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On condiagonalizable matrices

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Dedicated to Roger Horn on the occasion of his 65th birthday

Abstract

We call $A \in M_n(\mathbf{C})$ a condiagonalizable matrix if $A_{\mathrm{R}} = A\overline{A}$ (or, which is the same, $A_{\mathrm{L}} = \overline{A}A$) is diagonalizable by a conventional similarity transformation. Our main result is that any condiagonalizable matrix can be brought by a consimilarity transformation to a special block diagonal form with the diagonal blocks of orders one and two.

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1. Introduction

In this paper, the term "condiagonalizable matrices" is understood in the sense of the following definition.

Definition 1. A matrix $A \in M_n(\mathbf{C})$ is said to be condiagonalizable if $A_{\mathrm{R}} = A\overline{A}$ (or, which is the same, $A_{\mathrm{L}} = \overline{A}A$) is diagonalizable by a similarity transformation.

This goes against the familiar definition of condiagonalizability as it is given, for instance, in [1, Definition 4.6.2].

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Definition 2. A matrix $A \in M_n(\mathbf{C})$ is said to be condiagonalizable if there exists a nonsingular $S \in M_n(\mathbf{C})$ such that $S^{-1}A\overline{S}$ is diagonal.

The reason why we had to change the latter definition is as follows. According to Theorem 4.6.11 in [1], A is condiagonalizable in the sense of Definition 2 if $A_{\mathrm{R}} = A\overline{A}$ is a diagonalizable matrix with real nonnegative eigenvalues and rank $A_{\mathrm{R}} = \operatorname{rank} A$. Thus, it is highly unlikely that a randomly chosen $A \in M_n(\mathbf{C})$ is condiagonalizable. This is in sharp contrast with conventional diagonalizability, which is a generic property of square complex matrices.

Our main result in this paper is that any condiagonalizable matrix in the sense of Definition 1 can be brought by a consimilarity transformation to a special block diagonal form with the diagonal blocks of orders one and two. (Note that no assumptions apart from diagonalizability are made of A_{R}.) The more accurate formulation and the proof are given in Section 3 after the preliminaries have been presented in Section 2. As an illustration of the main theorem, we briefly touch in Section 4 two special matrix classes that we call conprojectors and coninvolutions. Our concluding remarks are given in Section 5.

2. Preliminaries

Matrices $A, B \in M_n(\mathbf{C})$ are said to be consimilar if $A = SB\overline{S}^{-1}$ for a nonsingular matrix $S \in M_n(\mathbf{C})$, where, as usual, $\overline{S}$ is the component-wise conjugate of S. A good exposition of the theory of consimilarity is given in Section 4.6 of [1] (see also the references therein).

The n eigenvalues of a matrix $A \in M_n(\mathbf{C})$ are its simplest (and most important) similarity invariants. We want to define analogous invariants with respect to consimilarity transformations. To this end, we introduce the matrices

$$A_{\mathrm{L}} = \overline{A}A \quad \text{and} \quad A_{\mathrm{R}} = A\overline{A}. \tag{1}$$

Although the products AB and BA need not to be similar in general, A_{L} is always similar to A_{R} (see [1, p. 246, Problem 9 in Section 4.6]). Therefore, in the subsequent discussion of their spectral properties, it will be sufficient to refer to one of them, say, A_{L}.

The spectrum of A_{L} has two remarkable properties:

1. It is symmetric with respect to the real axis. Moreover, the eigenvalues λ and $\bar{\lambda}$ are of the same multiplicity.
2. The negative eigenvalues of A_{L} (if any) are necessarily of even algebraic multiplicity.

For the proofs of these properties, we refer the reader to [1, p. 252–253].

