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Conjugate-normal matrices: A survey

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Abstract

Conjugate-normal matrices play the same important role in the theory of unitary congruence as the conventional normal matrices do with respect to unitary similarities. However, unlike the latter, the properties of conjugate-normal matrices are not widely known. Motivated by this fact, we give a survey of the properties of these matrices. In particular, a list of more than forty conditions is given, each of which is equivalent to A being conjugate-normal.

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1. Introduction

As justly noted in [7, Section 2.5], the class of normal matrices is important throughout matrix analysis. It is especially important in matters related to similarity transformations and, even more specifically, to unitary similarity transformations. Such significant matrix classes as Hermitian, skew-Hermitian, and unitary matrices are subclasses of normal matrices.

The importance of normal matrices explains the appearance of the survey [5] in 1987. It contains 70 conditions, each equivalent to the original definition of normality, i.e., to the relation

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$$AA^* = A^*A. \tag{1}$$

One decade later, about 20 additional criteria for normality were presented in [1]. For brevity, we refer to these two papers as the GJSW list and EI list, respectively.

If we are dealing with unitary congruences rather than unitary similarities, then normality is no longer a useful property, because it is not preserved by unitary congruence transformations. Is there a matrix class that can replace normal matrices in this new context?

The answer is yes, and the required matrix class are the so-called conjugate-normal matrices.

Definition 1. A matrix $A \in M_n(\mathbb{C})$ is said to be conjugate-normal if

$$AA^* = \overline{A^*A}. \tag{2}$$

In particular, complex symmetric, skew-symmetric, and unitary matrices are special subclasses of conjugate-normal matrices.

It seems that the term 'conjugate-normal matrices' was first introduced in [11], that is, 35 years ago. However, despite this respectable age, the properties of these matrices are not known as broadly as they, in our opinion, deserve to be.

Our intention in this paper is to give a survey of properties of conjugate-normal matrices. Moreover, we give a list of conditions equivalent to definition (2), which thus may be taken as condition 0 in this list. Citing [5], 'since we know of no similar list, our hope is that this will be generally useful to the matrix community.'

The paper is organized as follows. The necessary preliminary material is presented in Section 2. It includes a discussion of distinctions between the theories of two matrix relations: similarity and consimilarity. Section 3 deals with elementary properties of conjugate-normal matrices and their canonical forms with respect to unitary congruences. The next two sections follow the pattern given in [5]: first, about 40 conditions each equivalent to (2) are given, then brief proofs or comments to these conditions are given in Section 5. Section 6 contains our concluding remarks.

We use the following notation. If $A \in M_n(\mathbb{C})$, then A^{T}, $\overline{A}$, A^* and A^+ are the transpose, the entrywise conjugate, the Hermitian adjoint, and the Moore–Penrose inverse of A, respectively. The singular values of A are denoted by $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n$, its spectral radius by $\rho(A)$, and its spectrum by $\lambda(A)$. The symbol $C_k(A)$ stands for the kth compound matrix of A, [10, p. 16]. By $\mathcal{N}_n$ and $\mathcal{C}N_n$, we denote the classes of $n \times n$ normal and conjugate-normal matrices, respectively. The subscript is omitted if the order of A is clear from the context or not relevant. Depending on the context, the symbol $\|\cdot\|_2$ is used for the 2-norm of a vector or the spectral norm of a matrix; $\|\cdot\|_F$ denotes the Frobenius matrix norm.

2. Preliminaries

The most important quantities related to similarity transformations of a matrix are its eigenvalues, eigenvectors, and, more generally, its invariant subspaces. Now, that we deal with consimilarity transformations (and unitary congruences are a particular subclass of them), we should instead speak of 'con'-analogues of these quantities.

Recall that matrices $A, B \in M_n(\mathbb{C})$ are said to be consimilar if $B = SA\overline{S}^{-1}$ for a nonsingular matrix S (see [7, Section 4.6]). The coneigenvalues of A are the n scalars attributed to A that are preserved by any transformation of this kind. To give an exact definition, we introduce the matrices

$$A_L = \overline{A}A \quad \text{and} \quad A_R = A\overline{A} = \overline{A_L}. \tag{3}$$

Although the products AB and BA need not be similar in general, A_L is always similar to A_R (see [7, p. 246, Problem 9 in Section 4.6]). Therefore, in the subsequent discussion of their spectral properties, it will be sufficient to refer to one of them, say, A_L.

The spectrum of A_L has two remarkable properties:

1. It is symmetric with respect to the real axis. Moreover, the eigenvalues λ and $\bar{\lambda}$ are of the same multiplicity.
2. The negative eigenvalues of A_L (if any) are necessarily of even algebraic multiplicity.

For the proofs of these properties, we refer the reader to [7, pp. 252–253].

Let

$$\lambda(A_L) = \{\lambda_1, \ldots, \lambda_n\}$$

be the spectrum of A_L. The coneigenvalues of A are the n scalars $\mu_1, \ldots, \mu_n$ defined as follows: If $\lambda_i \in \lambda(A_L)$ does not lie on the negative real axis, then the corresponding coneigenvalue μ_i is defined as a square root of λ_i with nonnegative real part and the multiplicity of μ_i is set to that of λ_i:

$$\mu_i = \lambda_i^{1/2}, \qquad \mathrm{Re}\mu_i \geqslant 0.$$

With a real negative $\lambda_i \in \lambda(A_L)$, we associate two conjugate purely imaginary coneigenvalues

$$\mu_i = \pm\lambda_i^{\frac{1}{2}},$$

the multiplicity of each being half the multiplicity of λ_i.

