SIAM J. MATRIX ANAL. APPL.
Vol. 21, No. 2, pp. 646–667

MATRIX ALGEBRAS AND DISPLACEMENT DECOMPOSITIONS*

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Abstract. A class ξ of algebras of symmetric $n \times n$ matrices, related to Toeplitz-plus-Hankel structures and including the well-known algebra $\mathcal{H}$ diagonalized by the Hartley transform, is investigated. The algebras of ξ are then exploited in a general displacement decomposition of an arbitrary $n \times n$ matrix A. Any algebra of ξ is a 1-*space*, i.e., it is spanned by n matrices having as first rows the vectors of the canonical basis. The notion of 1-space (which generalizes the previous notions of $\mathcal{L}_1$ space [Bevilacqua and Zellini, *Linear and Multilinear Algebra*, 25 (1989), pp. 1–25] and Hessenberg algebra [Di Fiore and Zellini, *Linear Algebra Appl.*, 229 (1995), pp. 49–99]) finally leads to the identification in ξ of three new (non-Hessenberg) matrix algebras close to $\mathcal{H}$, which are shown to be associated with fast *Hartley-type* transforms. These algebras are also involved in new efficient centrosymmetric Toeplitz-plus-Hankel inversion formulas.

Key words. matrix algebras, displacement rank, Toeplitz-plus-Hankel matrices, inversion formulas, discrete Fourier transform, discrete Hartley transform

AMS subject classifications. 15A09, 65F05, 65T20

PII. S0895479896300747

1. Introduction.

It is well known that the inverse of any nonsingular Toeplitz matrix $T = (t_{i-j})_{i,j=1}^n$ can be represented using lower and upper triangular Toeplitz matrices L_m, U_m via the Gohberg–Semencul formula $T^{-1} = L_1U_1 + L_2U_2$ [23]. Kailath, Kung, and Morf [28] extended this result by showing that any $n \times n$ matrix A can be decomposed as

$$A = \sum_{m=1}^{\alpha} L_m U_m \tag{1.1}$$

with α equal to the *displacement rank* of A, i.e., $\alpha = \text{rank}(A - ZAZ^T)$, where $Z = (\delta_{i,\,j+1})_{i,\,j=1}^n$. On the basis of the ideas introduced in [28], different fast algorithms for the inversion or the factorization of structured matrices such as Toeplitz-like [27, 31, 33], Cauchy-like [19, 24], and polynomial Vandermonde-like matrices [29, 30] have been developed (see also [7, 25, 31]).

Besides the triangular Toeplitz used in [23, 28], other algebras have been exploited in *displacement formulas* of type (1.1), for example, ε-circulant [1, 18, 20], τ algebra [6, 16, 32], and algebras of dimension greater than n [8, 9]. In [16], most of these algebras appear as special instances of Hessenberg algebras, which allows one to regain the known displacement formulas in a more general context and to obtain new decompositions of high efficiency (especially if A is the inverse of a Toeplitz-plus-Hankel matrix) [16, 10, 17].

If A is a *Toeplitz-like* matrix, that is, A has a *small* displacement rank α, then the known displacement formulas let one compute the matrix-vector product $A\mathbf{f}$, $\mathbf{f} \in \mathbb{C}^n$, by means of a small number of fast discrete transforms (assuming preprocessing on A). These transforms are discrete Fourier transforms (DFT) in cases of formulas

*Received by the editors March 18, 1996; accepted for publication (in revised form) by R. Freund May 17, 1998; published electronically January 19, 2000. This work was partially supported by the Istituto Nazionale di Alta Matematica, Cittá Universitaria, Rome, Italy.
http://www.siam.org/journals/simax/21-2/30074.html

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involving triangular Toeplitz or ε-circulant matrices [1, 9, 17, 21, 22] and are sine or cosine transforms in cases of formulas involving τ or $\tau_{\varepsilon,\varphi}$ matrices [6, 10, 16, 17, 32], and therefore they are all associated with Hessenberg algebras [16].

In this paper we further extend the results of [6, 9, 10, 16, 17, 18, 20, 21, 22, 28, 32] in the sense that we introduce a new class of matrix algebras $\mathbb{L}$, including Hessenberg and other algebras of matrices diagonalized by means of Hartley [11, 12] or *Hartley-type* transforms, which have not been yet considered in displacement literature. This extension requires the study of matrix algebras containing the matrix $T^{\beta,\beta}_{\varepsilon,\varphi}$ displayed at the beginning of section 2. Notice that the algebra $\mathcal{H}$ of the matrices diagonalized by the Hartley transform (see [5]) contains the matrix $T^{1,1}_{0,0}$. The appropriate mechanism for capturing algebras $\mathbb{L}$ such that $\mathbb{L} \supset T^{\beta,\beta}_{\varepsilon,\varphi}$, which are generally not Hessenberg, is the notion of 1-*space* (which is an extension of the notions of $\mathcal{L}_1$ space [4] and Hessenberg algebra [16]).

A 1-space is a space of $n \times n$ matrices A spanned by n matrices J_k having as first rows the vectors of the canonical basis of $\mathbb{C}^n$. If $[z_1 z_2 \cdots z_n]$ is the first row of A, then each a_{ij} is a linear combination in $\mathbb{C}$ of $z_1, z_2, \dots, z_n$. Any space of matrices simultaneously diagonalized by a nonsingular matrix M whose first row has all nonzero entries can be easily checked to be a 1-space. This is the main reason why the introduction of 1-spaces allows one to extend the range of algebras which could be used, in principle, in (possibly) efficient displacement formulas. In particular, the algebra $\mathcal{H}$ diagonalized by the Hartley transform [5] is a 1-space even though it is not a Hessenberg algebra.

The results of this paper are now described in detail.

In section 2 we state some properties of commutative 1-spaces used throughout the paper. Then we define a class of symmetric 1-spaces $\xi(\varphi, \beta, \mathbf{p})$, φ, $\beta \in \mathbb{C}$, $\mathbf{p} \in \mathbb{C}^n - 1$ in terms of matrices of different dimensions from the algebra τ (τ is the algebra generated by $T^{0,0}_{0,0}$). The main result of section 2 is Theorem 2.5, where the symmetric 1-algebras (closed 1-spaces), including the matrix $T^{\beta,\beta}_{\varepsilon,\varphi}$, are shown to be the spaces $\xi(\varphi, \beta, \mathbf{p})$ with $\mathbf{p}$ running among the solutions of a linear system with coefficients depending upon φ, β, and ε.

In section 3 a *general* displacement formula for a matrix A in terms of 2α matrices from two arbitrary symmetric 1-algebras $\mathbb{L} \supset T^{\beta,\beta}_{\varepsilon,\varphi}$ and $\mathbb{L}' \supset T^{\beta',\beta'}_{\varepsilon',\varphi'}$ is obtained under the assumption that the rank of $AT^{\beta,\beta}_{\varepsilon,\varphi} - T^{\beta,\beta}_{\varepsilon,\varphi}A$ is α (see Theorem 3.2). This formula extends some formulas of [10] to the case of non-Hessenberg algebras.

In sections 4 and 5 the results of Theorems 2.5 and 3.2 are investigated and specialized. In particular it is shown that the Hartley algebra $\mathcal{H}$ introduced in [5] is an element of the class of 1-algebras ξ characterized in Theorem 2.5 and that there are at least three other algebras of ξ, called η, μ, and $\mathcal{K}$, which are associated with fast Hartley-type discrete transforms (see Theorem 5.2 and the following remark). Moreover, new decompositions of the inverse of an arbitrary centrosymmetric Toeplitz-plus-Hankel matrix $T + H = (t_{i-j} + h_{i+j} - 2)^n_{i,\,j} = 1$ in terms of matrices from $\mathcal{H}$, $\mathcal{K}$, η, and μ are obtained. In particular it is shown that there exist $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ such that

$$(T + H)^{-1} = [\mu(\mathbf{a}) + I]\eta(\mathbf{b}) - \mu(\mathbf{b})[\eta(\mathbf{a}) - I]. \tag{1.2}$$

(Here $\mathbb{L}(\mathbf{z})$ denotes the matrix of $\mathbb{L}$ whose first row is $\mathbf{z}^T$.) Under the assumption that the vectors $\mathbf{a}$ and $\mathbf{b}$ are known, formula (1.2) lets one calculate the matrix-vector product $(T + H)^{-1}\mathbf{f}$, $\mathbf{f} \in \mathbb{C}^n$, by means of 10 fast discrete transforms reducible to 8 in case $H = 0$, $[T^{-1}]_{11} \neq 0$, matching both best limits known so far [1, 10, 16]. In any case, the number of transforms reduces to 6 (as in [1, 10, 16, 21, 22]) if the transforms of vectors not depending upon $\mathbf{f}$ are included in the preprocessing stage, where $\mathbf{a}$ and $\mathbf{b}$ are computed.