Let

$$\lambda(A_{\mathrm{L}}) = \{\lambda_1, \ldots, \lambda_n\}$$

be the spectrum of A_{L}. The *coneigenvalues* of A are the n scalars $\mu_1, \ldots, \mu_n$ defined as follows:

If $\lambda_i \in \lambda(A_{\mathrm{L}})$ does not lie on the negative real axis, then the corresponding coneigenvalue μ_i is defined as a square root of λ_i with the nonnegative real part and the multiplicity of μ_i is set to that of λ_i

$$\mu_i = \lambda_i^{1/2}, \qquad \operatorname{Re}\mu_i \geqslant 0.$$

With a real negative $\lambda_i \in \lambda(A_{\mathrm{L}})$, we associate two conjugate purely imaginary coneigenvalues

$$\mu_i = \pm\lambda_i^{1/2}.$$

The multiplicity of each is set to half of that of λ_i.

Note that the definition of the coneigenvalues given above is similar or identical to the definitions in [2,3] and is different from the definition in [1]. In particular, the coneigenvalues as defined in [1] can exist only if A_{L} has real nonnegative eigenvalues. The coneigenvalues as defined above exist for any $n \times n$ complex matrix A.

Remark. If $A \in M_n(\mathbf{R})$, then each eigenvalue of A with a nonnegative real part is at the same time a coneigenvalue of this matrix. If an eigenvalue λ has a negative real part, then $\mu = -\lambda$ is a coneigenvalue of A.

The concept of a coninvariant subspace is a very important one for this paper.

Definition 3. A subspace $\mathscr{L}$ is said to be a coninvariant subspace of A (or A-coninvariant subspace) if

$$A\mathscr{L} \subset \overline{\mathscr{L}}. \tag{2}$$

Theorem 1. *Let $A \in M_n(\mathbf{C})$. Then A has a one- or two-dimensional coninvariant subspace.*

For the proof, see [4, Section 2].

The following proposition is almost obvious.

Proposition 1. *Any coninvariant subspace of A is an invariant subspace of A_{L}.*

Indeed, relation (2) implies that

$$A_{\mathrm{L}}\mathscr{L} = \overline{A}A\mathscr{L} \subset \overline{A}(\overline{\mathscr{L}}) \subset \mathscr{L}.$$

The question whether an invariant subspace $\mathscr{L}$ of A_{L} is at the same time a coninvariant subspace of A is more intricate. To answer it, we first show that $\mathscr{L}$ can be embedded into an A-coninvariant subspace.

Let $r = \dim \mathscr{L}$ and X be a base matrix of $\mathscr{L}$. By assumption,

$$\overline{A}AX = X\Lambda \tag{3}$$

for some r-by-r matrix Λ. Define

$$\overline{Y} = AX \tag{4}$$

and

$$Z = (X Y). \tag{5}$$

It follows from (3) that:

$$AY = A(\overline{AX}) = \overline{\overline{A}AX} = \overline{X\Lambda}. \tag{6}$$

Relations (4) and (6) imply that

$$A(XY) = \overline{(XY)}M, \tag{7}$$

where

$$M = \begin{pmatrix} 0 & \overline{\Lambda} \\ I_r & 0 \end{pmatrix}. \tag{8}$$

Equality (7) means that the subspace $\mathscr{M}$ spanned by the columns of Z is a coninvariant subspace of A. It obviously contains the original subspace $\mathscr{L}$.

The latter observation implies that the spectrum of the matrix

$$M_{\mathrm{L}} = \overline{M}M = \Lambda \oplus \bar{\Lambda} \tag{9}$$

contains the eigenvalues of A_{L} corresponding to its projection on $\mathscr{L}$ and, in addition to those eigenvalues, may contain only their conjugate numbers. This permits us to draw the following two important conclusions:

Corollary 1. *Let λ be a real eigenvalue of A_{L}. Then, the generalized eigenspace of A_{L} associated with λ is a coninvariant subspace of A.*

Corollary 2. *Let λ be a complex eigenvalue of A_{L}. Then, the direct sum of the generalized eigenspaces of A_{L} associated with λ and $\bar{\lambda}$ is a coninvariant subspace of A.*

For the proof of our main theorem, we also need the following assertions.