The set $\{\mu_1, \ldots, \mu_n\}$ is called the conspectrum of A and will be denoted by $c\lambda(A)$.

Coneigenvectors are not as important in the theory of consimilarity as eigenvectors are with respect to similarity transformations. The reason is that not every matrix in $M_n(\mathbb{C})$ has coneigenvectors; in fact, most matrices have no coneigenvectors. Let us expand on this point.

For a subspace $\mathscr{L} \in \mathbb{C}^n$ define

$$\overline{\mathscr{L}} = \{\bar{x} | x \in \mathscr{L}\},$$

where $\bar{x}$ is the component-wise conjugate of the column vector x.

Definition 2. $\mathscr{L}$ is a coninvariant subspace of A if

$$A\mathscr{L} \subset \overline{\mathscr{L}}. \tag{4}$$

In particular, if $\dim \mathscr{L} = 1$, then every nonzero vector $x \in \mathscr{L}$ is called a coneigenvector of A.

The fundamental fact on coninvariant subspaces is the following theorem.

Theorem 1. *Every matrix* $A \in M_n(\mathbb{C})$ $(n \geqslant 3)$ *has a one- or two-dimensional coninvariant subspace.*

To better explain how two-dimensional subspaces come about in Theorem 1, we reproduce its proof as given in [2].

Proof. Let x be an eigenvector of A_L; that is,

$$A_L x = \overline{A}Ax = \lambda x \tag{5}$$

for some $\lambda \in \mathbb{C}$. Define

$$y = \overline{Ax}. \tag{6}$$

Suppose that y and x are linearly dependent, i.e.,

$$Ax = \mu\bar{x} \tag{7}$$

for some $\mu \in \mathbb{C}$; then, x is a coneigenvector of A and $\mathscr{L} = \text{span}\{x\}$ is a one-dimensional coninvariant subspace. However, getting (7) is not a very likely event. Indeed, (7) would imply that

$$A_L x = \overline{A}Ax = |\mu|^2 x.$$

A comparison with (5) shows that the original eigenvalue λ must be nonnegative, whereas, for a randomly chosen A, the matrix A_L would hardly have real eigenvalues.

Thus, assume that y and x are linearly independent. Then, (6), rewritten as

$$Ax = \bar{y},$$

and (5), rewritten as

$$Ay = \bar{\lambda}\bar{x} \tag{8}$$

imply that $\mathscr{L} = \text{span}\{x, y\}$ is a two-dimensional coninvariant subspace of A. We can put it differently as the matrix relation

$$A[xy] = \overline{[xy]}\begin{bmatrix} 0 & \bar{\lambda} \\ 1 & 0 \end{bmatrix}. \qquad \square \tag{9}$$

The argument above begins with an eigenvector of A_L. It turns out that the vector y defined by (6) is also an eigenvector of A_L. Indeed, applying $\overline{A}$ to both sides of (8), we obtain

$$A_L y = \overline{A}Ay = \overline{\lambda Ax} = \bar{\lambda}y. \tag{10}$$

Observe that, while x is associated with the eigenvalue λ, the vector y corresponds to $\bar{\lambda}$. We conclude that the two-dimensional subspace $\mathscr{L}$ in Theorem 1 is an invariant subspace of A_L spanned by two of its eigenvectors corresponding to a pair of complex conjugate eigenvalues.

If $\lambda \neq 0$, we can make relation (9) to look more symmetrically. To this end, set $\mu = \sqrt{\lambda}$ (taking any of the two values of the square root) and, instead of (6), define y by the relation

$$y = \frac{1}{\bar{\mu}}\overline{Ax}. \tag{11}$$

Then, the same calculations as in Theorem 1 yield

$$Ax = \mu\bar{y}, \qquad Ay = \overline{\mu x},$$

and

$$A[xy] = \overline{[xy]}\begin{bmatrix} 0 & \bar{\mu} \\ \mu & 0 \end{bmatrix}. \tag{12}$$

An important relation often encountered in the theory of similarity is commutation. For instance, definition (1) of a normal matrix is just a requirement that A commute with its Hermitian adjoint A^*. Commutation relations are respected by simultaneous similarities in the sense that, if

$$AB = BA$$

and

$$\widetilde{A} = QAQ^{-1}, \qquad \widetilde{B} = QBQ^{-1},$$

then

$$\widetilde{A}\widetilde{B} = \widetilde{B}\widetilde{A}.$$

In the theory of consimilarity, commutation is often replaced by concommutation, i.e., by the relation

$$A\overline{B} = B\overline{A}. \tag{13}$$

This can even be considered as a general principle of consimilarity transformations.

Principle 1. *Suppose that a problem P concerning similarity involves a commutation relation. Then, in the corresponding problem $\overline{P}$ concerning consimilarity, look for a concommutation relation.*

As an illustration of this principle, consider the definition of a conjugate-normal matrix. Rewriting (2) as

$$A\overline{A}^{\mathrm{T}} = A^{\mathrm{T}}\overline{A}, \tag{14}$$

we see that this definition is nothing else than the requirement that A and A^{T} concommute. This example also reveals another general principle of consimilarity transformations.