2. A class of algebras of symmetric matrices. The main result of this section (Theorem 2.5) is a characterization of all spaces $\mathbb{L}$ of $n\times n$ matrices containing the matrix

$$(2.1)\qquad T^{\beta,\beta}_{\varepsilon,\varphi} = \begin{pmatrix} \varepsilon & 1 & 0 & \cdot & \cdot & 0 & \beta \\ 1 & 0 & 1 & . & & & 0 \\ 0 & 1 & . & . & . & & \cdot \\ . & . & . & . & . & . & \cdot \\ . & & . & . & . & 1 & 0 \\ 0 & & & . & 1 & 0 & 1 \\ \beta & 0 & \cdot & \cdot & 0 & 1 & \varphi \end{pmatrix}, \quad \varepsilon,\varphi,\beta\in\mathbb{C}$$

and satisfying the following three properties: $A=A^T$, $\forall\, A\in\mathbb{L}$; $AB\in\mathbb{L}$, $\forall\, A,\, B\in\mathbb{L}$; $\mathbb{L}$ is a 1-space (see Definition 2.1). Notice that the properties of symmetry and closure imply the commutativity of $\mathbb{L}$. Moreover, requiring $\mathbb{L}$ to be a 1-space essentially means that any matrix of $\mathbb{L}$ is determined once its first row is given.

The interest of matrix algebras including $T^{\beta,\beta}_{\varepsilon,\varphi}$ and of possible *displacement* decompositions involving them (see sections 3 and 4) is in the fact that for a Toeplitz-plus-Hankel matrix $T+H$, $[T+H]_{ij}=t_{i-j}+h_{i+j-2}$, $i,j=1,\dots,n$, the rank of $(T+H)T^{\beta,\beta}_{\varepsilon,\varphi}-T^{\beta,\beta}_{\varepsilon,\varphi}(T+H)$ is 4 for all values of ε, φ, β (see [26] for the case $\varepsilon=\varphi=\beta=0$). In section 5, this fact finally leads to efficient inversion formulas for $T+H$ involving Hartley-type matrix algebras. The appropriate mechanism with which to capture algebras including $T^{\beta,\beta}_{\varepsilon,\varphi}$ is the notion of 1-space introduced below.

Let $M_n(\mathbb{C})$ be the space of $n\times n$ matrices with entries in the complex field $\mathbb{C}$ and let $\mathbf{e}_k$, $k=1,\dots,n$, be the vectors of $\mathbb{C}^n$ $\mathbf{e}_k=[0\cdot\cdot 0\ \underset{k}{1}\ 0\cdot\cdot 0]^T$.

DEFINITION 2.1. *A subset $\mathbb{L}$ of $M_n(\mathbb{C})$ is a 1-space if there exist n $n\times n$ matrices $J_{\mathbf{k}}\in\mathbb{L}$, $k=1,\dots,n$, such that $\mathbb{L}=\{\sum_{k=1}^n a_kJ_k : a_k\in\mathbb{C}\}$ and*

$$\mathbf{e}_1^TJ_k=\mathbf{e}_k^T,\quad k=1,\dots,n.$$

Closed (under matrix multiplication) 1-spaces are also called 1-algebras.

Many significant classes of spaces of matrices have 1-space structure. Some examples are the group (or, more generally, hypergroup) matrix algebras [18, 3] and the intersection algebras of the association schemes [2, pp. 52–57]; a simple example is the space of all symmetric Toeplitz matrices (which is not a matrix algebra).

Moreover, every space $H_X=\{\sum_{k=1}^n a_kX^{k-1} : a_k\in\mathbb{C}\}$, where X is an $n\times n$ lower Hessenberg matrix, is a 1-space if the entries $[X]_{i,\,i+1}$ are all nonzero. In this case we also have that $H_X=\{A\in M_n(\mathbb{C}) : AX=XA\}$ because X is nonderogatory. In [16] H_X is called Hessenberg algebra (HA) and, for $\mathbf{z}=[z_1\cdots z_n]^T\in\mathbb{C}^n$, $H_X(\mathbf{z})$ denotes the matrix of H_X whose first row is $\mathbf{z}^T$. For our purposes it is useful to recall the HAs corresponding to the choices $X=T_{\varepsilon,\varphi}$ and $X=P_\beta$, where

$$(2.2)\quad T_{\varepsilon,\varphi}=\begin{pmatrix} \varepsilon & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & 1 & . & & & . \\ 0 & 1 & . & . & . & & . \\ . & . & . & . & . & . & . \\ . & & . & . & . & 1 & 0 \\ & & & . & 1 & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & \varphi \end{pmatrix} \quad\text{and}\quad P_\beta=\begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & 1 & . & & & . \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & . & \cdot \\ \cdot & & & & . & 1 & 0 \\ 0 & & & & & 0 & 1 \\ \beta & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

These HAs are denoted, respectively, by $\tau_{\varepsilon,\varphi}$ and C_β in conformity with [10, 16, 17, 20]. In fact the (non-Hessenberg) algebras containing $T^{\beta,\beta}_{\varepsilon,\varphi}$ studied in Theorem 2.5 and in section 4 are defined in terms of matrices from $\tau_{\varepsilon,\varphi}$ and C_β. Notice that the matrices of $\tau_{\varepsilon,\varphi}$ and of C_β are, respectively, symmetric and persymmetric, in particular $C_\beta(\mathbf{z}) = \sum_{i=1}^n z_i P_\beta^{i-1}$. C_β is the space of β-circulant matrices, and $C = C_1$ is the well-known space of circulant matrices [14].

Finally, observe that any space $\mathbb{L}$ defined as the set of all matrices diagonalized by a nonsingular matrix M is a 1-space if $[M]_{1,i} \neq 0\ \forall i$, because, in this case, $\mathbb{L} = \{Md(M^T\mathbf{z})d(M^T\mathbf{e}_1)^{-1}M^{-1} : \mathbf{z} \in \mathbb{C}^n\}$, where for $\mathbf{z} \in \mathbb{C}^n$ $d(\mathbf{z}) = \text{diag}(z_i, i = 1, \dots, n)$. As a consequence, the algebra $\mathcal{H}$ diagonalized by the Hartley transform (see [5]) is a 1-space even though it is not an HA. Recall that matrices from $\mathcal{H}$ are symmetric and that $\mathcal{H}$ contains the matrix $T^{1,1}_{0,0}$. Thus $\mathcal{H}$ is an example of a symmetric 1-algebra including $T^{\beta,\beta}_{\varepsilon,\varphi}$ for $\beta \neq 0$.

Following the notation used for HAs, if $\mathbb{L}$ is a 1-space and $\mathbf{z} \in \mathbb{C}^n$, $\mathbb{L}(\mathbf{z})$ denotes the matrix of $\mathbb{L}$ whose first row is $\mathbf{z}^T$, i.e., $\mathbb{L}(\mathbf{z}) = \sum_{i=1}^n z_i J_i$, where J_i are the matrices in Definition 2.1. Notice that $A \in \mathbb{L}$ iff $A = \mathbb{L}(A^T\mathbf{e}_1)$.

PROPOSITION 2.2. *Let $\mathbb{L}$ be a commutative 1-space. Then*

(i) *$\mathbb{L}$ is closed under matrix multiplication and $I \in \mathbb{L}$;*
(ii) $\mathbf{x}^T\mathbb{L}(\mathbf{y}) = \mathbf{y}^T\mathbb{L}(\mathbf{x}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$;
(iii) $\mathbb{L}(\mathbb{L}(\mathbf{x})^T\mathbf{y}) = \mathbb{L}(\mathbf{y})\mathbb{L}(\mathbf{x}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.