Proposition 2. *A matrix $A \in M_n(\mathbf{C})$ has the property that $A_{\mathrm{L}} = I$ if and only if there exists a nonsingular matrix $S \in M_n(\mathbf{C})$ such that $A = \overline{S}S^{-1}$.*

The proof of Proposition 2 is given in [1, Lemma 4.6.9]. We will use the idea of that proof in justifying the assertion below.

Proposition 3. *A matrix $A \in M_n(\mathbf{C})$ has the property that $A_{\mathrm{L}} = -I$ if and only if n is an even integer and there exists a nonsingular matrix $S \in M_n(\mathbf{C})$ such that $A = \overline{S}JS^{-1}$, where*

$$J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \quad \textit{and} \quad n = 2m.$$

Proof. The relation $A_{\mathrm{L}} = -I$ says that -1 is the only eigenvalue of A_{L}; thus, n is necessarily an even number. For a real θ, define

$$S_\theta = \mathrm{e}^{-\mathrm{i}\theta}\overline{A} + \mathrm{e}^{\mathrm{i}\theta}J.$$

Observe that $J^2 = -I_n$. Using this relation, we have

$$\begin{aligned} AS_\theta &= \mathrm{e}^{-\mathrm{i}\theta}A\overline{A} + \mathrm{e}^{\mathrm{i}\theta}AJ = -\mathrm{e}^{-\mathrm{i}\theta}I_n + \mathrm{e}^{\mathrm{i}\theta}AJ \\ &= \mathrm{e}^{-\mathrm{i}\theta}J^2 + \mathrm{e}^{\mathrm{i}\theta}AJ = (\mathrm{e}^{-\mathrm{i}\theta}J + \mathrm{e}^{\mathrm{i}\theta}A)J = \overline{S}_\theta J. \end{aligned}$$

Thus,

$$AS_\theta = \overline{S}_\theta J$$

for any real θ. Both matrices A and J are nonsingular; hence, the pencil $\alpha A + \beta J$ is regular. Choosing θ so that $-\mathrm{e}^{-2\mathrm{i}\theta}$ is different from the eigenvalues of this pencil, we obtain a nonsingular matrix S_θ. Then, the latter equality yields the desired result. □

The necessity part of the theorem is verified straightforwardly.

3. Main theorem

We first prove the following simple lemma.

Lemma 1. *Let x be an eigenvector of A_{L} corresponding to its eigenvalue λ. Then*:

(a) *$\bar{x}$ is an eigenvector of A_{R} corresponding to the eigenvalue $\bar{\lambda}$;*
(b) *$y = \overline{Ax}$ (if nonzero) is an eigenvector of A_{L} corresponding to $\bar{\lambda}$;*
(c) *$\bar{y} = Ax$ (if nonzero) is an eigenvector of A_{R} corresponding to λ.*

Proof. Proposition (a) is obvious, and Proposition (b) is an immediate implication of (a) and (c). Thus, we prove only (c) in the case $y \neq 0$. Since

$$\overline{A}Ax = \lambda x,$$

we have

$$A\overline{A}Ax = \lambda Ax,$$

that is,

$$A_{\mathrm{R}}\bar{y} = \lambda\bar{y}. \qquad \square$$

Now, we can state our main result.

Theorem 2. *Let $A \in M_n(\mathbf{C})$ be a condiagonalizable matrix. Then, A can be brought by a consimilarity transformation to its canonical form which is a direct sum of 1×1 and 2×2 blocks. The 1×1 blocks are the real nonnegative coneigenvalues of A, while each 2×2 block corresponds to a pair of complex conjugate coneigenvalues $\mu, \bar{\mu}$ and has the form*

$$\begin{pmatrix} 0 & \bar{\mu} \\ \mu & 0 \end{pmatrix}. \tag{10}$$

If A is singular and $k = \dim\ker A_{\mathrm{L}} - \dim\ker A > 0$, then the canonical form of A also contains k blocks of the form

