved by the authors in [8,9,11]. (Other solutions of this problem were proposed in [1,5–7,18].)

The normal Hankel problem (NHP) is the one of characterizing the matrices that are normal and Hankel at the same time. It turned out to be much harder than the NTP and was open for many years (see [2,3,10,12–17]). In this paper, we give a complete solution of this problem.

* Corresponding author. Address: Institute of Numerical Mathematics, Russian Academy of Sciences, ul.Gubkina 8, Moscow 119991, Russia.

E-mail address: vadim@bach.inm.ras.ru (V.N. Chugunov).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2010.01.021

Let $\mathcal{NH}_n$ be the set of normal Hankel matrices of order n. With each matrix $H \in \mathcal{NH}_n$, we associate the Toeplitz matrix

$$T = H\mathcal{P}_n, \tag{1}$$

where

$$\mathcal{P}_n = \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{pmatrix}$$

is the backward identity matrix of order n. One can easily verify the following proposition:

Proposition 1. *A Hankel matrix H is normal if and only if the matrix TT^* is real*; *that is,*

$$\operatorname{Im} TT^* = 0. \tag{2}$$

Proposition 1 implies that, instead of characterizing $\mathcal{NH}_n$, we may describe the corresponding Toeplitz matrices.

Suppose that matrix (1) is written in the algebraic form

$$T = T_1 + iT_2, \tag{3}$$

where

$$T_1 = \frac{T + \overline{T}}{2}, \quad T_2 = \frac{T - \overline{T}}{2i}. \tag{4}$$

As usual, the bar over the symbol of a matrix or a vector denotes the entry-wise complex conjugation.

Substituting (3) into (2), we obtain yet another normality condition for the original matrix H.

Proposition 2. *A Hankel matrix H is normal if and only if*

$$T_1T_2^t = T_2T_1^t. \tag{5}$$

Let $a_1, \ldots, a_{n-1}$ and $a_{-1}, \ldots, a_{-n+1}$ be the off-diagonal entries in the first row and the first column of T_1. Similarly, let $b_1, \ldots, b_{n-1}$ and $b_{-1}, \ldots, b_{-n+1}$ be the off-diagonal entries in the first row and the first column of T_2. Form the matrices

$$F = \begin{pmatrix} a_{n-1} & b_{n-1} \\ a_{n-2} & b_{n-2} \\ \vdots & \vdots \\ a_1 & b_1 \end{pmatrix} \tag{6}$$

and

$$G = \begin{pmatrix} a_{-1} & b_{-1} \\ a_{-2} & b_{-2} \\ \vdots & \vdots \\ a_{-n+1} & b_{-n+1} \end{pmatrix}. \tag{7}$$

In Section 2, we show that, if both matrices F and G are rank-deficient, then H must belong to one of the following four classes:

1. Arbitrary complex multiples of real Hankel matrices.
2. Matrices of the form

   $$\alpha\mathcal{P}_n + \beta H, \quad \alpha, \beta \in \mathbf{C},$$

   where H is an arbitrary real centrosymmetric Hankel matrix.

3. Block diagonal matrices of the form

$$\alpha H_1 \oplus \beta H_2, \quad \alpha, \beta \in \mathbf{C},$$

where H_1 is a real upper triangular Hankel matrix of order k (with $0 < k < n$) and H_2 is a real lower triangular Hankel matrix of order $l = n - k$. We call H_1 and H_2 an upper triangular and a lower triangular Hankel matrix, respectively, if

$$\{H_1\}_{ij} = 0 \quad \text{for } i + j > k + 1$$

and

$$\{H_2\}_{ij} = 0 \quad \text{for } i + j < l + 1.$$

4. Matrices of the form

$$\alpha H + \beta H^{-1}, \quad \alpha, \beta \in \mathbf{C},$$

where H a nonsingular real upper triangular (or lower triangular) Hankel matrix.

Now, assume that at least one of the matrices F and G has full rank. In Section 3, we show that, in this case, both F and G have rank two and obey the relation

$$G = FW \tag{8}$$

for some real 2×2 matrix

$$W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{9}$$

with a unit determinant:

$$\alpha\delta - \beta\gamma = 1. \tag{10}$$

In view of definitions (6) and (7), matrix equality (8) is equivalent to the scalar relations

$$a_{-i} = \alpha a_{n-i} + \gamma b_{n-i}, \quad b_{-i} = \beta a_{n-i} + \delta b_{n-i}, \quad 1 \leqslant i \leqslant n-1. \tag{11}$$

Writing the Toeplitz matrix (1) in the form

$$T = \begin{pmatrix} t_0 & t_1 & t_2 & \dots & t_{n-1} \\ t_{-1} & t_0 & t_1 & \dots & t_{n-2} \\ t_{-2} & t_{-1} & t_0 & \dots & t_{n-3} \\ \dots & \dots & \dots & \dots & \dots \\ t_{-n+1} & t_{-n+2} & t_{-n+3} & \dots & t_0 \end{pmatrix}, \tag{12}$$

we can replace real relations (11) with the complex formula

$$t_{-i} = \phi t_{n-i} + \psi \overline{t_{n-i}}, \quad 1 \leqslant i \leqslant n-1, \tag{13}$$

where

$$\phi = \frac{\alpha + \delta}{2} + i\frac{\beta - \gamma}{2}, \quad \psi = \frac{\alpha - \delta}{2} + i\frac{\beta + \gamma}{2}. \tag{14}$$

Then, relation (10) takes the complex form

$$|\phi|^2 - |\psi|^2 = 1. \tag{15}$$

The case $\psi = 0$, $|\phi| = 1$ corresponds to the well-known class of ϕ-circulants. For this reason, matrices defined by relation (13) for a fixed pair (ϕ, ψ) were called (ϕ, ψ)-circulants in [14].

Thus, beginning from Section 4, we deal only with various classes of (ϕ, ψ)-circulants. Each of these classes is specified by the corresponding matrix W (see (9)). In Section 4, we prove an important lemma that shows that the case of a general matrix W obeying relation (10) can be reduced to W having diagonal or Jordan form. Consequently, the following four cases must be distinguished:

1. The eigenvalues λ_1 and λ_2 of W are complex conjugate.
2. The eigenvalues λ_1 and λ_2 are real and distinct.
3. The eigenvalues λ_1 and λ_2 are identical, and W is diagonalizable.
4. The eigenvalues λ_1 and λ_2 are identical, and the Jordan form of W is a Jordan block of order two.

These four cases are dealt with in Sections 5–8.

2. Rank-deficient case

Suppose that one of the matrices T_1 and T_2 in (3) is diagonal and nonzero. For definiteness, assume that

$$T_1 = \alpha I_n.$$

Then, relation (5) says that T_2 is a (real) symmetric Toeplitz matrix. The corresponding matrix $H = T\mathcal{P}_n$ has the form

$$H = \alpha\mathcal{P}_n + iT_2\mathcal{P}_n$$

and, hence, belongs to class 2. Therefore, in what follows, we assume that neither T_1 nor T_2 is diagonal.

It will be convenient to isolate the diagonal parts in both T_1 and T_2:

$$T_1 = a_0 I_n + \widehat{T}_1, \quad T_2 = b_0 I_n + \widehat{T}_2. \tag{16}$$

Here, $\widehat{T}_1$ and $\widehat{T}_2$ have the zero principal diagonal.

Substituting (16) into (5), we obtain

$$\widehat{T}_2\widehat{T}_1^t - \widehat{T}_1\widehat{T}_2^t = a_0(\widehat{T}_2^t - \widehat{T}_2) - b_0(\widehat{T}_1^t - \widehat{T}_1). \tag{17}$$

This equation characterizes all the matrices T corresponding to the matrices in $\mathcal{NH}_n$. In particular, (17) implies that, for every such T, the matrix

$$\widehat{T}_2\widehat{T}_1^t - \widehat{T}_1\widehat{T}_2^t$$

must be Toeplitz. Let us discuss the consequences of this fact.