Principle 2. *In most of the relations concerning consimilarity, the Hermitian adjoint A^* is replaced by the transpose A^{T}.*

We will see a number of manifestations of both principles in the subsequent sections. Note that concommutation is respected by consimilarity transformations: if (13) is fulfilled and

$$\widetilde{A} = QA\overline{Q}^{-1}, \qquad \widetilde{B} = QB\overline{Q}^{-1},$$

then

$$\widetilde{A}\overline{\widetilde{B}} = \widetilde{B}\overline{\widetilde{A}}.$$

Now, from general consimilarity transformations, we turn to unitary congruences as the most interesting special case. There is an important theorem that plays the same role in the theory of unitary congruence as the Schur triangularization theorem does with respect to unitary similarities. This is the Youla theorem [13].

Theorem 2. *Any matrix $A \in M_n(\mathbb{C})$ can be brought by a unitary congruence transformation to a block triangular form with the diagonal blocks of orders 1 and 2. The 1×1 blocks correspond to real nonnegative coneigenvalues of A, while each 2×2 block corresponds to a pair of complex conjugate coneigenvalues. This block triangular matrix is called the Youla normal form of A. It can be upper or lower block triangular.*

In the next section, we will see that, for a conjugate-normal matrix A, the Youla normal form becomes a block diagonal matrix.

Two useful matrix decompositions used in connection with unitary similarities are the Toeplitz decomposition and the polar decomposition. The former is the representation of $A \in M_n(\mathbb{C})$ in the form

$$A = H + K, \qquad H = H^*, \quad K = -K^*,$$

and is uniquely determined by A:

$$H = \frac{1}{2}(A + A^*), \qquad K = \frac{1}{2}(A - A^*).$$

The polar decomposition

$$A = PU, \qquad P = P^* \geqslant 0, \quad UU^* = I$$

is determined uniquely for a nonsingular A. If A is singular, then the Hermitian positive semidefinite factor is still determined uniquely, but the unitary factor A is not.

For unitary congruences, the Toeplitz decomposition is replaced by the representation

$$A = S + K, \tag{15}$$

where S is symmetric and K is skew-symmetric. This SSS decomposition is uniquely determined by A:

$$S = \frac{1}{2}(A + A^{\mathrm{T}}), \qquad K = \frac{1}{2}(A - A^{\mathrm{T}}). \tag{16}$$

Instead of the polar decomposition, one uses the representation

$$A = SU, \qquad S = S^{\mathrm{T}}, \quad UU^* = I, \tag{17}$$

called an symmetric-unitary polar decomposition (SUPD). Representation (17) exists for every matrix $A \in M_n(\mathbb{C})$ but is never unique. More details on the SUPD can be found in [3].

3. Conjugate-normal matrices

There is a close kinship between the properties of normality and conjugate normality, which comes of no surprise considering how definitions (1) and (2) resemble each other. Some of the most straightforward relations between these properties are indicated in the following two propositions.

Theorem 3. *If $A \in \mathscr{C}N_n$ then $A_L \in \mathscr{N}_n$ and $A_R \in \mathscr{N}_n$.*

Proof. Using (14) and the conjugate equality

$$\overline{A}A^{\mathrm{T}} = A^*A,$$

we have

$$A_R A_R^* = (A\overline{A})(A^{\mathrm{T}}A^*) = A(\overline{A}A^{\mathrm{T}})A^* = A(A^*A)A^* = (AA^*)^2$$

and

$$A_R^* A_R = (A^{\mathrm{T}}A^*)(A\overline{A}) = A^{\mathrm{T}}(A^*A)\overline{A} = A^{\mathrm{T}}(\overline{A}A^{\mathrm{T}})\overline{A} = (A^{\mathrm{T}}\overline{A})^2 = (AA^*)^2.$$

Thus, A_R is normal. Hence, $A_L = \overline{A_R}$ is normal as well. □

Remark 1. The reverse implication

$$A_L, A_R \in \mathscr{N}_n \Rightarrow A \in \mathscr{C}N_n$$

is false. For instance, the matrices

$$A^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A^{(2)} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

are not conjugate-normal, although

$$A_L^{(1)} = 0 \qquad \text{and} \qquad A_R^{(2)} = 2I_2$$

are normal matrices. Indeed,

$$\{A^{(1)}A^{(1)*}\}_{11} = 1 \neq 0 = \{\overline{A^{(1)*}A^{(1)}}\}_{11}$$

and

$$\{A^{(2)}A^{(2)*}\}_{11} = 1 \neq 4 = \{\overline{A^{(2)*}A^{(2)}}\}_{11}.$$

To state the next proposition, we associate with each matrix $A \in M_n(\mathbb{C})$ the matrix

$$\widehat{A} = \begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}. \tag{18}$$

Since

$$\widehat{A}^* = \begin{bmatrix} 0 & A^{\mathrm{T}} \\ A^* & 0 \end{bmatrix}, \tag{19}$$

$$\widehat{A}\widehat{A}^* = AA^* \oplus \overline{A}A^{\mathrm{T}}, \tag{20}$$

and

$$\widehat{A}^*\widehat{A} = A^{\mathrm{T}}\overline{A} \oplus A^*A, \tag{21}$$

we arrive at the following result:

Theorem 4. *A matrix $A \in M_n(\mathbb{C})$ is conjugate-normal if and only if $\widehat{A}$ is normal.*