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{11}$$

Proof. Let us view $\mathbf{C}^n$ as the direct sum

$$\mathbf{C}^n = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_m, \tag{12}$$

where each $\mathscr{L}_i$ is either the eigenspace of A_{L} corresponding to its real eigenvalue or the direct sum of the two eigenspaces corresponding to a pair of complex conjugate eigenvalues. Choose a base matrix P_i for the subspace $\mathscr{L}_i$ ($1 \leqslant i \leqslant m$) and form the $n \times n$ matrix P as

$$P = (P_1 \cdots P_m). \tag{13}$$

By Corollaries 1 and 2, each $\mathscr{L}_i$ is an A-coninvariant subspace. This implies the existence of a matrix M_i with the appropriate size such that

$$AP_i = \overline{P}_i M_i, \quad i = 1, 2, \ldots, m.$$

It follows that:

$$AP = \overline{P}M,$$

where

$$M = M_1 \oplus \cdots \oplus M_m,$$

or

$$\overline{P}^{-1}AP = M.$$

Thus, we have shown that A can be brought by a consimilarity transformation to a block diagonal matrix such that each diagonal block has either a single real nonnegative coneigenvalue or a single pair of complex conjugate coneigenvalues. It remains to prove that, by further consimilarity transformations, every diagonal block can be made a direct sum of 1×1 and 2×2 matrices. We now explain how the current basis in each subspace $\mathscr{L}_i$ should be changed to achieve this goal.

Let $\mathscr{L}$ be any of those subspaces. We first assume that $\mathscr{L}$ corresponds to a positive coneigenvalue μ. Then, $\mathscr{L}$ is the eigenspace of A_{L} associated with $\lambda = \mu^2$. The projections Λ and M on $\mathscr{L}$ of A_{L} and A, respectively, satisfy the relations

$$\overline{M}M = \Lambda \quad \text{and} \quad \Lambda = \lambda I.$$

Then,

$$M_{\mathrm{L}} = \overline{M}M = \lambda I.$$

By Proposition 2, there exists a nonsingular matrix Q such that

$$M = \mu\overline{Q}Q^{-1}$$

or

$$\overline{Q}^{-1}MQ = \mu I.$$

Thus, the change of basis in $\mathscr{L}$ governed by the matrix Q makes M the scalar matrix μI.

Next, we examine the case where $\mathscr{L}$ corresponds to a pair of complex conjugate coneigenvalues $\mu, \bar{\mu}$ such that $\operatorname{Re}\mu > 0$. Then, $\mathscr{L}$ necessarily has an even dimension, say, $2k$. Viewing $\mathscr{L}$ as an A_{L}-invariant subspace, choose k linearly independent eigenvectors $x_1, \ldots, x_k$ corresponding to $\lambda = \mu^2$. Define

$$y_j = \frac{1}{\bar{\mu}}\overline{Ax_j}, \quad j = 1, \ldots, k. \tag{14}$$

The system $y_1, \ldots, y_k$ inherits the linear independence from $x_1, \ldots, x_k$. Indeed, assuming that

$$\alpha_1 y_1 + \cdots + \alpha_k y_k = 0,$$

we have

$$A(\bar{\alpha}_1 x_1 + \cdots + \bar{\alpha}_k x_k) = 0,$$

which means that a linear combination of eigenvectors corresponding to a nonzero eigenvalue λ of A_{L} belongs to $\ker A \subset \ker A_{\mathrm{L}}$; i.e., it is an eigenvector for the zero eigenvalue. It follows that this linear combination must be a zero vector. Since $x_1, \ldots, x_k$ are linearly independent, all the α_j $(1 \leqslant j \leqslant k)$ must be zero.

We claim that the augmented system $x_1, \ldots, x_k, y_1, \ldots, y_k$ is still linearly independent. Indeed, by Lemma 1, $y_1, \ldots, y_k$ are eigenvectors of A_{L} corresponding to $\bar{\lambda}$ and, hence, they are independent of the eigenvectors $x_1, \ldots, x_k$ corresponding to λ.