From the equalities

$$(\widehat{T}_2\widehat{T}_1^t - \widehat{T}_1\widehat{T}_2^t)_{i+1,j+1} = (\widehat{T}_2\widehat{T}_1^t - \widehat{T}_1\widehat{T}_2^t)_{i,j}, \quad i,j = 1,\ldots,n-1,$$

we derive

$$\sum_{k=1}^{n}(\widehat{T}_2)_{i+1,k}(\widehat{T}_1)_{j+1,k} - \sum_{k=1}^{n}(\widehat{T}_1)_{i+1,k}(\widehat{T}_2)_{j+1,k} - \sum_{k=1}^{n}(\widehat{T}_2)_{ik}(\widehat{T}_1)_{jk} + \sum_{k=1}^{n}(\widehat{T}_1)_{ik}(\widehat{T}_2)_{jk} = 0$$

or

$$\sum_{k=1}^{n} b_{k-i-1}a_{k-j-1} - \sum_{k=1}^{n} a_{k-i-1}b_{k-j-1} - \sum_{k=1}^{n} b_{k-i}a_{k-j} + \sum_{k=1}^{n} a_{k-i}b_{k-j} = 0,$$
$$i,j = 1,\ldots,n-1.$$

We change here the summation indices; namely, we set $m = k-1$ in the first and second sums and $m = k$ in the third and fourth sums. This yields

$$\sum_{m=0}^{n-1} b_{m-i}a_{m-j} - \sum_{m=0}^{n-1} a_{m-i}b_{m-j} - \sum_{m=1}^{n} b_{m-i}a_{m-j} + \sum_{m=1}^{n} a_{m-i}b_{m-j} = 0,$$

whence

$$a_{n-i}b_{n-j} - a_{n-j}b_{n-i} = a_{-i}b_{-j} - a_{-j}b_{-i}, \quad i,j = 1,\ldots,n-1. \tag{18}$$

Define

$$\Delta_{ij}^F = \det\begin{pmatrix} a_{n-i} & b_{n-i} \\ a_{n-j} & b_{n-j} \end{pmatrix} = a_{n-i}b_{n-j} - a_{n-j}b_{n-i}$$

and

$$\Delta_{ij}^G = \det\begin{pmatrix} a_{-i} & b_{-i} \\ a_{-j} & b_{-j} \end{pmatrix} = a_{-i}b_{-j} - a_{-j}b_{-i}.$$

Now, relation (18) take the form

$$\Delta_{ij}^F = \Delta_{ij}^G, \quad i,j = 1,\ldots,n-1. \tag{19}$$

So far, the ranks of the matrices F and G were not important. Now, for the rest of this section, we assume that

$$\operatorname{rank} F < 2 \quad \text{and} \quad \operatorname{rank} G < 2.$$

The case

$$\operatorname{rank} F = \operatorname{rank} G = 0$$

is clearly impossible since, otherwise, T_1 and T_2 would be diagonal matrices. For the other values of rank$F$ and rank$G$ we give separate analyses.

It will be convenient to use the following notation. For a vector $f = (f_1, f_2, \ldots, f_{n-1})^t$, the upper triangular Toeplitz matrix with the first row

$$(0\, f_1\, f_2\, \ldots\, f_{n-1})$$

is denoted by $\mathcal{T}(f)$. The symbols u_1 and u_2 stand for the columns of F, while the columns of G are denoted by l_1 and l_2.

2.1. $F \neq 0,\ G = 0$

In this case, we have $\Delta_{ij}^G = 0\ \forall i,j$. In view of (19), $\Delta_{ij}^F = 0$ for all i,j; hence, $\operatorname{rank} F = 1$.
Let

$$c = (c_{n-1}, c_{n-2}, \ldots, c_1)^t$$

be a real vector such that $u_1 = \alpha c$ and $u_2 = \beta c$ for real scalars α and β satisfying the condition

$$\alpha^2 + \beta^2 \neq 0. \tag{20}$$

Define the matrix $U = \mathcal{T}(\mathcal{P}_{n-1}c)$; then, $\widehat{T}_1 = \alpha U$ and $\widehat{T}_2 = \beta U$. Substituting these expressions into (17), we have

$$(a_0\beta - b_0\alpha)U^t - (a_0\beta - b_0\alpha)U = 0. \tag{21}$$

Since U is a nonzero strictly upper triangular matrix, equality (21) is equivalent to the relation

$$(a_0\beta - b_0\alpha)U = 0,$$

that is, to the relation $a_0\beta - b_0\alpha = 0$, which can be written in the form

$$\det\begin{pmatrix} a_0 & b_0 \\ \alpha & \beta \end{pmatrix} = 0.$$

In view of (20), there exists a (real) nonzero scalar κ such that

$$a_0 = \kappa\alpha, \quad b_0 = \kappa\beta.$$

Now, representations (16) take the form

$$T_1 = a_0 I_n + \widehat{T}_1 = \kappa\alpha I_n + \alpha U = \alpha(\kappa I_n + U),$$
$$T_2 = b_0 I_n + \widehat{T}_2 = \kappa\beta I_n + \beta U = \beta(\kappa I_n + U),$$

which says that T_1 and T_2 are scalar multiples of the same real upper triangular matrix. In terms of the original Hankel problem, this means that the normal matrix H is the product of a complex scalar and a real upper triangular Hankel matrix. In this case, H belongs to class 1.

2.2. $G \neq 0,\ F = 0$

This case comes under the analysis of the preceding subsection if F and G change places. The corresponding normal matrix H is the product of a complex scalar and a real lower triangular Hankel matrix. Thus, H again belongs to class 1.

2.3. $\operatorname{rank} F = 1,\ G \neq 0$

In view of relations (19), all the determinants $\Delta_{ij}^{G} = 0$; hence, $\operatorname{rank} G \leqslant 1$. Since $G \neq 0$, we have $\operatorname{rank} G = 1$.

Using, as before, the condition $\operatorname{rank} F = 1$, we find real scalars α and β satisfying relation (20) and a real vector

$$c = (c_{n-1}, c_{n-2}, \ldots, c_1)^t,$$

such that $u_1 = \alpha c$ and $u_2 = \beta c$. Using similarly the condition $\operatorname{rank} G = 1$, we conclude that there exist real scalars γ and δ satisfying the relation

$$\gamma^2 + \delta^2 \neq 0 \tag{22}$$

and a real vector

$$d = (d_{-1}, d_{-2}, \ldots, d_{-n+1})^t,$$

such that $l_1 = \gamma d$ and $l_2 = \delta d$. Define the strictly upper triangular matrix $U = \mathcal{T}(\mathcal{P}_{n-1}c)$ and the strictly lower triangular matrix $L = (\mathcal{T}(d))^t$; then,

$$\widehat{T}_1 = \alpha U + \gamma L, \quad \widehat{T}_2 = \beta U + \delta L. \tag{23}$$

Easy calculations yield

$$\widehat{T}_2\widehat{T}_1^t - \widehat{T}_1\widehat{T}_2^t = (\alpha\delta - \beta\gamma)(LU^t - UL^t). \tag{24}$$

Substituting (24) and (23) into (17), we find

$$\begin{aligned}(\alpha\delta - \beta\gamma)(LU^t - UL^t) = (a_0\beta - b_0\alpha)U^t - (a_0\delta - b_0\gamma)L \\ + (a_0\delta - b_0\gamma)L^t - (a_0\beta - b_0\alpha)U.\end{aligned} \tag{25}$$

It is easy to see that this equality is equivalent to the simpler relation

$$(\alpha\delta - \beta\gamma)UL^t = (a_0\beta - b_0\alpha)U - (a_0\delta - b_0\gamma)L^t. \tag{26}$$

To simplify our subsequent arguments, we define the quantities

$$\xi = \alpha\delta - \beta\gamma, \quad \xi_1 = a_0\beta - b_0\alpha, \quad \xi_2 = b_0\gamma - a_0\delta. \tag{27}$$

With the new notation, Eq. (26) takes the form

$$\xi UL^t = \xi_1 U + \xi_2 L^t. \tag{28}$$

The analysis of this equation will again be divided into several subcases.