Note that, if

$$c\lambda(A) = \{\mu_1, \ldots, \mu_n\} \tag{22}$$

is the conspectrum of an arbitrary matrix $A \in M_n(\mathbb{C})$, then

$$\lambda(\widehat{A}) = \{\mu_1, \ldots, \mu_n, -\mu_1, \ldots, -\mu_n\}. \tag{23}$$

Now, we look more closely at definition (2). It is well known that, for any matrix B, it holds that

$$\mathrm{im}(BB^*) = \mathrm{im}(B), \qquad \text{and} \qquad \ker(BB^*) = \ker(B^*),$$

where $\mathrm{im}(\cdot)$ and $\ker(\cdot)$ denote the range and the null space of the corresponding matrix. Applying this to (2) (or to (14)), we have

$$\mathrm{im}(A) = \mathrm{im}(AA^*) = \mathrm{im}(A^{\mathrm{T}}\overline{A}) = \mathrm{im}(A^{\mathrm{T}}(A^{\mathrm{T}})^*) = \mathrm{im}(A^{\mathrm{T}})$$

and

$$\ker(A^*) = \ker(AA^*) = \ker(A^{\mathrm{T}}\overline{A}) = \ker(\overline{A}),$$

which is the same as $\ker A = \ker A^{\mathrm{T}}$. We summarize this as

Theorem 5. *If $A \in \mathscr{C}N$, then*

$$\mathrm{im}(A) = \mathrm{im}(A^{\mathrm{T}}), \qquad \ker(A) = \ker(A^{\mathrm{T}}). \tag{24}$$

Below, we say more about the common properties of A and A^{T}. For the moment, we return to (2). Equating the entries on the left and the right, we find that

$$\{AA^*\}_{ij} = \sum_{k=1}^{n} a_{ik}\bar{a}_{jk} = \sum_{k=1}^{n} a_{ki}\bar{a}_{kj} = \{\overline{A^*A}\}_{ij}. \tag{25}$$

The result obtained can be stated as follows.

Theorem 6. *A matrix $A \in M_n(\mathbb{C})$ is conjugate-normal if and only if relation (25) is fulfilled for each pair (i, j), $1 \leqslant i, j \leqslant n$.*

In other words, $A \in \mathscr{C}N_n$ if and only if, for each pair (i, j), the scalar product of rows i and j is equal to the scalar product of columns i and j.

Corollary 1. *If $A \in \mathscr{C}N_n$, then, for each $i\,(1 \leqslant i \leqslant n)$, the 2-norm of row i is equal to the 2-norm of column i.*

Corollary 2. *Let $A_{i_1,\ldots,i_k}$ be the submatrix formed of rows $i_1, \ldots, i_k (i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k)$ of a matrix $A \in M_n(\mathbb{C})$. If $A \in \mathscr{C}N_n$, then*

$$||A_{i_1,\ldots,i_k}||_F = ||A^{\mathrm{T}}_{i_1,\ldots,i_k}||_F. \tag{26}$$

Recall that a normal matrix A shares its invariant subspaces with the Hermitian adjoint A^*. We can prove an analogous fact for conjugate-normal matrices.

Theorem 7. *Every coninvariant subspace of $A \in \mathscr{C}N_n$ is also a coninvariant subspace of A^{T}.*

Proof. Let $\mathscr{L}_k$ be a k-dimensional coninvariant subspace of $A(1 \leqslant k \leqslant n)$. Choose an orthonormal basis $q_1, \ldots, q_k$ in $\mathscr{L}_k$, complement it to a basis $q_1, \ldots, q_k, q_{k+1}, \ldots, q_n$ of the entire space $\mathbb{C}^n$, and define

$$Q = [\underbrace{Q_1}_{k}\ \underbrace{Q_2}_{n-k}] = [q_1 \cdots q_n],$$

$$B = Q^{\mathrm{T}}AQ. \tag{27}$$

Partition B in conformity with Q:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Thus, B_{11} is a $k \times k$ block. Rewrite (27) as

$$AQ = \overline{Q}B. \tag{28}$$

Then, the definition of a coninvariant subspace implies that

$$B_{21} = 0.$$

Since $B \in \mathscr{C}N$, relation (26) yields

$$B_{12} = 0.$$

From (27), we derive

$$B^{\mathrm{T}} = Q^{\mathrm{T}}A^{\mathrm{T}}Q$$

and

$$A^{\mathrm{T}} Q_1 = \overline{Q_1} B_{11}^T. \tag{29}$$

Equality (29) means that $\mathscr{L}_k$ is a coninvariant subspace of A^{T}. □

Remark 2. The proof presented above gives more than just the assertion of Theorem 7. Indeed, since $B_{12} = 0$, relation (28) yields

$$A Q_2 = \overline{Q_2} B_{22}.$$

Thus, $\mathscr{L}^{\perp} = \operatorname{span}\{q_{k+1}, \ldots, q_n\}$ is a coninvariant subspace of A.

We state this result as

Theorem 8. *If $\mathscr{L}$ is a coninvariant subspace of a conjugate-normal matrix A, then $\mathscr{L}^{\perp}$ is also a coninvariant subspace of A.*

Remark 3. Theorems 7 and 8 correspond to items 8 and 9 in the GJSW list. However, unlike the latter, they are not criteria, because the conditions in these theorems do not imply that A is conjugate-normal.