Proof. As $J_kJ_s = J_sJ_k\ \forall s, k$, we have that $\mathbf{e}_k^TJ_s = \mathbf{e}_s^TJ_k\ \forall s, k$. Consequently, $J_1 \equiv \mathbb{L}(\mathbf{e}_1)$ is the identity matrix I. Moreover, for all i, j, $\mathbf{e}_i^T(\sum_{r=1}^n [J_s]_{kr}J_r)\mathbf{e}_j = \sum_{r=1}^n [J_s]_{kr}[J_r]_{ij} = \sum_{r=1}^n [J_s]_{kr}[J_i]_{rj} = [J_sJ_i]_{kj} = [J_iJ_s]_{kj} = [J_kJ_s]_{ij}$ and thus

$$J_kJ_s = \sum_{r=1}^n [J_s]_{kr}J_r \quad \forall\, s, k,$$

that is, assertion (i) holds. For (iii) observe that, by (i), both $\mathbb{L}(\mathbf{y})\mathbb{L}(\mathbf{x})$ and $\mathbb{L}(\mathbb{L}(\mathbf{x})^T\mathbf{y})$ are in $\mathbb{L}$ and have $\mathbf{y}^T\mathbb{L}(\mathbf{x})$ as first row. Finally, for (ii) use (iii) and the commutativity of $\mathbb{L}$. □

Proposition 2.2 and the following notation are used throughout the paper. The symbol I_j^i, $1 \le i, j \le n$, denotes the $(|j-i|+1) \times n$ $(0,1)$ matrix, which maps a vector $\mathbf{z} = [z_1 \cdots z_n]^T \in \mathbb{C}^n$ into the vector $I_j^i\mathbf{z} = [z_i \cdots z_j]^T \in \mathbb{C}^{|j-i|+1}$. Thus $I = I_n^1$ and $J = I_1^n$ are, respectively, the $n \times n$ identity and the reversion matrix. I and J also denote, respectively, identity and reversion matrices of dimensions different from n. Also, set $e_k = I_{n-1}^1\mathbf{e}_k$, $k = 1, \dots, n-1$, and $\hat{\mathbf{z}} = [z_k \cdots z_1]^T = J\mathbf{z}$ if $\mathbf{z} \in \mathbb{C}^k$.

Now we state Theorem 2.5, where the symmetric closed 1-spaces containing $T^{\beta,\beta}_{\varepsilon,\varphi}$ are shown to be the spaces $\xi(\varphi, \beta, \mathbf{p})$ in Definition 2.4 obtained by choosing as $\mathbf{p}$ the solutions of (2.6). As a consequence (see section 4) for given ε, φ, β, there are as many symmetric 1-algebras including $T^{\beta,\beta}_{\varepsilon,\varphi}$ as the solutions of equation (2.6), i.e., none, an infinite number, or only one, depending upon the values of ε, φ, β. A preliminary Lemma 2.3 follows.

LEMMA 2.3. (i) *Let A be an $n \times n$ matrix and $\mathbf{x}_m$ and $\mathbf{y}_m$, $m = 1, \dots, \alpha$, vectors of $\mathbb{C}^n$ such that $AT_{\varepsilon,0} - T_{\varepsilon,0}A = \sum_{m=1}^\alpha \mathbf{x}_m\mathbf{y}_m^T$. Then*

$$A = \sum_{m=1}^{\alpha} \begin{pmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \\ 0 & \tau(I_n^2\mathbf{x}_m) & \end{pmatrix} \Omega_\varepsilon(\mathbf{y}_m) + \Omega_\varepsilon(A^T\mathbf{e}_1), \tag{2.3}$$

where $\tau = \tau_{0,0}$ and $\Omega_\varepsilon = \tau_{\varepsilon,0}$.

(ii) *In particular, for* $\mathbf{z} \in \mathbb{C}^n$,

$$\Omega_\varepsilon(\mathbf{z}) = \tau(\mathbf{z}) - \varepsilon \begin{pmatrix} 0 & \cdot\;\cdot\;\cdot\;\cdot & 0 \\ \vdots & & \\ \vdots & \tau(I_n^2\mathbf{z}) & \\ 0 & & \end{pmatrix}. \tag{2.4}$$

Proof. For (i) see [16]. (ii) follows from the identities

$$\tau(\mathbf{z})T_{\varepsilon,0} - T_{\varepsilon,0}\tau(\mathbf{z}) = \tau(\mathbf{z})T_{0,0} - T_{0,0}\tau(\mathbf{z}) + \varepsilon[\tau(\mathbf{z})\mathbf{e}_1\mathbf{e}_1^T - \mathbf{e}_1\mathbf{e}_1^T\tau(\mathbf{z})] = \varepsilon(\mathbf{z}\mathbf{e}_1^T - \mathbf{e}_1\mathbf{z}^T)$$

and from assertion (i) for $A = \tau(\mathbf{z})$. □

DEFINITION 2.4. *For* φ, $\beta \in \mathbb{C}$, $\mathbf{p} \in \mathbb{C}^{n-1}$, *define the space of* $n \times n$ *matrices*

$$\begin{aligned} \xi &\equiv \xi(\varphi, \beta, \mathbf{p}) \\ &= \left\{ \tau(\mathbf{z}) - \begin{pmatrix} 0 & \cdot\;\cdot\;\cdot\;\cdot & 0 \\ \vdots & \tau(I_{n-1}^2\mathbf{z}) & \vdots \\ 0 & \cdot\;\cdot\;\cdot\;\cdot & 0 \end{pmatrix} (\varphi I + \beta J) + \begin{pmatrix} 0 & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & 0 \\ \vdots & & \\ 0 & J\Omega_\varphi(I_2^n\mathbf{z})\Omega_\varphi(\mathbf{p})\,J & \end{pmatrix} : \mathbf{z} \in \mathbb{C}^n \right\} \end{aligned} \tag{2.5}$$

and denote by $\xi(\mathbf{z})$ *the matrix of* ξ *whose first row is* $\mathbf{z}^T$.

THEOREM 2.5. *If* $\mathbb{L}$ *is a symmetric closed* 1*-space containing the matrix* $T_{\varepsilon,\varphi}^{\beta,\beta}$ *for some* ε, φ, $\beta \in \mathbb{C}$, *then* $\mathbb{L} = \xi(\varphi, \beta, \mathbf{p})$ *with* $\mathbf{p}$ *such that*

$$\Omega_\varphi(\beta e_1 + e_{n-1})\mathbf{p} = (\varphi - \varepsilon)e_1. \tag{2.6}$$

Conversely, every space of matrices $\xi(\varphi, \beta, \mathbf{p})$ *with* $\mathbf{p}$ *solving* (2.6) *for some* $\varepsilon \in \mathbb{C}$ *is a symmetric closed* 1*-space containing the matrix* $T_{\varepsilon,\varphi}^{\beta,\beta}$; *moreover,* $\xi(\varphi, \beta, \mathbf{p}) = \{A \in M_n(\mathbb{C}) : AT_{\varepsilon,\varphi}^{\beta,\beta} = T_{\varepsilon,\varphi}^{\beta,\beta}A$ *and* $A\xi(\mathbf{e}_n) = \xi(\mathbf{e}_n)A\}$.

Proof. Let $\mathbb{L}$ be a symmetric closed 1-space containing the matrix $T_{\varepsilon,\varphi}^{\beta,\beta}$ and let A be an arbitrary element of $\mathbb{L}$. Notice that $AT_{\varepsilon,\varphi}^{\beta,\beta} = T_{\varepsilon,\varphi}^{\beta,\beta}A$ and therefore $I_n^2A\mathbf{e}_1(e_1 + \beta e_{n-1})^T + BT_{0,\varphi}^{()} = (e_1 + \beta e_{n-1})(I_n^2A\mathbf{e}_1)^T + T_{0,\varphi}^{()}B$, where B and $T_{0,\varphi}^{()}$ are the $(n-1) \times (n-1)$ lower-right submatrices of A and $T_{\varepsilon,\varphi}^{\beta,\beta}$, respectively. Right- and left-multiply this equality by the matrix J to obtain

$$JBJT_{\varphi,0}^{()} - T_{\varphi,0}^{()}JBJ = (\beta e_1 + e_{n-1})(I_2^nA\mathbf{e}_1)^T - (I_2^nA\mathbf{e}_1)(\beta e_1 + e_{n-1})^T \tag{2.7}$$