2.3.1. $\xi = 0$

This condition means that the matrix

$$R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

has a zero determinant. Since R has no zero rows (see (20) and (22)), there exists a (real) nonzero scalar κ such that

$$\gamma = \kappa\alpha, \quad \delta = \kappa\beta. \tag{29}$$

Substituting (29) into (23) yields

$$\widehat{T}_1 = \alpha U + \kappa\alpha L = \alpha(U + \kappa L), \quad \widehat{T}_2 = \beta U + \kappa\beta L = \beta(U + \kappa L).$$

Thus, $\widehat{T}_1$ and $\widehat{T}_2$ are scalar multiples of the same matrix $T_3 = U + \kappa L$:

$$\widehat{T}_1 = \alpha T_3, \quad \widehat{T}_2 = \beta T_3.$$

Substituting these expressions into (17), we obtain

$$(a_0\beta - b_0\alpha)(T_3^t - T_3) = 0. \tag{30}$$

In particular, this relation is fulfilled if T_3 is a symmetric matrix. Then (see (16)), we have

$$T_1 = a_0 I_n + \alpha T_3, \quad T_2 = b_0 I_n + \beta T_3 \tag{31}$$

and T is a linear combination (with complex coefficients) of the identity matrix and the real symmetric Toeplitz matrix T_3. Being a symmetric Toeplitz matrix, T is centrosymmetric, and the last property is preserved when we turn to the matrix H. Thus, in the case under discussion, H is a linear combination (with complex coefficients) of the backward identity matrix $\mathcal{P}_n$ and a real centrosymmetric Hankel matrix; that is, H belongs to class 2.

If T_3 is nonsymmetric, then (30) converts into the equality $a_0\beta - b_0\alpha = 0$, which can be interpreted as a singularity requirement for the matrix

$$Z = \begin{pmatrix} \alpha & \beta \\ a_0 & b_0 \end{pmatrix}. \tag{32}$$

From (20), we conclude that there exists a (real) nonzero scalar η such that $a_0 = \eta\alpha$ and $b_0 = \eta\beta$. Then (see (31)), we have

$$T_1 = \eta\alpha I_n + \alpha T_3 = \alpha(\eta I_n + T_3), \quad T_2 = \eta\beta I_n + \beta T_3 = \beta(\eta I_n + T_3).$$

The corresponding H is a complex multiple of a real Hankel matrix; that is, H belongs to class 1.

2.3.2. $\xi \neq 0$

Define

$$\mu_1 = -\xi_1/\xi, \quad \mu_2 = -\xi_2/\xi \tag{33}$$

and rewrite (28) by dividing both sides by ξ:

$$UL^t = -\mu_1 U - \mu_2 L^t. \tag{34}$$

We give this equality the form

$$(U + \mu_2 I_n)(L^t + \mu_1 I_n) = \mu_1\mu_2 I_n. \tag{35}$$

The analysis of Eq. (35) is divided into several subcases corresponding to various values of the coefficients μ_1 and μ_2.

2.3.2.1. $\mu_1 = 0, \mu_2 = 0$. According to the definition of ξ_1 and ξ_2, we have the system of equations

$$\begin{cases} a_0\beta - b_0\alpha = 0, \\ a_0\delta - b_0\gamma = 0, \end{cases}$$

with respect to a_0 and b_0 with the nonzero determinant $-\xi$. Therefore,

$$a_0 = b_0 = 0.$$

Equality (35) takes the form $UL^t = 0$. Recall that both factors here are strictly upper triangular Toeplitz matrices. Denote by k_U the index of the first nonzero superdiagonal in U and by k_L the analogous index for L^T. Then, U and L can be represented in the block form

$$U = \begin{pmatrix} O_{(n-k_U+1)(k_U-1)} & \widetilde{U} \\ O_{(k_U-1)(k_U-1)} & O_{(k_U-1)(n-k_U+1)} \end{pmatrix},$$

$$L = \begin{pmatrix} O_{(k_L-1)(n-k_L+1)} & O_{(k_L-1)(k_L-1)} \\ \widetilde{L} & O_{(n-k_L+1)(k_L-1)} \end{pmatrix},$$

where $\widetilde{U}$ and $\widetilde{L}$ are square matrices and the symbol $0_{n_1 n_2}$ stands for the zero matrix of size $n_1 \times n_2$.

The condition $UL^t = 0$ implies the relation

$$k_U + k_L \geqslant n + 2$$

or

$$k_U - 1 \geqslant n - k_L + 1.$$

It follows that the column groups containing the submatrices $\widetilde{U}$ and $\widetilde{L}$ have nonoverlapping index sets. This means that the corresponding Hankel matrix H is a complex linear combination of the form

$$H = \alpha H_1 \oplus \beta H_2,$$

where H_1 and H_2 are real Hankel matrices; moreover, H_1 is upper triangular, while H_2 is lower triangular. In other words, H belongs to class 3.

2.3.2.2. $\mu_1 = 0, \mu_2 \neq 0$. In this case, Eq. (35) simplifies to the form

$$(U + \mu_2 I_n)L^t = 0. \tag{36}$$

The upper triangular Toeplitz matrix $U + \mu_2 I_n$ with the nonzero diagonal entry μ_2 is nonsingular; hence, (36) implies that $L = 0$. From the condition $\mu_1 = \xi_1 = 0$, we deduce the existence of a (real) nonzero scalar κ such that

$$a_0 = \kappa\alpha, \quad b_0 = \kappa\beta.$$

Using these relations, formula (23), and the equality $L = 0$ along with representation (16), we obtain

$$T_1 = a_0 I_n + \widehat{T}_1 = \kappa\alpha I_n + \alpha U = \alpha(\kappa I_n + U),$$
$$T_2 = b_0 I_n + \widehat{T}_2 = \kappa\beta I_n + \beta U = \beta(\kappa I_n + U).$$

The corresponding H is a complex multiple of a real upper triangular Hankel matrix; that is, H belongs to class 1.

2.3.2.3. $\mu_1 \neq 0, \mu_2 = 0$. This case is analogous to the preceding one up to changing the roles of μ_1 and μ_2. Eq. (35) takes the form

$$U(L^t + \mu_1 I_n) = 0.$$

Since $L + \mu_1 I_n$ is a nonsingular matrix, we have $U = 0$. From the condition $\mu_2 = \xi_2 = 0$, we deduce the existence of a (real) nonzero scalar κ such that

$$a_0 = \kappa\gamma, \quad b_0 = \kappa\delta.$$

Using these relations, formulas (23), and the equality $U = 0$ along with representation (16), we obtain

$$T_1 = a_0 I_n + \widehat{T}_1 = \kappa\gamma I_n + \gamma L = \gamma(\kappa I_n + L),$$
$$T_2 = b_0 I_n + \widehat{T}_2 = \kappa\delta I_n + \delta L = \delta(\kappa I_n + L).$$

The corresponding H is a complex multiple of a real lower triangular Hankel matrix; that is, H belongs to class 1.