Now, we examine the Youla form B of a conjugate-normal matrix A. Reasoning as in the proof of Theorem 7, we conclude that all the off-diagonal blocks in B are zero. Thus, the Youla form of $A \in \mathscr{C}N$ is a block diagonal matrix with 1×1 and 2×2 diagonal blocks.

Several canonical forms for conjugate-normal matrices were derived by various authors independently of the Youla theorem. Since 1×1 blocks can always be made real nonnegative scalars (which are the real coneigenvalues of A), the canonical matrices differ from each other only in the form of 2×2 blocks. Recall that each 2×2 block corresponds to a pair of complex conjugate coneigenvalues $\mu = a + \mathrm{i}b = \varrho e^{\mathrm{i}\varphi}$ and $\bar{\mu}$. For the canonical form derived in [11], 2×2 blocks are chosen as

$$\begin{bmatrix} 0 & \varrho \\ \varrho e^{\mathrm{i}2\varphi} & 0 \end{bmatrix}.$$

In the Wigner canonical form (see [12]), 2×2 blocks are Hermitian matrices of the type

$$\begin{bmatrix} 0 & \bar{\mu} \\ \mu & 0 \end{bmatrix}. \tag{30}$$

Finally, in [2], 2×2 blocks are real normal matrices of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \tag{31}$$

In the latter case, the entire canonical matrix B is real and normal. It follows that a conjugate-normal matrix A is unitarily congruent to some real normal matrix. Since unitary congruences preserve conjugate normality, the reverse statement is also true; namely, any unitary congruence transformation of a real normal matrix yields a conjugate-normal matrix. We thus have the following proposition:

Theorem 9. *A matrix $A \in M_n(\mathbb{C})$ is conjugate-normal if and only if A is unitarily congruent to a real normal matrix.*

Theorems 4, 6, and 9 can be looked at as criteria for conjugate normality. As such, they are considered to be items 1–3 in the list of conditions equivalent to conjugate normality. This list will be considerably extended in the next section.

4. Conditions

In this section, we present a list of about forty conditions on a matrix $A \in M_n(\mathbb{C})$, each of which is equivalent to A being conjugate-normal. The first condition in this list is numbered 4 for the reasons explained at the end of the previous section.

Some conditions involve an additional requirement of nonsingularity. In each case, such a restriction is indicated explicitly.

Most of the conditions in our list are counterparts of the appropriate conditions in the GJSW or EI lists, and we indicate the locations of the latter in those lists.

4. A^{T} is conjugate-normal.
5. $\overline{A}$ is conjugate-normal.
6. A^* is conjugate-normal.
7. A^{-1} is conjugate-normal (for invertible A).
   (Cf. condition 2 in the GJSW list.)
8. $A^{-1}A^{\mathrm{T}}$ is unitary (for invertible A).
   (Cf. condition 3 in the GJSW list.)
9. $\overline{A} = A^{-\mathrm{T}}AA^*$ (for invertible A). Thus, a matrix $A \in \mathscr{C}N$ is consimilar to $\overline{A}$, the transformation being specified by A^*.
   (Cf. condition 4 in the GJSW list.)
10. $AA^*A = A^{\mathrm{T}}\overline{A}A$ (or $A\overline{A}A^{\mathrm{T}} = AA^*A$).
    (Cf. condition 74 in the EI list.)
11. $A\overline{C} = CA$, where $C = AA^* - A^{\mathrm{T}}\overline{A}$.
    (Cf. condition 73 in the EI list. The matrix C is an analogue of the self-commutator $[A, A^*] = AA^* - A^*A$.)
12. $A\overline{B} = B\overline{A}$ implies that $A^{\mathrm{T}}\overline{B} = BA^*$. In words, if A concommutes with some matrix B, then A^{T} concommutes with B as well.
    (Cf. condition 6 in the GJSW list.)
13. $U^{\mathrm{T}}AU$ is conjugate-normal for any (or for some) unitary U.
    (Cf. condition 7 in the GJSW list.)
14. $p(A\overline{A})A = Ap(\overline{A}A)$ is conjugate-normal for any polynomial p with real coefficients.
    (Cf. condition 1 in the GJSW list.)
15. There exists a polynomial $p(z)$ with real coefficients such that
    $$A^{\mathrm{T}} = p(A\overline{A})A = Ap(\overline{A}A).$$
    (Cf. condition 17 in the GJSW list.)
16. The matrix $A^*A - \overline{AA^*}$ is semidefinite.
    (Cf. condition 20 in the GJSW list.)

The following eight conditions refer to the SSS decomposition of A (see (16)).

17. $S\overline{K} = K\overline{S}$.
    (Cf. condition 21 in the GJSW list.)

18. $A\overline{S} = S\overline{A}$.
(Cf. condition 22 in the GJSW list.)
19. $A\overline{S} + SA^* = 2S\overline{S}$ (or $\overline{S}A + A^*S = 2\overline{S}S$).
(Cf. condition 23 in the GJSW list.)
20. $A\overline{K} = K\overline{A}$.
(Cf. condition 24 in the GJSW list.)
21. $A\overline{K} - KA^* = 2K\overline{K}$ (or $\overline{K}A - A^*K = 2\overline{K}K$).
(Cf. condition 25 in the GJSW list.)
22. $S^{-1}A + A^*\overline{S}^{-1} = 2I$ (or $AS^{-1} + \overline{S}^{-1}A^* = 2I$), as long as S is nonsingular.
(Cf. condition 26 in the GJSW list.)
23. $K^{-1}A - A^*\overline{K}^{-1} = 2I$ (or $AK^{-1} - \overline{K}^{-1}A^* = 2I$), as long as K is nonsingular.
(Cf. condition 27 in the GJSW list.)
24. $S\overline{S} - K\overline{K} = AA^*$.
(Cf. condition 75 in the EI list.)