($T_{\varphi,0}^{()} = JT_{0,\varphi}^{()}J$). The identity (2.7) and Lemma 2.3(i) (with n replaced by $n-1$) yield

$$JBJ = \begin{pmatrix} 0 & \cdot\;\cdot & 0 \\ \vdots & & \\ 0 & J & \end{pmatrix} \Omega_\varphi(I_2^nA\mathbf{e}_1) - \begin{pmatrix} 0 & \cdot\;\cdot\;\cdot\;\cdot & 0 \\ \vdots & & \\ 0 & \tau(I_2^{n-1}A\mathbf{e}_1) & \end{pmatrix} \Omega_\varphi(\beta e_1 + e_{n-1}) + \Omega_\varphi(I_2^nA\mathbf{e}_n).$$

Therefore,

$$B = \begin{pmatrix} & J & 0 \\ & & \vdots \\ 0 & \cdot\;\cdot\;\cdot & 0 \end{pmatrix} J\Omega_\varphi(I_2^nA\mathbf{e}_1)J - \begin{pmatrix} \tau(I_2^{n-1}A\mathbf{e}_1) & 0 \\ & \vdots \\ 0 \;\cdot\;\cdot\;\cdot\;\cdot & 0 \end{pmatrix} J\Omega_\varphi(\beta e_1 + e_{n-1})J + J\Omega_\varphi(I_2^nA\mathbf{e}_n)J$$

and, by the equality (2.4),

$$B = \begin{pmatrix} & & 0 \\ & J & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \tau(I_2^n A\mathbf{e}_1) - \begin{pmatrix} \tau(I_2^{n-1}A\mathbf{e}_1) & 0 \\ & \vdots \\ 0 \quad \cdots & 0 \end{pmatrix} J$$

$$(2.8) \qquad - \beta \begin{pmatrix} \tau(I_2^{n-1}A\mathbf{e}_1) & 0 \\ & \vdots \\ 0 \quad \cdots & 0 \end{pmatrix} + J\Omega_\varphi(I_2^n A\mathbf{e}_n)J.$$

As a consequence of (2.8) we have

$$\begin{pmatrix} 0 & (I_n^2 A\mathbf{e}_1)^T \\ \vdots & B \\ 0 & \end{pmatrix} = J \begin{pmatrix} 0 & \cdots \quad 0 \\ \vdots & \tau(I_2^n A\mathbf{e}_1) \\ 0 & \end{pmatrix} - \beta \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \tau(I_2^{n-1}A\mathbf{e}_1) & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \tau(I_2^{n-1}A\mathbf{e}_1) & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots \quad 0 \\ \vdots & J \\ 0 & \end{pmatrix}$$

$$(2.9) \qquad + \begin{pmatrix} 0 & \cdots \quad 0 \\ \vdots & J\Omega_\varphi(I_2^n A\mathbf{e}_n)\, J \\ 0 & \end{pmatrix}.$$

As $A = \mathbb{L}(A\mathbf{e}_1)$, by Proposition 2.2(ii), $A\mathbf{e}_n = \mathbb{L}(\mathbf{e}_n)A\mathbf{e}_1$, i.e., $A\mathbf{e}_n = (J + \left(\begin{smallmatrix} 0 & \cdots & 0 \\ \vdots & Q & \\ 0 & & \end{smallmatrix}\right))A\mathbf{e}_1$, for a certain $(n-1) \times (n-1)$ matrix Q not depending upon A. Thus $I_2^n A\mathbf{e}_n = I_{n-1}^1 A\mathbf{e}_1 + JQJI_2^n A\mathbf{e}_1$ and (2.9) becomes

$$\begin{pmatrix} 0 & (I_n^2 A\mathbf{e}_1)^T \\ \vdots & B \\ 0 & \end{pmatrix} = J \begin{pmatrix} 0 & \cdots \quad 0 \\ \vdots & \tau(I_2^n A\mathbf{e}_1) \\ 0 & \end{pmatrix} - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \tau(I_2^{n-1}A\mathbf{e}_1) & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots \quad 0 \\ \vdots & J \\ 0 & \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \cdots \quad 0 \\ \vdots & \tau(I_{n-1}^1 A\mathbf{e}_1) \\ 0 & \end{pmatrix}$$

$$- \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \tau(I_{n-1}^2 A\mathbf{e}_1) & \vdots \\ 0 & \cdots & 0 \end{pmatrix} (\varphi I + \beta J) + \begin{pmatrix} 0 & \cdots \quad 0 \\ \vdots & J\Omega_\varphi(JQJI_2^n A\mathbf{e}_1)J \\ 0 & \end{pmatrix}.$$
(2.10)

Notice that the sum of the first three matrices on the right-hand side of (2.10) plus $A\mathbf{e}_1\mathbf{e}_1^T$ is the matrix $\tau(A\mathbf{e}_1)$. In fact the identity $\tau(A\mathbf{e}_1)T_{0,0} = T_{0,0}\tau(A\mathbf{e}_1)$ implies that (2.7) holds for $\varphi = \beta = 0$ and for B ($T_{0,0}^{()}$) the $(n-1) \times (n-1)$ lower-right submatrix of $\tau(A\mathbf{e}_1)$ ($T_{0,0}$); the thesis follows from (2.10), which then holds for $\varphi = \beta = 0$ and $Q = 0$. Thus we have an explicit expression of $A \in \mathbb{L}$:

$$A = \tau(A\mathbf{e}_1) - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \tau(I_{n-1}^2 A\mathbf{e}_1) & \vdots \\ 0 & \cdots & 0 \end{pmatrix} (\varphi I + \beta J) + \begin{pmatrix} 0 & \cdots \quad 0 \\ \vdots & J\Omega_\varphi(JQJI_2^n A\mathbf{e}_1)\, J \\ 0 & \end{pmatrix}.$$

By exploiting it for $A = \mathbb{L}(\mathbf{e}_n) = J + \left(\begin{smallmatrix} 0 & \cdots & 0 \\ \vdots & Q & \\ 0 & & \end{smallmatrix}\right)$, we realize that $JQJ = \Omega_\varphi(JQJe_1)$ or, equivalently, that $JQJ = \Omega_\varphi(\mathbf{p})$ for some $\mathbf{p} \in \mathbb{C}^{n-1}$ not depending upon A.

Therefore, by Proposition 2.2(iii), the generic matrix A of a symmetric closed 1-space containing $T^{\beta,\beta}_{\varepsilon,\varphi}$ has the expression

$$A=\tau(A\mathbf{e}_1)-\begin{pmatrix}0 & \cdots & 0\\ \vdots\ \tau(I^2_{n-1}A\mathbf{e}_1) & \vdots\\ 0 & \cdots & 0\end{pmatrix}(\varphi I+\beta J)+\begin{pmatrix}0 & \cdots\cdots & 0\\ \vdots & J\Omega_\varphi(I^n_2A\mathbf{e}_1)\Omega_\varphi(\mathbf{p})\,J & \\ 0 & & \end{pmatrix} \tag{2.11}$$

for some $\mathbf{p}\in\mathbb{C}^{n-1}$. In particular, (2.11) must be verified for $A=T^{\beta,\beta}_{\varepsilon,\varphi}$ and thus $\mathbf{p}$ must verify (2.6).

Now let us prove the second part of Theorem 2.5. Consider the space $\xi=\xi(\varphi,\beta,\mathbf{p})$ in Definition 2.4 and assume that $\mathbf{p}$ solves equation $\Omega_\varphi(\beta e_1+e_{n-1})\mathbf{p}=(\varphi-\varepsilon)e_1$ for some $\varepsilon\in\mathbb{C}$. The matrix $T^{\beta,\beta}_{\varepsilon,\varphi}$ is an element of ξ; in fact, by Proposition 2.2(iii), $\xi(\varepsilon\mathbf{e}_1+\mathbf{e}_2+\beta\mathbf{e}_n)=T^{\beta,\beta}_{\varepsilon,\varphi}$. Obviously, ξ is a symmetric 1-space. Thus we have to prove only that ξ is equal to the space $\mathbb{A}$ defined as

$$\mathbb{A}=\left\{A\in M_n(\mathbb{C}):AT^{\beta,\beta}_{\varepsilon,\varphi}=T^{\beta,\beta}_{\varepsilon,\varphi}A\quad\text{and}\quad A\xi(\mathbf{e}_n)=\xi(\mathbf{e}_n)A\right\} \tag{2.12}$$

since the closure of ξ follows from the closure of $\mathbb{A}$. Observe that $\mathbb{A}$ is a linear space whose dimension is not greater than n. In fact, let A_i, $i=1,\dots,k$, be k linearly independent matrices of $\mathbb{A}$. If $k>n$, then there exist k elements of $\mathbb{C}$, z_i, $i=1,\dots,k$, not all null and such that $\sum_{i=1}^k z_i\mathbf{e}_1^TA_i=\mathbf{0}^T$. The matrix $\sum_{i=1}^k z_iA_i$ is an element of $\mathbb{A}$ and $\mathbf{e}_1^T(\sum_{i=1}^k z_iA_i)=\mathbf{0}^T$. However, if a matrix $A\in\mathbb{A}$, then it satisfies the identities

$$\begin{aligned}&\mathbf{e}_1^TAT^{\beta,\beta}_{\varepsilon,\varphi}=\varepsilon\mathbf{e}_1^TA+\mathbf{e}_2^TA+\beta\mathbf{e}_n^TA,\qquad \mathbf{e}_1^TA\xi(\mathbf{e}_n)=\mathbf{e}_n^TA,\\ &\mathbf{e}_i^TAT^{\beta,\beta}_{\varepsilon,\varphi}=\mathbf{e}_{i-1}^TA+\mathbf{e}_{i+1}^TA,\quad i=2,\dots,n-1.\end{aligned} \tag{2.13}$$