2.3.2.4. $\mu_1 \neq 0, \mu_2 \neq 0$. Rewrite (35) in the form

$$\left(\frac{1}{\mu_2}U + I_n\right)\left(\frac{1}{\mu_1}L^t + I_n\right) = I_n \tag{37}$$

and define the matrix

$$R = \frac{1}{\mu_2}U + I_n. \tag{38}$$

The triangular matrix R has the unit diagonal and, hence, is invertible. Thus, we find from (38) and (37) that

$$U = \mu_2(R - I_n), \quad L = \mu_1(R^{-1} - I_n)^t.$$

Substituting these expressions into (23) yields

$$\widehat{T}_1 = \alpha\mu_2(R - I_n) + \gamma\mu_1(R^{-1} - I_n)^t,$$
$$\widehat{T}_2 = \beta\mu_2(R - I_n) + \delta\mu_1(R^{-1} - I_n)^t,$$

whence

$$T_1 = \alpha\mu_2 R + \gamma\mu_1 R^{-t} + (a_0 - \alpha\mu_2 - \gamma\mu_1)I_n,$$
$$T_2 = \beta\mu_2 R + \delta\mu_1 R^{-t} + (b_0 - \beta\mu_2 - \delta\mu_1)I_n.$$

We show that the coefficients of the identity matrices in these formulas are equal to zero. According to (33) and (27), we have

$$\mu_1 = -\frac{\xi_1}{\xi} = \frac{\alpha b_0 - \beta a_0}{\alpha\delta - \beta\gamma}, \quad \mu_2 = -\frac{\xi_2}{\xi} = \frac{\delta a_0 - \gamma b_0}{\alpha\delta - \beta\gamma}.$$

Consequently,

$$a_0 - \alpha\mu_2 - \gamma\mu_1 = \frac{\alpha\delta a_0 - \beta\gamma a_0 - \alpha\delta a_0 + \alpha\gamma b_0 - \alpha\gamma b_0 + \beta\gamma a_0}{\alpha\delta - \beta\gamma} = 0$$

and

$$b_0 - \beta\mu_2 - \delta\mu_1 = \frac{\alpha\delta b_0 - \beta\gamma b_0 - \beta\delta a_0 + \beta\gamma b_0 - \alpha\delta b_0 + \beta\delta a_0}{\alpha\delta - \beta\gamma} = 0.$$

Thus,

$$T_1 = \alpha\mu_2 R + \gamma\mu_1 R^{-t}, \quad T_2 = \beta\mu_2 R + \delta\mu_1 R^{-t}.$$

The corresponding H is a linear combination (with complex coefficients) of the nonsingular real upper triangular Hankel matrix $H_1 = R\mathcal{P}_n$ and the lower triangular Hankel matrix H_1^{-1}; that is, H belongs to class 4.

3. Full-rank case

Suppose that one of the matrices F and G has rank two. Then, equalities (19) imply that, in fact, both F and G are full-rank matrices. Moreover, these equalities say that F and G have the same second compound matrix or, in other terms, they define the same bivector. Geometrically, this fact means that the columns of F span the same subspace as the columns of G (see [4, Chapter X, Section 3]). Consequently, there exists a (real) 2×2 matrix

$$W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

such that

$$G = FW. \tag{39}$$

Choose any pair of indices i and j such that

$$\Delta_{ij}^F \neq 0.$$

Then, equalities (19) and (39) imply that

$$\det W = 1. \tag{40}$$

Let W be a fixed real matrix obeying (40). Define the complex scalars ϕ and ψ by formulas (14) and consider all the Toeplitz matrices satisfying relations (13). These are exactly matrices for which equality (39) holds with a chosen matrix W. As already said in Section 1, such matrices are called (ϕ, ψ)-circulants. The corresponding Hankel matrices $H = T\mathcal{P}_n$ will be called Hankel (ϕ, ψ)-circulants.

Below, we denote the set of (ϕ, ψ)-circulants corresponding to a fixed pair (ϕ, ψ) by the symbol $\mathcal{C}(\phi, \psi)$. In particular, $\mathcal{C}(1, 0)$ and $\mathcal{C}(-1, 0)$ are the classes of conventional circulants and skew-circulants, respectively.

4. Basic lemma

Let

$$V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \tag{41}$$

be a fixed real nonsingular 2×2 matrix. We say that the class $\mathcal{C}(\phi, \psi)$ undergoes the V-transformation if every matrix

$$T = T_1 + iT_2, \quad T \in \mathcal{C}(\phi, \psi), \tag{42}$$

is replaced by

$$\widetilde{T} = \widetilde{T}_1 + i\widetilde{T}_2 = (v_{11}T_1 + v_{21}T_2) + i(v_{12}T_1 + v_{22}T_2). \tag{43}$$

Lemma 1. *Let $\mathcal{C}(\phi, \psi)$ be the class of (ϕ, ψ)-circulants associated with the matrix W. Then, the V-transformation of this class is the class of $(\tilde{\phi}, \tilde{\psi})$-circulants associated with the matrix*

$$\widetilde{W} = V^{-1}WV. \tag{44}$$

If $T \in \mathcal{C}(\phi, \psi)$ produces the normal Hankel matrix $H = T\mathcal{P}_n$, then the same is true of its V-transformation $\widetilde{T} \in \mathcal{C}(\tilde{\phi}, \tilde{\psi})$.

Proof. We denote by $\tilde{a}_i, \tilde{a}_{-i}, \tilde{b}_i, \tilde{b}_{-i}, \widetilde{F}$, and $\widetilde{G}$ the counterparts of the values $a_i, a_{-i}, b_i, b_{-i}, F$, and G related to T_1 and T_2. From definition (43), we derive

$$\widetilde{F} = FV$$

and

$$\widetilde{G} = GV.$$

Taking (39) into account, we have

$$\widetilde{G} = GV = FWV = \widetilde{F}V^{-1}WV = \widetilde{F}\widetilde{W}.$$

Thus, $\widetilde{T} \in \mathcal{C}(\tilde{\phi}, \tilde{\psi})$, where $(\tilde{\phi}, \tilde{\psi})$ is the pair associated with $\widetilde{W}$. Conversely, every matrix $\widetilde{T} = \widetilde{T}_1 + i\widetilde{T}_2 \in \mathcal{C}(\tilde{\phi}, \tilde{\psi})$ can be obtained by the V-transformation of the matrix $T = T_1 + iT_2 \in \mathcal{C}(\phi, \psi)$, where

$$T_1 = u_{11}\widetilde{T}_1 + u_{21}\widetilde{T}_2,$$
$$T_2 = u_{12}\widetilde{T}_1 + u_{22}\widetilde{T}_2$$

and

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = V^{-1}.$$

Let $T = T_1 + iT_2 \in \mathcal{C}(\phi, \psi)$ satisfy condition (5). Then, we have

$$\begin{aligned}\widetilde{T}_1\widetilde{T}_2^t &= (v_{11}T_1 + v_{21}T_2)(v_{12}T_1 + v_{22}T_2)^t \\ &= v_{11}v_{12}T_1T_1^t + v_{21}v_{22}T_2T_2^t + v_{11}v_{22}T_1T_2^t + v_{12}v_{21}T_2T_1^t\end{aligned}$$

and

$$\widetilde{T}_2\widetilde{T}_1^t = v_{11}v_{12}T_1T_1^t + v_{21}v_{22}T_2T_2^t + v_{11}v_{22}T_2T_1^t + v_{12}v_{21}T_1T_2^t,$$

whence

$$\widetilde{T}_1\widetilde{T}_2^t - \widetilde{T}_2\widetilde{T}_1^t = \det V \cdot (T_1T_2^t - T_2T_1^t) = 0.$$

The lemma is proved. □

It follows from Lemma 1 that the analysis of the normal Hankel problem for a general matrix W obeying relation (40) can be reduced to the analysis for W having diagonal or Jordan form. In the subsequent sections, we conduct such analyses for four spectrally different situations listed in Section 1.