In the following conditions, $c\lambda(A) = \{\mu_, \ldots, \mu_n\}, c\lambda(S) = \{\zeta_1, \ldots, \zeta_n\}$, and $c\lambda(K) = \{i\eta_1, \ldots, i\eta_n\}$ ($\eta_j \in \mathbb{R}$, $1 \leqslant j \leqslant n$) are the conspectra of A, S, and K, respectively.

25. $|\mu_1|^2 + \cdots + |\mu_n|^2 = ||A||_F^2$.
(Cf. condition 53 in the GJSW list.)
26. $(\mathrm{Re}\mu_1)^2 + \cdots + (\mathrm{Re}\mu_n)^2 = \zeta_1^2 + \cdots + \zeta_n^2$.
(Cf. condition 54 in the GJSW list.)
27. $(\mathrm{Im}\mu_1)^2 + \cdots + (\mathrm{Im}\mu_n)^2 = \eta_1^2 + \cdots + \eta_n^2$.
(Cf. condition 55 in the GJSW list.)
28. There exists a permutation $\delta \in S_n$ such that

$$c\lambda(A) = \{\zeta_j + i\eta_{\delta_j} | j = 1, \ldots, n\}.$$

(Cf. condition 34 in the GJSW list.)
29. $\mathrm{Re}c\lambda(A) = \{\zeta_1, \ldots, \zeta_n\}$.
(Cf. condition 35 in the GJSW list.)
30. $\mathrm{Im}c\lambda(A) = \{\eta_1, \ldots, \eta_n\}$.
(Cf. condition 36 in the GJSW list.)

The following conditions refer to symmetric-unitary polar decompositions (SUPDs) introduced in Section 2 (see (17)).

31. A and A^{T} admit SUPDs $A = SU$ and $A^{\mathrm{T}} = SV$ with the same symmetric factor S.
(Cf. condition 71 in the EI list.)
32. There exists an SUPD of A such that $A\overline{S} = S\overline{A}$.
(Cf. condition 39 in the GJSW list.)
33. There exists an SUPD of A such that $U\overline{S}S = \overline{S}SU$.
(Cf. condition 37 in the GJSW list.)
34. There exists an SUPD of A such that $UA^*A = A^*AU$.
(Cf. condition 38 in the GJSW list.)
35. $\lambda(AA^*) = \{|\mu_1|^2, \ldots, |\mu_n|^2\}$.
(Cf. condition 57 in the GJSW list.)

36. The singular values of A are $|\mu_1|, \ldots, |\mu_n|$.
(Cf. condition 58 in the GJSW list.)
37. The coneigenvalues of the symmetric matrix S in any SUPD of A are $|\mu_1|, \ldots, |\mu_n|$.
(Cf. condition 47 in the GJSW list.)
38. Suppose that the coneigenvalues $\mu_1, \ldots, \mu_n$ of A are numbered so that $|\mu_1| \geqslant |\mu_2| \geqslant \cdots \geqslant |\mu_n|$. Then,

$$s_1 \cdots s_k = |\mu_1 \cdots \mu_k|, \quad 1 \leqslant k \leqslant n.$$

(Cf. condition 59 in the GJSW list.)
39. A^+ is conjugate normal.
(Cf. condition 60 in the GJSW list.)
40. A^+A^{T} is a partial isometry.
(Cf. condition 61 in the GJSW list.)
41. $(Ax, Ay) = (A^{\mathrm{T}}x, A^{\mathrm{T}}y)$ for all $x, y \in \mathbb{C}^n$.
(Cf. condition 62 in the GJSW list.)
42. $(Ax, Ax) = (A^{\mathrm{T}}x, A^{\mathrm{T}}x)$ for all $x \in \mathbb{C}^n$.
(Cf. condition 63 in the GJSW list.)
43. $||Ax||_2 = ||A^{\mathrm{T}}x||_2$ for all $x \in \mathbb{C}^n$.
(Cf. condition 64 in the GJSW list.)
44. $A^{\mathrm{T}} = UA$ for some unitary U.
(Cf. condition 65 in the GJSW list.)

For a matrix A with the conspectrum $c\lambda(A) = \{\mu_1, \ldots, \mu_n\}$, define the conspectral radius as

$$c\rho(A) = \max_{1 \leqslant i \leqslant n} |\mu_i|.$$

45. $||C_k(A)||_2 = c\rho(C_k(A)), \quad k = 1, 2, \ldots, n-1.$
(Cf. condition 88 in the EI list.)

5. Proofs and comments

Some of the conditions presented in Section 4 are obvious or can be verified by straightforward calculations. Several conditions are just restatements of the others in the same list.

The main technical tool in proving most of the nontrivial conditions is Theorem 4. The following approach is used: for a given condition on A, find the appropriate condition on $\widehat{A}$ in the GJSW or EI list. Since the latter condition is equivalent to normality of $\widehat{A}$, the former is equivalent to A being conjugate-normal.

Let us illustrate this approach by several examples.