If, moreover, $\mathbf{e}_1^TA=\mathbf{0}^T$ from (2.13), it follows that $A=0$. Thus the matrix $\sum_{i=1}^k z_iA_i$ above must be null and the A_i's are linearly dependent, that is, a contradiction. Now we show that $\xi\subset\mathbb{A}$. As a consequence of this fact and of the inequalities $\dim\xi=n$ and $\dim\mathbb{A}\le n$, we have that $\xi=\mathbb{A}$.

For $\mathbf{z}\in\mathbb{C}^n$, set

$$M(\mathbf{z})=\tau(\mathbf{z})-\begin{pmatrix}0 & \cdots & 0\\ \vdots\ \tau(I^2_{n-1}\mathbf{z}) & \vdots\\ 0 & \cdots & 0\end{pmatrix}(\varphi I+\beta J),\quad N(\mathbf{z})=\begin{pmatrix}0 & \cdots\cdots & 0\\ \vdots & J\Omega_\varphi(I^n_2\mathbf{z})\Omega_\varphi(\mathbf{p})\,J & \\ 0 & & \end{pmatrix}$$

and notice that $\xi(\mathbf{z})=M(\mathbf{z})+N(\mathbf{z})$. By exploiting the equality $T^{\beta,\beta}_{\varepsilon,\varphi}=T^{\beta,\beta}_{\varphi,\varphi}+(\varepsilon-\varphi)\mathbf{e}_1\mathbf{e}_1^T$, as well as the fact that the first row and the first column of $N(\mathbf{z})$ are null, and the equality $M(\mathbf{z})T^{\beta,\beta}_{\varphi,\varphi}=T^{\beta,\beta}_{\varphi,\varphi}M(\mathbf{z})$ (the proof of this identity is obvious and mechanical and thus is omitted), we have

$$\xi(\mathbf{z})T^{\beta,\beta}_{\varepsilon,\varphi}-T^{\beta,\beta}_{\varepsilon,\varphi}\xi(\mathbf{z})=(\varepsilon-\varphi)(\mathbf{z}\mathbf{e}_1^T-\mathbf{e}_1\mathbf{z}^T)+N(\mathbf{z})T^{\beta,\beta}_{\varphi,\varphi}-T^{\beta,\beta}_{\varphi,\varphi}N(\mathbf{z}).$$

As

$$N(\mathbf{z})T^{\beta,\beta}_{\varphi,\varphi}-T^{\beta,\beta}_{\varphi,\varphi}N(\mathbf{z})=\begin{pmatrix}0 & -\mathbf{p}^T\Omega_\varphi(\beta e_1+e_{n-1})\Omega_\varphi(I^n_2\mathbf{z})J\\ J\Omega_\varphi(I^n_2\mathbf{z})\Omega_\varphi(\beta e_1+e_{n-1})\mathbf{p} & O\end{pmatrix}$$

the assumption $\Omega_\varphi(\beta e_1 + e_{n-1})\mathbf{p} = (\varphi - \varepsilon)e_1$ yields

$$N(\mathbf{z})T^{\beta,\beta}_{\varphi,\varphi} - T^{\beta,\beta}_{\varphi,\varphi}N(\mathbf{z}) = (\varphi - \varepsilon)(\mathbf{z}\mathbf{e}_1^T - \mathbf{e}_1\mathbf{z}^T),$$

and therefore $\xi(\mathbf{z})T^{\beta,\beta}_{\varepsilon,\varphi} = T^{\beta,\beta}_{\varepsilon,\varphi}\xi(\mathbf{z}) \ \ \forall \mathbf{z} \in \mathbb{C}^n$.

Now set $Q = \xi(\mathbf{z})\xi(\mathbf{e}_n) - \xi(\mathbf{e}_n)\xi(\mathbf{z})$. Notice that $\mathbf{e}_1^T Q = \mathbf{z}^T N(\mathbf{e}_n) - \mathbf{e}_n^T N(\mathbf{z}) = \mathbf{0}^T$. Therefore, as $Q^T = -Q$, the first row and the first column of Q are null. Moreover, $QT^{\beta,\beta}_{\varepsilon,\varphi} = T^{\beta,\beta}_{\varepsilon,\varphi}Q$, which implies

$$Q = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \tau_{0,\varphi}(\mathbf{x}) & \\ 0 & & \end{pmatrix}$$

for some $\mathbf{x} \in \mathbb{C}^{n-1}$. Thus Q is simultaneously symmetric and skewsymmetric; therefore, $Q = \xi(\mathbf{z})\xi(\mathbf{e}_n) - \xi(\mathbf{e}_n)\xi(\mathbf{z}) = 0 \ \forall \mathbf{z} \in \mathbb{C}^n$. □

3. 1-algebras and displacement formulas. The algebras characterized in Theorem 2.5 are now involved in a *general* decomposition formula (see Theorem 3.2 below) which leads, in the next section, to new significant displacement decompositions corresponding to special choices of these matrix algebras. A preliminary Lemma 3.1 generalizing related results on HAs [16, 10, 17] follows below. The role of this lemma in the proof of Theorem 3.2 is analogous to the role of *orthogonality relations* in the proof of displacement decompositions involving group matrices [18]. In Lemma 3.1 and Theorem 3.2 A denotes an arbitrary $n \times n$ matrix.

LEMMA 3.1. *Let $\mathbb{L}$ be a commutative 1-space and let $X \in \mathbb{L}$. If $\mathbf{x}_m$, $\mathbf{y}_m \in \mathbb{C}^n$, $m = 1, \ldots, \alpha$, are such that $AX - XA = \sum_{m=1}^{\alpha} \mathbf{x}_m\mathbf{y}_m^T$, then $\sum_{m=1}^{\alpha} \mathbf{x}_m^T \mathbb{L}(\mathbf{y}_m)^T = \mathbf{0}^T$.*

Proof. By Proposition 2.2(ii), for $r = 1, \ldots, n$,

$$\sum_{m=1}^{\alpha} \mathbf{x}_m^T \mathbb{L}(\mathbf{y}_m)^T \mathbf{e}_r = \sum_{m=1}^{\alpha} \mathbf{x}_m^T J_r^T \mathbf{y}_m = \sum_{m=1}^{\alpha}\sum_{i,j=1}^{n} [\mathbf{x}_m]_i[\mathbf{y}_m]_j[J_r^T]_i$$
$$= \sum_{i,j=1}^{n} [AX - XA]_{ij}[J_r]_{ji} = \sum_{i=1}^{n}[(AX - XA)J_r]_{ii} = \sum_{i=1}^{n}[(AJ_r)X - X(AJ_r)]_{ii} = 0. \quad \square$$

THEOREM 3.2. *Let $\mathbb{L}$ and $\mathbb{L}'$ be two symmetric closed 1-spaces containing the matrices $T^{\beta,\beta}_{\varepsilon,\varphi}$ and $T^{\beta',\beta'}_{\varepsilon',\varphi'}$, respectively. If $AT^{\beta,\beta}_{\varepsilon,\varphi} - T^{\beta,\beta}_{\varepsilon,\varphi}A = \sum_{m=1}^{\alpha} \mathbf{x}_m\mathbf{y}_m^T$, then*

$$\begin{aligned}(3.1)\qquad &(\varepsilon - \varepsilon')A + (\beta - \beta')(A\mathbb{L}(\mathbf{e}_n) + \mathbb{L}'(\mathbf{e}_n)A) + (\varphi - \varphi')\mathbb{L}'(\mathbf{e}_n)A\mathbb{L}(\mathbf{e}_n) \\ &= \sum_{m=1}^{\alpha} \mathbb{L}'(\mathbf{x}_m)\mathbb{L}(\mathbf{y}_m) + \mathbb{L}'((\varepsilon - \varepsilon')\mathbf{e}_1 + (\beta - \beta')\mathbf{e}_n)\mathbb{L}(A^T\mathbf{e}_1) \\ &\quad + \mathbb{L}'((\beta - \beta')\mathbf{e}_1 + (\varphi - \varphi')\mathbf{e}_n)\mathbb{L}(A^T\mathbf{e}_n).\end{aligned}$$