5. Different real eigenvalues

Consider the class $\mathcal{C}(\phi, \psi)$ whose associated matrix W has real and distinct eigenvalues. Since $\lambda_1\lambda_2 = \det W = 1$, we have

$$\lambda_2 = \lambda_1^{-1} \quad \text{and} \quad \lambda_1 \neq \lambda_2. \tag{45}$$

The matrix W can be diagonalized by a real similarity transformation; that is, there exists a real nonsingular 2×2 matrix U such that

$$U^{-1}WU = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} = \Lambda. \tag{46}$$

According to the basic lemma, the U-transformation of $\mathcal{C}(\phi, \psi)$ is the class $\mathcal{C}(\tilde{\phi}, \tilde{\psi})$ associated with the diagonal matrix Λ. For this class, we have

$$\tilde{\phi} = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right), \quad \tilde{\psi} = \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right), \quad \alpha = \lambda_1. \tag{47}$$

It remains to identify the matrices in $\mathcal{C}(\tilde{\phi}, \tilde{\psi})$ that generate normal Hankel matrices. For this class, relations (11) take the form

$$a_{-i} = \alpha a_{n-i}, \quad b_{-i} = \alpha^{-1}b_{n-i}, \quad 1 \leqslant i \leqslant n-1,$$

and can be combined into the single relation

$$t_{-j} = \alpha a_{n-j} + i\alpha^{-1}b_{n-j}, \quad j = 1, 2, \ldots, n-1. \tag{48}$$

This says that the matrix T in (3) must have an α-circulant as its real part T_1 and an α^{-1}-circulant as its imaginary part T_2. For this reason, we call such a matrix T a *separable circulant*. The corresponding matrix $H = T\mathcal{P}_n$ is called a *separable Hankel circulant.* In the remaining part of this section, we prove the existence of normal separable Hankel circulants for every $\alpha \neq 0$ and describe the techniques for constructing such circulants.

Let ν be an nth root of α. Define

$$W_\alpha = \operatorname{diag}(1,\ \nu,\ \nu^2,\ \ldots,\ \nu^{n-1}). \tag{49}$$

Then, the real α-circulant T_1 in (3) can be written as

$$T_1 = W_\alpha C_1 W_\alpha^{-1}, \tag{50}$$

where C_1 is a conventional circulant. Similarly, the real α^{-1}-circulant T_2 in (3) can be written as

$$T_2 = W_\alpha^{-1} C_2 W_\alpha \tag{51}$$

for some circulant C_2. Note that both C_1 and C_2 can be complex.

The basic matrix relation (5) says that $T_1 T_2^t$ must be a symmetric matrix. Using representations (50) and (51), we have

$$T_1 T_2^t = W_\alpha C_1 W_\alpha^{-1} (W_\alpha^{-1} C_2 W_\alpha)^t = W_\alpha C_1 W_\alpha^{-1} W_\alpha C_2^t W_\alpha^{-1} = W_\alpha C_1 C_2^t W_\alpha^{-1} = W_\alpha C W_\alpha^{-1}, \tag{52}$$

where

$$C = C_1 C_2^t \tag{53}$$

is a circulant.

Lemma 2. *The matrix $T_1 T_2^t$ in* (52) *is symmetric if and only if C in* (53) *is a scalar matrix.*

Proof. The sufficiency part is obvious because, along with C, $T_1 T_2^t$ is itself a scalar matrix. Conversely, assume that $T_1 T_2^t$ is symmetric; that is,

$$(T_1 T_2^t)_{kl} = (T_1 T_2^t)_{lk} \quad \forall k, l.$$

Denoting the entries of C by C_{kl}, we have

$$\nu^{k-l} C_{kl} = \nu^{l-k} C_{lk}. \tag{54}$$

We set $k = j + 1$ $(j \geqslant 1)$, $l = 1$ and use the fact that C is a circulant; thus,

$$C_{kl} = C_{j+1,1} = c_{-j} = c_{n-j},$$
$$C_{lk} = C_{1,j+1} = c_j.$$

Then, (54) yields

$$c_{n-j} = \nu^{-2j} c_j. \tag{55}$$

Next, we set $k = n - j + 1$, $l = 1$ in (54), which produces the equality

$$c_j = \nu^{-2(n-j)} c_{n-j}. \tag{56}$$

Combining (55) and (56), we obtain

$$c_j = \nu^{-2n} c_j = \alpha^{-2} c_j,$$

which is only possible if $c_j = 0$ $(j = 1, 2, \ldots, n-1)$. Consequently, C is a diagonal matrix. Being Toeplitz, C must be a scalar matrix. □

Thus, the matrices C_1 and C_2 in (50) and (51) must satisfy the relation

$$C_1 C_2^t = \kappa I_n \tag{57}$$

for some real scalar κ. This implies that

$$T_1 T_2^t = \kappa I_n. \tag{58}$$

If $\kappa \neq 0$ and T_1 is an appropriate α-circulant, then we can take any nonzero real multiple of T_1^{-t} as a matrix T_2. In this case, the only thing left is to ensure the choice of C_1 such that (50) produces a real matrix.

If $\kappa = 0$, then the circulants C_1 and C_2 must obey the relation

$$C_1 C_2^t = 0. \tag{59}$$

In addition, we must ensure that both (50) and (51) are real matrices.

Therefore, we address ourselves to the question for which circulants C the formula

$$T = W_\alpha C W_\alpha^{-1} \tag{60}$$

yields a real matrix T.

If α is positive and ν in (49) is chosen as the positive nth root of α, then W_α is real and matrix (60) is real exactly when C is a real circulant. The same is true if α is negative, n is an odd integer, and ν in (60) is chosen as the negative nth root of α.

Thus, we assume that α is negative, n is an even integer, and ν in (49) is chosen as the principal nth root of α, that is, the root whose argument is equal to π/n.

Let

$$F = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \epsilon & \epsilon^2 & \dots & \epsilon^{n-1} \\ 1 & \epsilon^2 & \epsilon^4 & \dots & \epsilon^{2(n-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \epsilon^{n-1} & \epsilon^{2(n-1)} & \dots & \epsilon^{(n-1)^2} \end{pmatrix} \tag{61}$$

be the DFT matrix of order n. Here, $\epsilon = \exp\left(\frac{2\pi i}{n}\right)$ is the principal nth root of unity. Then, the circulant C in (60) can be written as

$$C = F^* D F = \overline{F} D F \tag{62}$$

(since F is symmetric). Now, for T in (60) to be a real matrix, we must have

$$W_\alpha F^* D F W_\alpha^{-1} = \overline{W_\alpha} F \overline{D} F^* \overline{W_\alpha^{-1}}. \tag{63}$$

Multiplying (63) on the left by $F W_\alpha^{-1}$ and on the right by $\overline{W_\alpha} F$, we obtain

$$D(F W_\alpha^{-1} \overline{W_\alpha} F) = (F W_\alpha^{-1} \overline{W_\alpha} F) \overline{D}. \tag{64}$$

Lemma 3

$$F W_\alpha^{-1} \overline{W_\alpha} F = \mathcal{P}_2 \oplus \mathcal{P}_{n-2}. \tag{65}$$

Proof. We have

$$(F W_\alpha^{-1} \overline{W_\alpha} F)_{lm} = \frac{1}{n} \sum_{j=1}^{n} \epsilon^{(l-1)(j-1)} \frac{1}{|\alpha|^{\frac{j-1}{n}}} \epsilon^{-\frac{j-1}{2}} |\alpha|^{\frac{j-1}{n}} \epsilon^{-\frac{j-1}{2}} \epsilon^{(j-1)(m-1)} = \frac{1}{n} \sum_{j=1}^{n} \epsilon^{(j-1)(l+m-3)}.$$

The sum on the right-hand side is different from zero and equal to one if and only if

$$l + m = 3$$

or

$$l + m = n + 3.$$

This proves the lemma. □

Returning to relation (64), we conclude that the diagonal entries of D must obey the relations

$$d_1 = \overline{d_2}, d_3 = \overline{d_n}, d_4 = \overline{d_{n-1}}, \dots, d_{n/2+1} = \overline{d_{n/2+2}}. \tag{66}$$

Thus, in the case $\kappa \neq 0$, any diagonal matrix D satisfying (66) (and only such a matrix) can be used to produce a circulant C that generates a real matrix T in formula (60). Taking this matrix as T_1 in relation (3), we can then set T_2 equal to any nonzero real multiple of T_1^{-t}. This gives us a required separable circulant. Moreover, any separable circulant for the case $\kappa \neq 0$ can be obtained in this way. By reversing the order of its columns, we get a separable Hankel circulant.