Observe that, for each j $(1 \leqslant j \leqslant k)$, the pair x_j, y_j spans a two-dimensional coninvariant subspace of A. Indeed, (14) implies that

$$Ay_j = \frac{1}{\bar{\mu}} A\overline{Ax_j} = \frac{1}{\bar{\mu}} \overline{A_{\mathrm{L}} x_j} = \frac{1}{\bar{\mu}} \overline{\lambda x_j} = \frac{1}{\bar{\mu}} \bar{\mu}^2 \bar{x}_j = \bar{\mu}\bar{x}_j.$$

In combination with (14), this means that

$$A(x_j y_j) = \overline{(x_j y_j)} N,$$

where N is matrix (10).

Now, changing the basis in $\mathscr{L}$ to $x_1, y_1, x_2, y_2, \ldots, x_k, y_k$, we obtain the desired block diagonal form with the 2×2 diagonal blocks (10) for the projection of A on $\mathscr{L}$.

Next, we consider the case where $\mathscr{L}$ corresponds to a pair of purely imaginary coneigenvalues $\mu, -\mu$. Being an eigenspace of A_{L} for the negative eigenvalue $\lambda = \mu^2$, $\mathscr{L}$ necessarily has an even dimension $2k$. The projections Λ and M on $\mathscr{L}$ of A_{L} and A, respectively, satisfy the relations

$$\overline{M} M = \Lambda \quad \text{and} \quad \Lambda = \lambda I.$$

Then,

$$M_{\mathrm{L}} = \overline{M} M = \lambda I.$$

By Proposition 3, there exists a nonsingular matrix Q such that

$$M = |\mu| \overline{Q} J Q^{-1}$$

or

$$\overline{Q}^{-1} M Q = |\mu| J.$$

Thus, the change of basis in $\mathscr{L}$ governed by the matrix Q transforms M into the matrix

$$|\mu| J = \begin{pmatrix} 0 & -|\mu| I_k \\ |\mu| I_k & 0 \end{pmatrix}.$$

By an obvious permutation of the vectors in the new basis, we can make the above matrix to be the direct sum of 2×2 blocks of the form

$$\begin{pmatrix} 0 & -|\mu| \\ |\mu| & 0 \end{pmatrix}. \tag{15}$$

Finally, multiplying the vectors in the current basis by $\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$, we replace each block (15) by the corresponding block (10).

It remains to consider the case where $\mathscr{L}$ corresponds to the zero coneigenvalue. Then, $\mathscr{L}$ is the null space of A_{L}. If $\ker A_{\mathrm{L}} = \ker A$, then the projection M on $\mathscr{L}$ of the matrix A is a zero matrix, and we are done. Therefore, we assume that $k = \dim\ker A_{\mathrm{L}} - \dim\ker A > 0$. It follows that there exist k linearly independent vectors $z_1, \ldots, z_k$ of the form

$$z_i = Ax_i, \tag{16}$$

where

$$x_i \in \mathscr{L}, \quad i = 1, \ldots, k. \tag{17}$$

Relations (16) and (17) imply that

$$\overline{A} z_i = 0, \quad i = 1, \ldots, k.$$

Define the vectors

$$y_i = \bar{z}_i = \overline{Ax_i}, \quad i = 1, \ldots, k. \tag{18}$$

Then,

$$Ay_i = 0, \quad i = 1, \ldots, k. \tag{19}$$

We claim that the system $x_1, \ldots, x_k, y_1, \ldots, y_k$ is linearly independent. Indeed, assume that

$$\alpha_1 x_1 + \cdots + \alpha_k x_k + \beta_1 y_1 + \cdots + \beta_k y_k = 0. \tag{20}$$

Multiplying both sides by A and using (16) and (19), we obtain

$$\alpha_1 z_1 + \cdots + \alpha_k z_k = 0.$$

Since $z_1, \ldots, z_k$ are linearly independent, we have

$$\alpha_i = 0, \quad i = 1, \ldots, k.$$

Then, (20) implies that

$$\beta_i = 0, \quad i = 1, \ldots, k.$$

Observe that, for each j $(1 \leqslant j \leqslant k)$, the pair x_j, y_j spans a two-dimensional coninvariant subspace of A. Indeed, (18) and (19) show that

$$A(x_j y_j) = \overline{(x_j y_j)} N,$$

where N is matrix (11).