Proof. Let X be a symmetric $n \times n$ matrix such that if $AX = XA$ and $\mathbf{e}_1^T A = \mathbf{e}_n^T A = \mathbf{0}^T$, then $A = 0$. Set $[X]_{1n} = [X]_{n1} = \beta$, $[X]_{11} = \varepsilon$, and $[X]_{nn} = \varphi$ and let X' be the $n \times n$ matrix defined by $X = X' + (\varepsilon - \varepsilon')\mathbf{e}_1\mathbf{e}_1^T + (\beta - \beta')(\mathbf{e}_1\mathbf{e}_n^T + \mathbf{e}_n\mathbf{e}_1^T) + (\varphi - \varphi')\mathbf{e}_n\mathbf{e}_n^T$. The assertion of Theorem 3.2 is now shown for X and X' instead of for $T^{\beta,\beta}_{\varepsilon,\varphi}$ and $T^{\beta',\beta'}_{\varepsilon',\varphi'}$, respectively. The thesis will follow because $T^{\beta,\beta}_{\varepsilon,\varphi}$ and $T^{\beta',\beta'}_{\varepsilon',\varphi'}$ satisfy the hypotheses on X and X'. (The simple proof of this fact is left to the reader.)

Let M and N be the matrices on the left-hand side and on the right-hand side in equality (3.1), respectively. We shall prove that if $AX - XA = \sum_{m=1}^{\alpha} \mathbf{x}_m\mathbf{y}_m^T$, then

$(M-N)X = X(M-N)$ and $\mathbf{e}_1^T(M-N) = \mathbf{e}_n^T(M-N) = \mathbf{0}^T$, and therefore, by the hypothesis on X, $M = N$.

The equality $\mathbf{e}_1^T M = \mathbf{e}_1^T N$ is easily verifiable by exploiting Lemma 3.1. The equalities $(M-N)X = X(M-N)$ and $\mathbf{e}_n^T M = \mathbf{e}_n^T N$ are equivalent to the equalities

$$[(\beta-\beta')\mathbf{e}_1 + (\varphi-\varphi')\mathbf{e}_n]\left\{\sum_{m=1}^{\alpha} \mathbf{x}_m^T \mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{y}_m)\right\}$$
$$= [(\beta-\beta')\mathbf{e}_1 + (\varphi-\varphi')\mathbf{e}_n]\{(\varepsilon-\varepsilon')[\mathbf{e}_n^T A - \mathbf{e}_1^T A\mathbb{L}(\mathbf{e}_n)]$$
$$+ \mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)[(\beta-\beta')(A-\mathbb{L}(A^T\mathbf{e}_1)) + (\varphi-\varphi')(A\mathbb{L}(\mathbf{e}_n) - \mathbb{L}(A^T\mathbf{e}_n))]\}$$

and

$$\sum_{m=1}^{\alpha} \mathbf{x}_m^T\mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{y}_m) = (\varepsilon-\varepsilon')[\mathbf{e}_n^T A - \mathbf{e}_1^T A\mathbb{L}(\mathbf{e}_n)]$$
$$+ \mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)[(\beta-\beta')(A-\mathbb{L}(A^T\mathbf{e}_1)) + (\varphi-\varphi')(A\mathbb{L}(\mathbf{e}_n) - \mathbb{L}(A^T\mathbf{e}_n))], \tag{3.2}$$

respectively. The proof of the second equivalence is simple. Let us prove the first one.

$$NX - XN = \sum_{m=1}^{\alpha}[\mathbb{L}'(\mathbf{x}_m)X - X\mathbb{L}'(\mathbf{x}_m)]\mathbb{L}(\mathbf{y}_m) + (\beta-\beta')[\mathbb{L}'(\mathbf{e}_n)X - X\mathbb{L}'(\mathbf{e}_n)]\mathbb{L}(A^T\mathbf{e}_1)$$
$$+ (\varphi-\varphi')[\mathbb{L}'(\mathbf{e}_n)X - X\mathbb{L}'(\mathbf{e}_n)]\mathbb{L}(A^T\mathbf{e}_n).$$

For the sake of simplicity, set $Q = \mathbb{L}'(\mathbf{e}_n)X - X\mathbb{L}'(\mathbf{e}_n)$ and then exploit the equality $X = X' + (\varepsilon-\varepsilon')\mathbf{e}_1\mathbf{e}_1^T + (\beta-\beta')(\mathbf{e}_1\mathbf{e}_n^T + \mathbf{e}_n\mathbf{e}_1^T) + (\varphi-\varphi')\mathbf{e}_n\mathbf{e}_n^T$ to obtain

$$NX - XN = \sum_{m=1}^{\alpha}\{(\varepsilon-\varepsilon')(\mathbf{x}_m\mathbf{e}_1^T - \mathbf{e}_1\mathbf{x}_m^T) + (\beta-\beta')$$
$$\times [\mathbf{x}_m\mathbf{e}_n^T + \mathbb{L}'(\mathbf{e}_n)\mathbf{x}_m\mathbf{e}_1^T - \mathbf{e}_1\mathbf{x}_m^T\mathbb{L}'(\mathbf{e}_n) - \mathbf{e}_n\mathbf{x}_m^T]$$
$$+ (\varphi-\varphi')[\mathbb{L}'(\mathbf{e}_n)\mathbf{x}_m\mathbf{e}_n^T - \mathbf{e}_n\mathbf{x}_m^T\mathbb{L}'(\mathbf{e}_n)]\}\mathbb{L}(\mathbf{y}_m)$$
$$+ (\beta-\beta')Q\mathbb{L}(A^T\mathbf{e}_1) + (\varphi-\varphi')Q\mathbb{L}(A^T\mathbf{e}_n)$$
$$= \sum_{m=1}^{\alpha}\{(\varepsilon-\varepsilon')[\mathbf{x}_m\mathbf{y}_m^T - \mathbf{e}_1\mathbf{x}_m^T\mathbb{L}(\mathbf{y}_m)]$$
$$+ (\beta-\beta')[\mathbf{x}_m\mathbf{y}_m^T\mathbb{L}(\mathbf{e}_n) + \mathbb{L}'(\mathbf{e}_n)\mathbf{x}_m\mathbf{y}_m^T$$
$$- \mathbf{e}_1\mathbf{x}_m^T\mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{y}_m) - \mathbf{e}_n\mathbf{x}_m^T\mathbb{L}(\mathbf{y}_m)]$$
$$+ (\varphi-\varphi')[\mathbb{L}'(\mathbf{e}_n)\mathbf{x}_m\mathbf{y}_m^T\mathbb{L}(\mathbf{e}_n) - \mathbf{e}_n\mathbf{x}_m^T\mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{y}_m)]\}$$
$$+ (\beta-\beta')Q\mathbb{L}(A^T\mathbf{e}_1) + (\varphi-\varphi')Q\mathbb{L}(A^T\mathbf{e}_n).$$