If $\kappa = 0$, then we must take two diagonal matrices D_1 and D_2 satisfying relations (66) and the additional conditions

$$d_i^{(1)} d_i^{(2)} = 0, \quad i = 1, 2, \ldots, n,$$

resulting from (59). Let C_1 and C_2 be the corresponding circulants (see (62)). Substituting them into (50) and (51), we obtain two real matrices that can be used as T_1 and T_2 in (3). The resulting matrix T is a separable circulant, and any separable circulant for the case $\kappa = 0$ can be generated in this way. This completes the analysis of this section.

6. Complex conjugate eigenvalues

Consider the class $\mathcal{C}(\phi, \psi)$ whose associated matrix W has the complex conjugate eigenvalues

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta, \ \beta \neq 0. \tag{67}$$

Since $\lambda_1\lambda_2 = |\lambda_1|^2 = \det W = 1$, we have

$$\alpha^2 + \beta^2 = 1. \tag{68}$$

Form the real 2×2 matrix

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{69}$$

It has the same eigenvalues λ_1 and λ_2. Consequently, there exists a real nonsingular 2×2 matrix U such that

$$A = U^{-1} W U. \tag{70}$$

According to the basic lemma, the U-transformation of $\mathcal{C}(\phi, \psi)$ is the class $\mathcal{C}(\tilde{\phi}, 0)$ associated with the matrix A. For this class, we have

$$\tilde{\phi} = \alpha + i\beta. \tag{71}$$

It remains to identify the matrices in $\mathcal{C}(\tilde{\phi}, 0)$ that generate normal Hankel matrices.

Theorem 1. *A matrix $T \in \mathcal{C}(\tilde{\phi}, 0)$ generates a normal Hankel matrix H by formula* (1) *if and only if T* (*and, hence, H*) *is a scalar multiple of a unitary matrix*.

Proof. Let $T \in \mathcal{C}(\tilde{\phi}, 0)$, and let

$$T = SU$$

be the polar decomposition of T, where S is the polar modulus of T. Thus, S is the unique positive semidefinite square root of TT^*. According to Proposition 1, TT^* must be a real matrix. On the other hand, since $\tilde{\phi}$-circulants constitute an algebra closed under the Hermitian adjoint operation, TT^* must be a $\tilde{\phi}$-circulant. These two requirements can only be fitted if TT^* is a (real) scalar matrix. The same is true of S. Then, up to a scalar, T is identical to its unitary factor U. □

Remark. The condition on T given in Theorem 1 can be verified by a straightforward calculation.

7. Equal eigenvalues: diagonalizable case

Let the class $\mathcal{C}(\phi, \psi)$ be associated with a diagonalizable matrix W whose eigenvalues are identical; thus, $\lambda_1 = \lambda_2 = \lambda$. Since $\lambda^2 = \det W = 1$, we have

$$\lambda = 1 \quad \text{or} \quad \lambda = -1. \tag{72}$$

The first case corresponds to the matrix

$$W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \tag{73}$$

while the second case corresponds to the matrix

$$W = -I_2. \tag{74}$$

The class $\mathcal{C}(\phi, \psi)$ associated with matrix (73) is the set of conventional circulants; that is, $\phi = 1, \psi = 0$. The appropriate matrices of this class are described by the following theorem.

Theorem 2. *Let $T \in \mathcal{C}(1, 0)$, and let*

$$T = F^*DF \tag{75}$$

be the spectral decomposition of T (see (62)). Then, T generates a normal Hankel circulant H by formula (1) if and only if the matrix D in (75) satisfies the relations

$$|d_m| = |d_{n+2-m}|, \quad m = 2, 3, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor. \tag{76}$$

Proof. For H to be a normal matrix, we must satisfy relation (2). Using (75), we can transform the equality

$$TT^* = \overline{TT^*} \tag{77}$$

into

$$D\overline{D}F^2 = F^2 D\overline{D}. \tag{78}$$

Now, F^2 is a matrix of a very special form; namely,

$$F^2 = 1 \oplus \mathcal{P}_{n-1}. \tag{79}$$

Indeed, we have

$$(F^2)_{ml} = \frac{1}{n}\sum_{j=1}^{n} \epsilon^{(m-1)(j-1)}\epsilon^{(j-1)(l-1)} = \frac{1}{n}\sum_{j=1}^{n} \epsilon^{(j-1)(m+l-2)},$$

which is different from zero and equal to one if and only if $m = l = 1$ or $m + l = n + 2$.

Returning to (78), we conclude that the diagonal matrix D must obey relations (76). □

The class $\mathcal{C}(\phi, \psi)$ associated with matrix (74) is the set of conventional skew-circulants; that is, $\phi = -1, \psi = 0$. To describe the appropriate matrices of class $\mathcal{C}(-1, 0)$, we use the spectral decomposition

$$T = W_{-1}F^*DFW_{-1}^* \tag{80}$$

of the skew-circulant T. Here, W_{-1} is given by (49) with

$$\nu = e^{i\frac{\pi}{n}}.$$

Theorem 3. *A matrix $T \in \mathcal{C}(-1, 0)$ generates a normal Hankel skew-circulant H by formula (1) if and only if the matrix D in (80) satisfies the relations*

$$|d_1| = |d_2|, \quad |d_3| = |d_n|, \quad |d_4| = |d_{n-1}|, \ldots \tag{81}$$

Proof. Using representation (80), we can transform (77) into the commutation relation

$$F\overline{W_{-1}}^2 FD\overline{D} = D\overline{D}F\overline{W_{-1}}^2 F. \tag{82}$$

Observe that Lemma 3 holds true for $\alpha = -1$; thus,

$$F\overline{W_{-1}}^2F = \mathcal{P}_2 \oplus \mathcal{P}_{n-2}.$$

It follows that the diagonal matrix D must obey relations (81). □

8. Equal eigenvalues: nondiagonalizable case

Let the class $\mathcal{C}(\phi, \psi)$ be associated with a nondiagonalizable matrix W whose eigenvalues are identical; thus, $\lambda_1 = \lambda_2 = \lambda$. As in Section 7, we conclude that

$$\lambda = 1 \quad \text{or} \quad \lambda = -1. \tag{83}$$

Let U be a real nonsingular 2×2 matrix such that

$$U^{-1}WU = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = J_1 \tag{84}$$

or

$$U^{-1}WU = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = J_2. \tag{85}$$

According to the basic lemma, the U-transformation of $\mathcal{C}(\phi, \psi)$ is the class $\mathcal{C}(\tilde{\phi}, \tilde{\psi})$ associated with J_1 or J_2. It remains to identify the matrices in the last two classes that generate normal Hankel matrices.