Change the order in the system $x_1, \ldots, x_k, y_1, \ldots, y_k$ to $x_1, y_1, x_2, y_2, \ldots, x_k, y_k$. If $l = \dim \ker A - k > 0$, we augment this system with l linearly independent vectors $v_1, \ldots, v_l$ to obtain a basis in $\mathscr{L}$. In this basis, the desired block diagonal form is attained with the 2×2 diagonal blocks (11) and, perhaps, with the l zero diagonal entries. The theorem is proved. □

4. Conprojectors and coninvolutions

Definition 4. A matrix $A \in M_n(\mathbf{C})$ is called a conprojector if A_{L} (or A_{R}) is an ordinary projector.

It follows from this definition that any conprojector A satisfies the equality:

$$\overline{A} A \overline{A} A = \overline{A} A. \tag{21}$$

It is well known that a projector may have only two eigenvalues, namely, 1 and 0. It follows that any conprojector may have only two coneigenvalues, that is, again, 1 and 0. Now, Theorem 2 implies the following assertion.

Theorem 3. *Let $A \in M_n(\mathbf{C})$ be a conprojector. Then, A can be brought by a consimilarity transformation to its canonical form which is a direct sum of 1×1 and 2×2 blocks. The 1×1 blocks are either ones or zeros, while each 2×2 block has the form*

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{22}$$

The blocks of form (22) can indeed appear in the canonical form. For instance,

$$A = 1 \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is a conprojector because

$$A_{\mathrm{L}} = A^2 = 1 \oplus 0 \oplus 0$$

is a projector (and even an orthoprojector).

Corollary 3. *The class of conprojectors can now be described as the set of matrices of the form*

$$A = P\Lambda\overline{P}^{-1},$$

where $P \in M_n(\mathbf{C})$ is an arbitrary nonsingular matrix and Λ is one of the canonical forms specified in Theorem 3.

Definition 5. A matrix $A \in M_n(\mathbf{C})$ is called a coninvolution if A_{L} (or A_{R}) is an ordinary involution.

It follows from this definition that any coninvolution A satisfies the equality:

$$\overline{A}A\overline{A}A = I. \tag{23}$$

It is well known that an involution may have only two eigenvalues, namely, 1 and -1. It follows that any coninvolution may have only three coneigenvalues, that is, 1, i and $-$i. Now, Theorem 2 implies the following assertion.

Theorem 4. *Let $A \in M_n(\mathbf{C})$ be a coninvolution. Then, A can be brought by a consimilarity transformation to its canonical form which is a direct sum of* 1×1 *and* 2×2 *blocks. The* 1×1 *blocks are just ones, while each* 2×2 *block has the form*

$$\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Corollary 4. *The class of coninvolutions can now be described as the set of matrices of the form*

$$A = P\Lambda\overline{P}^{-1},$$

where $P \in M_n(\mathbf{C})$ is an arbitrary nonsingular matrix and Λ is one of the canonical forms specified in Theorem 4.

5. Concluding remarks

There exists a canonical form with respect to consimilarity transformations that is very similar to the classical Jordan canonical form. The appropriate theorem can be found, for instance, in [5]. Our main theorem could be easily derived from this powerful result. However, in textbooks on linear algebra, the matters related to diagonalizability usually appear much earlier than the Jordan form (which often does not appear at all). Our motivation in this paper was to give a description of condiagonalizable matrices that would be more elementary than the use of the canonical Jordan-like form.

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