By exploiting the assumption $AX - XA = \sum_{m=1}^{\alpha}\mathbf{x}_m\mathbf{y}_m^T$ and Lemma 3.1, the last expression becomes

$$(\varepsilon-\varepsilon')(AX - XA) + (\beta-\beta')$$
$$\times \left[(AX-XA)\mathbb{L}(\mathbf{e}_n) + \mathbb{L}'(\mathbf{e}_n)(AX-XA) - \mathbf{e}_1\sum_{m=1}^{\alpha}\mathbf{x}_m^T\mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{y}_m)\right]$$

$$
\begin{aligned}
&+ (\varphi - \varphi') \left[\mathbb{L}'(\mathbf{e}_n)(AX - XA)\mathbb{L}(\mathbf{e}_n) - \mathbf{e}_n \sum_{m=1}^{\alpha} \mathbf{x}_m^T \mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{y}_m) \right] \\
&+ (\beta - \beta')Q\mathbb{L}(A^T\mathbf{e}_1) + (\varphi - \varphi')Q\mathbb{L}(A^T\mathbf{e}_n) \\
= \;& (\varepsilon - \varepsilon')(AX - XA) + (\beta - \beta') \\
&\times \left[A\mathbb{L}(\mathbf{e}_n)X - XA\mathbb{L}(\mathbf{e}_n) + \mathbb{L}'(\mathbf{e}_n)AX - X\mathbb{L}'(\mathbf{e}_n)A - QA - \mathbf{e}_1 \sum_{m=1}^{\alpha} \mathbf{x}_m^T \mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{y}_m) \right] \\
&+ (\varphi - \varphi') \Bigg[\mathbb{L}'(\mathbf{e}_n)A\mathbb{L}(\mathbf{e}_n)X - X\mathbb{L}'(\mathbf{e}_n)A\mathbb{L}(\mathbf{e}_n) - QA\mathbb{L}(\mathbf{e}_n) \\
&- \mathbf{e}_n \sum_{m=1}^{\alpha} \mathbf{x}_m^T \mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{y}_m) \Bigg] + (\beta - \beta')Q\mathbb{L}(A^T\mathbf{e}_1) + (\varphi - \varphi')Q\mathbb{L}(A^T\mathbf{e}_n) \\
= \;& MX - XM + (\beta - \beta')Q[\mathbb{L}(A^T\mathbf{e}_1) - A] + (\varphi - \varphi')Q[\mathbb{L}(A^T\mathbf{e}_n) - A\mathbb{L}(\mathbf{e}_n)] \\
&- [(\beta - \beta')\mathbf{e}_1 + (\varphi - \varphi')\mathbf{e}_n] \sum_{m=1}^{\alpha} \mathbf{x}_m^T \mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{y}_m).
\end{aligned}
$$

By replacing $\mathbf{x}_m$ with $\mathbf{e}_n$ in the expression of $\mathbb{L}'(\mathbf{x}_m)X - X\mathbb{L}'(\mathbf{x}_m)$ obtained above, we have $Q = (\varepsilon - \varepsilon')(\mathbf{e}_n\mathbf{e}_1^T - \mathbf{e}_1\mathbf{e}_n^T) + (\beta - \beta')[\mathbb{L}'(\mathbf{e}_n)\mathbf{e}_n\mathbf{e}_1^T - \mathbf{e}_1\mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)] + (\varphi - \varphi')[\mathbb{L}'(\mathbf{e}_n)\mathbf{e}_n\mathbf{e}_n^T - \mathbf{e}_n\mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)]$. Thus

$$
\begin{aligned}
NX - XN = \;& MX - XM + (\beta - \beta')\{[(\beta - \beta')\mathbf{e}_1 + (\varphi - \varphi')\mathbf{e}_n]\mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)[A - \mathbb{L}(A^T\mathbf{e}_1)] \\
&+ [(\varepsilon - \varepsilon')\mathbf{e}_1 - (\varphi - \varphi')\mathbb{L}'(\mathbf{e}_n)\mathbf{e}_n][\mathbf{e}_n^T A - \mathbf{e}_1^T A\mathbb{L}(\mathbf{e}_n)]\} \\
&+ (\varphi - \varphi')\{[(\beta - \beta')\mathbf{e}_1 + (\varphi - \varphi')\mathbf{e}_n]\mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)[A\mathbb{L}(\mathbf{e}_n) - \mathbb{L}(A^T\mathbf{e}_n)] \\
&+ [(\varepsilon - \varepsilon')\mathbf{e}_n + (\beta - \beta')\mathbb{L}'(\mathbf{e}_n)\mathbf{e}_n][\mathbf{e}_n^T A - \mathbf{e}_1^T A\mathbb{L}(\mathbf{e}_n)]\} \\
&- [(\beta - \beta')\mathbf{e}_1 + (\varphi - \varphi')\mathbf{e}_n] \sum_{m=1}^{\alpha} \mathbf{x}_m^T \mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{y}_m) \\
= \;& MX - XM + [(\beta - \beta')\mathbf{e}_1 + (\varphi - \varphi')\mathbf{e}_n]\{(\varepsilon - \varepsilon')[\mathbf{e}_n^T A - \mathbf{e}_1^T A\mathbb{L}(\mathbf{e}_n)] \\
&+ \mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)[(\beta - \beta')(A - \mathbb{L}(A^T\mathbf{e}_1)) + (\varphi - \varphi')(A\mathbb{L}(\mathbf{e}_n) - \mathbb{L}(A^T\mathbf{e}_n))]\} \\
&- [(\beta - \beta')\mathbf{e}_1 + (\varphi - \varphi')\mathbf{e}_n] \left\{ \sum_{m=1}^{\alpha} \mathbf{x}_m^T \mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{y}_m) \right\}
\end{aligned}
$$

and the first equivalence is proved. Now we have to prove (3.2) and the proof of Theorem 3.2 will be complete, because then the equality preceding (3.2)—identical to (3.2), but a factor—is satisfied. For $s = 1, \ldots, n$

$$
\begin{aligned}
&\sum_{m=1}^{\alpha} \mathbf{x}_m^T \mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{y}_m)\mathbf{e}_s = \sum_{m=1}^{\alpha} \mathbf{x}_m^T \mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{e}_s)\mathbf{y}_m = \sum_{m=1}^{\alpha} \sum_{i,\,j=1}^{n} [\mathbf{x}_m]_i [\mathbf{y}_m]_j [\mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{e}_s)]_{ij} \\
&\quad = \sum_{i,\,j=1}^{n} [AX - XA]_{ij} [\mathbb{L}(\mathbf{e}_s)\mathbb{L}'(\mathbf{e}_n)]_{ji} = \sum_{i=1}^{n} [A\mathbb{L}(\mathbf{e}_s)X\mathbb{L}'(\mathbf{e}_n) - XA\mathbb{L}(\mathbf{e}_s)\mathbb{L}'(\mathbf{e}_n)]_{ii}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}[-A\mathbb{L}(\mathbf{e}_s)(\mathbb{L}'(\mathbf{e}_n)X - X\mathbb{L}'(\mathbf{e}_n))]_{ii} \\
&= \sum_{i=1}^{n}[-A\mathbb{L}(\mathbf{e}_s)\{(\varepsilon-\varepsilon')(\mathbf{e}_n\mathbf{e}_1^T - \mathbf{e}_1\mathbf{e}_n^T) + (\beta-\beta')[\mathbb{L}'(\mathbf{e}_n)\mathbf{e}_n\mathbf{e}_1^T - \mathbf{e}_1\mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)] \\
&\quad + (\varphi-\varphi')[\mathbb{L}'(\mathbf{e}_n)\mathbf{e}_n\mathbf{e}_n^T - \mathbf{e}_n\mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)]\}]_{ii} \\
&= -(\varepsilon-\varepsilon')\sum_{i=1}^{n}\mathbf{e}_1^T[A\mathbb{L}(\mathbf{e}_n)\mathbf{e}_s\mathbf{e}_1^T - A\mathbf{e}_s\mathbf{e}_n^T]\mathbf{e}_i - (\beta-\beta')\sum_{i=1}^{n}\mathbf{e}_i^T[A\mathbb{L}(\mathbf{e}_s)\mathbb{L}'(\mathbf{e}_n)\mathbf{e}_n\mathbf{e}_1^T \\
&\quad - A\mathbf{e}_s\mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)]\mathbf{e}_i - (\varphi-\varphi')\sum_{i=1}^{n}\mathbf{e}_i^T[A\mathbb{L}(\mathbf{e}_s)\mathbb{L}'(\mathbf{e}_n)\mathbf{e}_n\mathbf{e}_n^T - A\mathbb{L}(\mathbf{e}_n)\mathbf{e}_s\mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)]\mathbf{e}_i \\
&= (\varepsilon-\varepsilon')[\mathbf{e}_n^T A\mathbf{e}_s - \mathbf{e}_1^T A\mathbb{L}(\mathbf{e}_n)\mathbf{e}_s] + (\beta-\beta')[\mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)A\mathbf{e}_s - \mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{e}_s)A^T\mathbf{e}_1] \\
&\quad + (\varphi-\varphi')[\mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)A\mathbb{L}(\mathbf{e}_n)\mathbf{e}_s - \mathbf{e}_n^T\mathbb{L}'(\mathbf{e}_n)\mathbb{L}(\mathbf{e}_s)A^T\mathbf{e}_n],
\end{aligned}
$$