In what follows, we use some additional notation. Let L be a strictly lower triangular Toeplitz matrix with the first column $(0, a_1, a_2, \ldots, a_{n-1})^t$. Then, the symbol L^c stands for the lower triangular Toeplitz matrix with the first column $(0, a_{n-1}, a_{n-2}, \ldots, a_1)^t$. In addition to $\mathcal{P}_n$, we introduce two special permutation matrices

$$\Omega_n = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}, \tag{86}$$

$$\Theta_n = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & -1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}. \tag{87}$$

Lemma 4. *Let C be a real circulant with the first column $(a_0, a_1, a_2, \ldots, a_{n-1})^t$, and let L be the strictly lower triangular Toeplitz matrix whose subdiagonal entries are identical to the subdiagonal entries of C. Then, $CL^t - LC^t$ is a circulant if and only if C is an orthogonal circulant.*

Proof. Since

$$C = a_0I_n + L + L^{ct},$$

we have

$$CL^t - LC^t = (a_0L + LL^c)^t - (a_0L + LL^c).$$

It follows that $CL^t - LC^t$ is a circulant if and only if

$$(L(a_0I_n + L^c))^c + (L(a_0I_n + L^c)) = 0. \tag{88}$$

Define the vector y by the formula

$$y = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ a_1 & 0 & 0 & \dots & 0 & 0 \\ a_2 & a_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_1 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ a_1 \end{pmatrix}. \tag{89}$$

Now, relation (88) is equivalent to the conditions

$$y_j + y_{n+2-j} = 0, \quad j = 2, \dots, n. \tag{90}$$

We set

$$r = \left\lceil \frac{n-1}{2} \right\rceil.$$

Observe that equalities (90) remain true if j is replaced by $n+2-j$. Hence, it suffices to consider the relations

$$y_j + y_{n+2-j} = 0, \quad j = 2, \dots, r+1. \tag{91}$$

It easily follows from (89) that

$$y_j = a_{j-1}a_0 + \sum_{k=1}^{j-2} a_k a_{n-j+k+1}, \quad j = 2, \dots, n, \tag{92}$$

and

$$y_{n+2-j} = a_{n+1-j}a_0 + \sum_{k=1}^{n-j} a_k a_{j+k-1}, \quad j = 2, \dots, n. \tag{93}$$

Substituting these expressions into (91), we have

$$a_{j-1}a_0 + \sum_{k=1}^{j-2} a_k a_{n-j+k+1} + a_{n+1-j}a_0 + \sum_{k=1}^{n-j} a_k a_{j+k-1} = 0, \quad j = 2, \dots, r+1.$$

By rearranging the terms, we obtain

$$a_{j-1}a_0 + \sum_{k=1}^{n-j} a_{j+k-1}a_k + a_0 a_{n+1-j} + \sum_{k=1}^{j-2} a_k a_{n-j+k+1} = 0, \quad j = 2, \dots, r+1.$$

Note that the first summand can be considered as the term of the first sum corresponding to $k = 0$. Similarly, the third summand can be considered as the term of the second sum for $k = 0$. Consequently, we can write

$$\sum_{k=0}^{n-j} a_{j+k-1}a_k + \sum_{k=0}^{j-2} a_k a_{n-j+k+1} = 0, \quad j = 2, \dots, r+1. \tag{94}$$

Define the $r \times n$ matrix

$$Q = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-2} & a_{n-1} & a_0 \\ a_2 & a_3 & a_4 & \dots & a_{n-1} & a_0 & a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ a_r & a_{r+1} & a_{r+2} & \dots & a_{r-3} & a_{r-2} & a_{r-1} \end{pmatrix}$$

and the n-dimensional

$$x = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix}.$$

Then, equalities (94) can be rewritten as the matrix–vector relation

$$Qx = 0.$$

Using the cyclic permutation matrix Ω_n, we can reformulate this relation in terms of the scalar products

$$(\Omega_n^{n-j}x, x) = 0, \quad j = 1, \ldots, r. \tag{95}$$

Since $\Omega_n^n = I_n$, conditions (95) are equivalent to the equalities

$$(x, \Omega_n^j x) = 0, \quad j = 1, \ldots, r, \tag{96}$$

which mean that the original circulant C is orthogonal. Indeed, the scalar product of the kth and the mth columns of C, where $m > k$, is

$$(\Omega_n^{k-1}x, \Omega_n^{m-1}x) = (x, \Omega_n^{m-k}x).$$

For $m - k \leqslant r$, equalities (96) imply $(\Omega_n^{k-1}x, \Omega_n^{m-1}x) = 0$. If $m - k > r$, then

$$(\Omega_n^{k-1}x, \Omega_n^{m-1}x) = (x, \Omega_n^{m-k}x) = (x, \Omega_n^{n-(m-k)}x) = 0.$$

The lemma is proved. □

An analog of Lemma 4 holds for skew-circulants.

Lemma 5. *Let C be a real skew-circulant with the first column $(a_0, -a_1, -a_2, \ldots, -a_{n-1})^t$, and let L be the strictly lower triangular Toeplitz matrix whose subdiagonal entries are the negatives of the corresponding subdiagonal entries in C. Then, $CL^t - LC^t$ is a skew-circulant if and only if C is an orthogonal skew-circulant.*

Proof. We go along the same lines as in the proof of Lemma 4. Using the representation

$$C = a_0 I_n - L + L^{ct},$$

we can write the requirement that $CL^t - LC^t$ be a skew-circulant in the form

$$(L(a_0 I_n + L^c))^c - (L(a_0 I_n + L^c)) = 0. \tag{97}$$

This is equivalent to the relations

$$y_j - y_{n+2-j} = 0, \quad j = 2, \ldots, n,$$

where y is the vector defined by (89). Using formulas (92) and (93) and conducting transformations similar to those in Lemma 4, we obtain the equality

$$a_{j-1}a_0 - \sum_{k=1}^{n-j} a_{j+k-1}a_k - a_0 a_{n+1-j} + \sum_{k=1}^{j-2} a_k a_{n-j+k+1} = 0, \quad j = 2, \ldots, r+1. \tag{98}$$

The $r \times n$ matrix Q and the n-dimensional x are now defined as

$$Q = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & a_0 \\ a_2 & a_3 & a_4 & \cdots & a_{n-1} & a_0 & -a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ a_r & a_{r+1} & a_{r+2} & \cdots & -a_{r-3} & -a_{r-2} & -a_{r-1} \end{pmatrix}$$

and

$$x = \begin{pmatrix} a_0 \\ -a_1 \\ -a_2 \\ \vdots \\ -a_{n-1} \end{pmatrix}.$$

Equalities (98) are again rewritten as the matrix-vector relation

$$Qx = 0,$$

which is the same as

$$(\Theta_n^{n-j}x, x) = 0, \quad j = 1, \ldots, r.$$

Then, the same argument as in Lemma 4 proves that the original skew-circulant C is orthogonal. □

We are now able to describe the classes $\mathcal{C}(\tilde{\phi}, \tilde{\psi})$ associated with J_1 and J_2.

We begin with the matrix J_1. In this case, relations (11) show that, for $T \in \mathcal{C}(\tilde{\phi}, \tilde{\psi})$, the matrix T_1 in representation (3) is a (real) circulant (which we now denote by C_1), while T_2 is the sum of a (real) circulant C_2 and a (real) lower triangular Toeplitz matrix L_1 whose subdiagonal entries are identical to the subdiagonal entries of C_1. Using Proposition 2, we can write

$$C_1(C_2 + L_1)^t = (C_2 + L_1)C_1^t \tag{99}$$

or

$$C_1L_1^t - L_1C_1^t = C_2C_1^t - C_1C_2^t. \tag{100}$$

The matrix on the right-hand side of this equality is a circulant; hence, the matrix

$$C_3 = C_1L_1^t - L_1C_1^t \tag{101}$$

must be a circulant as well. By Lemma 4, C_1 must be an orthogonal circulant.

Let us show how to choose the appropriate matrices C_1 and C_2. Take an arbitrary real orthogonal circulant C_1. This determines the matrix L_1 and, hence, C_3. Now, C_2 can be found as a solution to the equation

$$C_2C_1^t - C_1C_2^t = C_3. \tag{102}$$

Let

$$C_1 = F^*D_1F, \quad C_2 = F^*D_2F, \quad C_3 = F^*D_3F \tag{103}$$

be the spectral decompositions of the circulants C_1, C_2 and C_3 (see (62)). Since C_1 and C_3 are given, the diagonal matrices D_1 and D_3 are known, and it remains to determine D_2.