Proof of condition 12. Sufficiency is shown by setting $B = A$. Since $A\overline{B} = B\overline{A}$ is trivially true for $B = A$, we must have $A^{\mathrm{T}}\overline{A} = AA^*$, i.e., A is conjugate-normal. Now, let $A \in \mathscr{C}N$, and let B concommute with A:

$$A\overline{B} = B\overline{A}. \tag{32}$$

For

$$\widehat{A} = \begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix} \quad \text{and} \quad \widehat{B} = \begin{bmatrix} 0 & B \\ \overline{B} & 0 \end{bmatrix},$$

we have

$$\widehat{A}\widehat{B} = A\overline{B} \oplus \overline{A}B, \quad \widehat{B}\widehat{A} = B\overline{A} \oplus \overline{B}A.$$

In view of (32), $\widehat{A}$ and $\widehat{B}$ commute. Since $\widehat{A}$ is normal, condition 6 in the GJSW list yields

$$\widehat{A}^*\widehat{B} = \widehat{B}\widehat{A}^*,$$

which is equivalent to

$$A^{\mathrm{T}}\overline{B} = BA^*. \qquad \square$$

In this proof, the approach outlined above was used only in one direction. Let us consider a couple of examples where this approach is exploited in full.

Proof of condition 11. Formulas (18)–(21) show that the self-commutator of $\widehat{A}$ has the form

$$[\widehat{A}, \widehat{A}^*] = C \oplus \overline{C},$$

where $C = AA^* - A^{\mathrm{T}}\overline{A}$. According to condition 73 in the EI list, $\widehat{A}$ is normal if and only if $\widehat{A}$ commutes with its self-commutator. Since

$$\widehat{A}\begin{bmatrix} C & 0 \\ 0 & \overline{C} \end{bmatrix} = \begin{bmatrix} 0 & A\overline{C} \\ \overline{A}C & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C & 0 \\ 0 & \overline{C} \end{bmatrix}\widehat{A} = \begin{bmatrix} 0 & CA \\ \overline{CA} & 0 \end{bmatrix},$$

the commutation condition amounts to the relation

$$A\overline{C} = CA.$$

This relation is equivalent to A being conjugate-normal. $\square$

Proof of condition 40. It is easily verified that the Moore–Penrose inverse of $\widehat{A}$ is given by

$$\widehat{A}^+ = \begin{bmatrix} 0 & \overline{A^+} \\ A^+ & 0 \end{bmatrix}.$$

One possible way is to check four Moore–Penrose conditions defining the pseudoinverse.

Condition 61 in the GJSW list says that $\widehat{A}$ is normal if and only if $\widehat{A}^+\widehat{A}^*$ is a partial isometry. Recall that a matrix B is called a partial isometry if B^*B is an orthoprojector. Since

$$\widehat{A}^+\widehat{A}^* = \begin{bmatrix} \overline{A^+A^{\mathrm{T}}} & 0 \\ 0 & A^+A^{\mathrm{T}} \end{bmatrix},$$

we conclude that A^+A^* is a partial isometry if and only if $\widehat{A}^+\widehat{A}^{\mathrm{T}}$ is. Thus, this condition is equivalent to the conjugate normality of A. $\square$

Now that the approach involving $\widehat{A}$ is clear, we discuss a few conditions for which this approach is inefficient.

Proof of condition 14. Sufficiency is easy: just set $p(z) \equiv 1$; then, $p(A\overline{A})A = A$ is conjugate-normal. To prove necessity, consider a canonical form of the conjugate-normal matrix A. For definiteness, let B be the Wigner canonical form of A. We should verify that $p(B\overline{B})B$ is a matrix of the same block diagonal form as B. Each 1×1 block, i.e., a nonnegative scalar μ produces a scalar $p(\mu^2)\mu$. The new scalars can be negative, but this is easily mended by an additional congruence with a unitary diagonal transformation matrix. Each 2×2 block in B (see (30)) transforms into

$$\begin{bmatrix} 0 & p(\bar{\mu}^2)\bar{\mu} \\ p(\mu^2)\mu & 0 \end{bmatrix}. \tag{33}$$

Thus, $p(B\overline{B})B$ is a canonical Wigner matrix, and $p(B\overline{B})B$ (hence, $p(A\overline{A})A$) is conjugate-normal.

The assumption that p has real coefficients is important because, otherwise, the scalars $p(\mu^2)\mu$ and $p(\bar{\mu}^2)\bar{\mu}$ may not be conjugate. What is more important, they may have different moduli, which contradicts Corollary 1. □

We are now going to prove condition 15, which corresponds to condition 17 in the GJSW list. The latter reads 'there exists a polynomial p such that $A^* = p(A)$.' It is not required that p have real coefficients. However, as shown by Laffey [9], one can always find a real polynomial p for the relation $A^* = p(A)$ with a normal A.

Proof of condition 15. *Necessity.* It is convenient to seek the desired polynomial p using the Wigner canonical form B of A. Thus, we want to have

$$B^{\mathrm{T}} = p(B\overline{B})B.$$

This yields

$$p(\mu^2)\mu = \mu \tag{34}$$

for each real μ, which is automatically fulfilled if $\mu = 0$ and is a nontrivial condition for each positive coneigenvalue μ.

For 2×2 blocks, we require that

$$\begin{bmatrix} 0 & p(\bar{\mu}^2)\bar{\mu} \\ p(\mu^2)\mu & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mu \\ \bar{\mu} & 0 \end{bmatrix}$$

(see (30)), which reduces to two scalar relations

$$p(\mu^2)\mu = \bar{\mu} \tag{35}$$

and

$$p(\bar{\mu}^2)\bar{\mu} = \mu. \tag{36}$$

Now, relations (34) written for all distinct positive μ and relations (35) and (36) written for all distinct complex conjugate pairs $(\mu, \bar{\mu})$ constitute a Vandermonde-like system of linear equations with respect to the coefficients of the polynomial p. Thus, the system is solvable. Its solution is unique if we restrict the degree of p by $m - 1$, where m is the number of equations.

The special feature of our system is that, along with an equation with complex coefficients, it contains the one with the conjugate coefficients. Since the solution is unique, it must be real.

Sufficiency. From $A^{\mathrm{T}} = p(A\overline{A})A$, we derive

$$AA^* = A\overline{A^{\mathrm{T}}} = A\overline{p(A\overline{A})A} = Ap(\overline{A}A)\overline{A}$$

(note that the last equality uses the fact that p has real coefficients) and

$$\overline{A^*A} = A^{\mathrm{T}}\overline{A} = p(A\overline{A})A\overline{A} = Ap(\overline{A}A)\overline{A}.$$

Thus, $A \subset \mathscr{C}N$. □

Proof of condition 31. *Sufficiency.* Let $A = SU$ and $A^{\mathrm{T}} = SV$, where U and V are unitary. Then,

$$AA^* = (SU)(U^*\overline{S}) = S\overline{S}$$

and

$$\overline{A^*A} = A^{\mathrm{T}}\overline{A} = (SV)(V^*\overline{S}) = S\overline{S}.$$

Hence, A is conjugate-normal.

Necessity. Perform a unitary congruence that transforms the conjugate-normal matrix A into a real normal matrix B. Observe that, in the polar decomposition of B, the symmetric factor P and the orthogonal factor W commute:

$$B = PW = WP.$$

It follows that

$$B^{\mathrm{T}} = PW^{\mathrm{T}}.$$

Performing the reverse congruence transformation, we obtain SUPDs for A and A^{T} having the same symmetric factor S. □

Note that conditions 32–34 can be proved following the same pattern. The details can be found in [3].

Proof of condition 44. *Sufficiency.* If $A^{\mathrm{T}} = UA$ for a unitary U, then

$$(A^{\mathrm{T}}x, A^{\mathrm{T}}y) = (UAx, UAy) = (Ax, Ay) \quad \forall x, y \in \mathbb{C}^n.$$

By condition 41, $A \in \mathscr{C}N$.

Necessity. First, bring A and A^{T} by a unitary congruence transformation to their Wigner canonical forms B and B^{T}:

$$B = QAQ^{\mathrm{T}}, \qquad B^{\mathrm{T}} = QA^{\mathrm{T}}Q^{\mathrm{T}}. \tag{37}$$

Then for each 2×2 block

$$\begin{bmatrix} 0 & \varrho \mathrm{e}^{-\mathrm{i}\varphi} \\ \varrho e^{i\varphi} & 0 \end{bmatrix}$$

in B (see (30)) form the diagonal matrix

$$\begin{bmatrix} e^{i2\varphi} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}2\varphi} \end{bmatrix}. \tag{38}$$

The direct sum of matrices of type (38) complemented by an identity matrix (which corresponds to the 1×1 blocks in B) yields a diagonal unitary matrix D such that

$$B^{\mathrm{T}} = DB.$$

Finally, we reverse transformation (37):

$$A^{\mathrm{T}} = Q^*B^{\mathrm{T}}\overline{Q} = Q^*DB\overline{Q} = (Q^*DQ)(Q^*B\overline{Q}) = UA.$$

Thus, $U = Q^*DQ$ is the desired unitary matrix. □

Proof of condition 45. The coneigenvalues of $C_k(A)$ squared are the ordinary eigenvalues of

$$C_k(A)\overline{C_k(A)} = C_k(A)C_k(\overline{A}) = C_k(A\overline{A}). \tag{39}$$

If $\mu_1, \ldots, \mu_n$ ($|\mu_1| \geqslant |\mu_2| \geqslant \cdots \geqslant |\mu_n|$) are the coneigenvalues of A, then the eigenvalues of matrix (39) are all the possible k-products

$$\mu_{i_1}^2 \mu_{i_2}^2 \cdots \mu_{i_k}^2$$

with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ (see [10, p. 24]). It follows that

$$c\rho(C_k(A)) = |\mu_1| \cdots |\mu_k|.$$

On the other hand,

$$||C_k(A)||_2 = s_1 \cdots s_k.$$

Thus, condition 45 is just a disguised form of condition 38. □

6. Concluding remarks

The discussion presented above is not exhaustive. Several interesting properties of conjugate-normal matrices were left outside the scope of this paper. Here, we briefly mention one of these properties.

It is well known that normal matrices are perfectly conditioned with respect to the problem of calculating their eigenvalues. A similar fact can be established for the coneigenvalues of a conjugate-normal matrix. More on this can be found in [4].

It was shown in Remark 1 that the normality of A_L and A_R does not imply that A is conjugate-normal. Matrices defined by the requirement that A_L (or A_R) be normal were introduced in [6] and were called congruence-normal matrices there. This is an interesting matrix class standing farther from normal matrices than conjugate-normal matrices. We intend to discuss this class in a separate paper.

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