Substituting decompositions (103) into (102), we have

$$F^*D_2FFD_1F^* - F^*D_1FFD_2F^* = F^*D_3F.$$

Multiplying this equation on the left by F and on the right by F^3 and using the relation $F^4 = I_n$, we obtain

$$D_2F^2D_1F^2 - D_1F^2D_2F^2 = D_3. \tag{104}$$

Recall that C_1 and C_2 are real matrices. It follows that

$$F^2D_1F^2 = \overline{D_1}, \quad F^2D_2F^2 = \overline{D_2}. \tag{105}$$

Using relations (105) in (104), we find that

$$D_2\overline{D_1} - D_1\overline{D_2} = D_3. \tag{106}$$

The fact that C_1 and C_2 are real also implies that

$$D_1 = \mathrm{diag}(u_1 + iv_1, u_2 + iv_2, \ldots, u_n + iv_n),$$

and

$$D_2 = \mathrm{diag}(x_1 + iy_1, x_2 + iy_2, \ldots, x_n + iy_n)$$

must satisfy the relations

$$v_1 = 0,$$
$$u_j = u_{n+2-j}, \quad v_j = -v_{n+2-j}, \quad j = 2, \ldots, n,$$

and

$$y_1 = 0,$$
$$x_j = x_{n+2-j}, \quad y_j = -y_{n+2-j}, \quad j = 2, \ldots, n.$$

The skew-symmetric matrix C_3 has a purely imaginary spectrum; thus,

$$D_3 = \mathrm{diag}(ic_1, ic_2, \ldots, ic_n). \tag{107}$$

On the other hand, C_3 is real, which, in combination with (107), yields the relations

$$c_1 = 0,$$
$$c_j = -c_{n+2-j}, \quad j = 2, \ldots, n.$$

Now, the matrix equation (106) reduces to the scalar equations

$$u_j y_j - v_j x_j = c_j/2, \quad j = 2, \ldots, r+1, \quad r = [n/2]. \tag{108}$$

Observe that, for each j, u_j and v_j cannot vanish simultaneously because $u_j + iv_j$ is an eigenvalue of the orthogonal matrix C_1. Consequently, from (108), one of the values x_j and y_j can be expressed as a (linear) function of the other.

Summing up, we see that the appropriate matrices D_2 constitute a real linear manifold of dimension $n - r$. Using the middle formula in (103), we can construct all the matrices C_2. This determines the corresponding matrices $T \in \mathcal{C}(\tilde{\phi}, \tilde{\psi})$.

The case of the matrix J_2 is treated very similarly, although there are some slight distinctions. For $T \in \mathcal{C}(\tilde{\phi}, \tilde{\psi})$, the matrix T_1 in representation (3) is now a (real) skew-circulant (denoted by C_1), while T_2 is the sum of a (real) skew-circulant C_2 and a (real) lower triangular Toeplitz matrix L_1 whose subdiagonal entries are the negatives of the corresponding subdiagonal entries of C_1.

The use of Proposition 2 again leads to Eq. (99). Lemma 5 says that, for this equation to have a skew-circulant solution, the matrix C_1 must be orthogonal. Then, as before, C_2 is determined by Eq. (102). To find the appropriate solutions, we use the spectral decompositions of C_1, C_2 and C_3. They are now given by formulas of type (80) rather than of type (62). Substituting these decompositions into (102) and performing calculations similar to those in the case of J_1, we obtain Eq. (106).

Using the above representations for the diagonal matrices D_1 and D_2 and the fact that the skew-circulants C_1 and C_2 are real, we have

$$u_j = u_{n+1-j}, \quad v_j = -v_{n+1-j}, \quad j = 1, \ldots, n,$$

and

$$x_j = x_{n+1-j}, \quad y_j = -y_{n+1-j}, \quad j = 1, \ldots, n.$$

Since C_3 is real and skew-symmetric (see 101), we conclude that the diagonal entries $ic_1, ic_2, \ldots, ic_n$ of D_3 must satisfy the relations

$$c_j = -c_{n+1-j}, \quad j = 1, \ldots, n.$$

This observation reduces matrix equation (106) to scalar equations (108), where j now runs over the values 1 to $s = [n/2]$.

The appropriate matrices D_2 constitute a real linear manifold of dimension $n - s$. The manifold of the appropriate matrices C_2 is of the same dimension. From C_1 and C_2, we construct the corresponding matrices $T \in \mathcal{C}(\tilde{\phi}, \tilde{\psi})$.

Acknowledgments

We thank the referee for the careful reading of our manuscript and several useful remarks.

The first author acknowledges the support of the Russian Foundation for Basic Research (project no. 08-01-00115) and the Program for Basic Research of Mathematics Division of Russian Academy of Sciences "Computational and Informational Problems in Solving Large-Scale Problems" (project "Matrix methods in integral and differential equations").

References

[1] A. Arimoto, A simple proof of the classification of normal Toeplitz matrices, Electron. J. Linear Algebra 9 (2002) 108–111.
[2] V.N. Chugunov, Kh.D. Ikramov, A contribution to the normal Hankel problem, Linear Algebra Appl. 430 (2009) 2094–2101.
[3] V.N. Chugunov, On two particular cases in solving the normal Hankel problem, Comput. Math. Math. Phys. 49 (2009) 1–8.
[4] N.V. Efimov, E.R. Rozendorn, Linear Algebra and Multidimensional Geometry, Mir Publishers, Moscow, 1975.
[5] D.R. Farenick, M. Krupnik, N. Krupnik, W.Y. Lee, Normal Toeplitz matrices, SIAM J. Matrix Anal. Appl. 17 (1996) 1037–1043.
[6] V.I. Gel'fgat, A criterion for the normality of a Toeplitz matrix, Comput. Math. Math. Phys. 35 (1995) 1147–1150.
[7] G. Gu, L. Patton, Commutation relations for Toeplitz and Hankel matrices, SIAM J. Matrix Anal. Appl. 24 (2003) 728–746.
[8] Kh.D. Ikramov, Describing normal Toeplitz matrices, Comput. Math. Math. Phys. 34 (1994) 399–404.
[9] Kh.D. Ikramov, On the classification of normal Toeplitz matrices with real entries, Mat. Zametki 57 (1995) 670–680.
[10] Kh.D. Ikramov, On the problem of characterizing normal Hankel matrices, Fundam. Appl. Math. 3 (1997) 809–819.
[11] Kh.D. Ikramov, V.N. Chugunov, A criterion for the normality of a complex Toeplitz matrix, Comput. Math. Math. Phys. 36 (1996) 131–138.
[12] Kh.D. Ikramov, V.N. Chugunov, On the skew-symmetric part of the product of Toeplitz matrices, Math. Notes 63 (1998) 124–127.
[13] Kh.D. Ikramov, V.N. Chugunov, On a new class of normal Hankel matrices, Moscow Univ. Comp. Math. Cybernet. (Vestnik Moskov. Univ. Ser. XV. Vychisl. Mat. Kibernet.) (1) (2007) 138–141.
[14] Kh.D. Ikramov, V.N. Chugunov, On normal Hankel matrices, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 346 (2007) 63–80. (Chisl. Metody i Vopr. Organ. Vychisl. 20).
[15] Kh. D. Ikramov, V.N. Chugunov, On normal Hankel matrices of low orders, Math. Notes 84 (2008) 197–206.
[16] Kh. D. Ikramov, V.N. Chugunov, Classifying normal Hankel matrices, Dokl. Math. 79 (2009) 114–117.
[17] Kh.D. Ikramov, V.N. Chugunov, On the reduction of the normal Hankel problem to two particular cases, Math. Notes 85 (2009) 664–671.
[18] K. Ito, Every normal Toeplitz matrix is either of type I or of type II, SIAM J. Matrix Anal. Appl. 17 (1996) 998–1006.