that is, (3.2) holds. ∎

Remark. It is clear that Theorem 3.2 holds unchanged if $T^{\beta,\beta}_{\varepsilon,\varphi} = T_{\varepsilon,\varphi} + \beta(\mathbf{e}_1\mathbf{e}_n^T + \mathbf{e}_n\mathbf{e}_1^T)$ is replaced by $M = Y + \beta(\mathbf{e}_1\mathbf{e}_n^T + \mathbf{e}_n\mathbf{e}_1^T)$, where Y is a generic symmetric tridiagonal matrix having at least $n-2$ nonzero entries $[Y]_{i,\,i+1}$, and $T^{\beta',\beta'}_{\varepsilon',\varphi'}$ is replaced by $M' = Y' + \beta'(\mathbf{e}_1\mathbf{e}_n^T + \mathbf{e}_n\mathbf{e}_1^T)$, where $Y' = Y + (\varepsilon'-\varepsilon)\mathbf{e}_1\mathbf{e}_1^T + (\varphi'-\varphi)\mathbf{e}_n\mathbf{e}_n^T$ (set $[Y]_{11} = \varepsilon$, $[Y]_{nn} = \varphi$). (In fact M and M' satisfy the assumptions on X and X' at the beginning of the proof.) This result includes Theorems 3.2 ($i = 1$, $i = n$) and 3.4 of [10].

For $\beta = \beta' = 0$ the result stated in Theorem 3.2 leads to displacement decompositions exploiting symmetric HAs that include, as special instances, some of the most significant formulas stated in [10] (see Corollaries 4.1 and 4.2 in [10]). In the next section it is shown that formula (3.1) also leads to significant decompositions exploiting 1-spaces which are not HAs. More specifically, in these last decompositions some Hartley-type matrix algebras will be involved.

4. The algebras $\eta, \mu, \mathcal{H}, \mathcal{K}$. Theorem 3.2 can be exploited in order to obtain—as special cases of formula (3.1)—effective displacement decompositions of a generic matrix A. To this end we need to know if, and under what assumptions, there exist symmetric 1-algebras containing matrices of the form $T^{\beta,\beta}_{\varepsilon,\varphi}$. Theorem 2.5 relates the existence of such spaces $\xi(\varphi,\beta,\mathbf{p})$ to the existence of vectors $\mathbf{p}$ solving the linear system

$$I_{\beta\varphi}\mathbf{p} = (\varphi-\varepsilon)e_1, \tag{4.1}$$

where $I_{\beta\varphi}$ is the $(n-1)\times(n-1)$ matrix

$$I_{\beta\varphi} = \Omega_\varphi(\beta e_1 + e_{n-1}) = \begin{pmatrix} \beta & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \beta \end{pmatrix} + \begin{pmatrix} & & & & 1 \\ & & & 1 & -\varphi \\ & & \iddots & \iddots & \\ & 1 & \iddots & & \\ 1 & -\varphi & & & \end{pmatrix}. \tag{4.2}$$

Equation (4.1) may have no solution, infinite solutions, or only one solution; these three cases and the corresponding 1-spaces $\xi(\varphi,\beta,\mathbf{p})$ are studied in Proposition 4.2. For the sake of simplicity, for U, V $n\times n$ matrices, set $\mathcal{C}_V(U) = UV - VU$.

LEMMA 4.1. *$I_{\beta\varphi}$ is nonsingular iff $\exists$ $\mathbf{z} \in \mathbb{C}^{n-1}$ and $\delta \in \mathbb{C}$, $\delta \neq 0$, such that $\mathbf{z}^T I_{\beta\varphi} = \delta e_1^T$. In this case $I_{\beta,\varphi}^{-1} = \delta^{-1}\Omega_\varphi(\mathbf{z})$.*

Proof. The assertion holds for any matrix A of a commutative 1-space $\mathbb{L}$; in fact, if $\mathbf{z}^T A = \delta e_1^T$, then $e_i^T \mathbb{L}(\mathbf{z})A = \mathbf{z}^T \mathbb{L}(e_i)A = \mathbf{z}^T A\mathbb{L}(e_i) = \delta e_i^T$, $i = 1, \ldots, n-1$, that is, $\delta^{-1}\mathbb{L}(\mathbf{z})A = I$. □

PROPOSITION 4.2. *We have the following three cases.*

(i) *$I_{\beta\varphi}$ singular and $\varepsilon \neq \varphi$: There is no symmetric 1-algebra containing $T_{\varepsilon,\varphi}^{\beta,\beta}$.*

(ii) *$I_{\beta\varphi}$ singular and $\varepsilon = \varphi$: There are infinite symmetric 1-algebras containing $T_{\varphi,\varphi}^{\beta,\beta}$ and therefore $T_{\varphi,\varphi}^{\beta,\beta}$ is derogatory. More specifically, these spaces are the $\xi(\varphi, \beta, \mathbf{p})$ (in (2.5)) where $\mathbf{p}$ is such that $I_{\beta\varphi}\mathbf{p} = \mathbf{0}$, and they can be represented as*

$$\xi(\varphi, \beta, \mathbf{p}) = \left\{A \in M_n(\mathbb{C}) : \mathcal{C}_A\left(T_{\varphi,\varphi}^{\beta,\beta}\right) = \mathcal{C}_A(\xi(\mathbf{e}_n)) = 0\right\}. \tag{4.3}$$

Only one of them is also persymmetric, and we call it $\tau_{\varphi,\varphi}^{\beta,\beta}$. We have

$$\tau_{\varphi,\varphi}^{\beta,\beta} = \xi(\varphi, \beta, \mathbf{0}) = \left\{A \in M_n(\mathbb{C}) : \mathcal{C}_A\left(T_{\varphi,\varphi}^{\beta,\beta}\right) = \mathcal{C}_A(J) = 0\right\}. \tag{4.4}$$

(iii) *$I_{\beta\varphi}$ nonsingular: For any $\varepsilon \in \mathbb{C}$ there exists a unique symmetric 1-algebra containing $T_{\varepsilon,\varphi}^{\beta,\beta}$. Moreover, if $\tau_{\varepsilon,\varphi}^{\beta,\beta}$ denotes such a space, we have*

$$\tau_{\varepsilon,\varphi}^{\beta,\beta} = \xi\left(\varphi, \beta, (\varphi - \varepsilon)I_{\beta\varphi}^{-1}e_1\right) = \left\{A \in M_n(\mathbb{C}) : \mathcal{C}_A\left(T_{\varepsilon,\varphi}^{\beta,\beta}\right) = 0\right\}. \tag{4.5}$$

Therefore, $T_{\varepsilon,\varphi}^{\beta,\beta}$ is nonderogatory and $\tau_{\varepsilon,\varphi}^{\beta,\beta}$ is the set of all polynomials in $T_{\varepsilon,\varphi}^{\beta,\beta}$.

Proof of Proposition 4.2(i). Assume that $I_{\beta\varphi}$ is singular and that $\varepsilon \neq \varphi$. By the first part of Theorem 2.5, a symmetric 1-algebra containing $T_{\varepsilon,\varphi}^{\beta,\beta}$ is equal to $\xi(\varphi, \beta, \mathbf{p})$, where $\mathbf{p}$ is such that $I_{\beta\varphi}\mathbf{p} = (\varphi - \varepsilon)e_1$. Then, by Lemma 4.1, $I_{\beta\varphi}$ is invertible, that is, a contradiction.

Proof of Proposition 4.2(ii). Assume that $I_{\beta\varphi}$ is singular and that $\varepsilon = \varphi$. Then the vectors $\mathbf{p} \in \mathbb{C}^{n-1}$ satisfying the equality $I_{\beta\varphi}\mathbf{p} = (\varphi - \varepsilon)e_1 = \mathbf{0}$ are infinite and, by the second part of Theorem 2.5, every space $\xi(\varphi, \beta, \mathbf{p})$ is a symmetric 1-algebra containing $T_{\varphi,\varphi}^{\beta,\beta}$, and it can be represented as in (4.3). The matrix $T_{\varphi,\varphi}^{\beta,\beta}$ is derogatory, because otherwise the set of all polynomials in $T_{\varphi,\varphi}^{\beta,\beta}$ should be an n-dimensional subspace of each $\xi(\varphi, \beta, \mathbf{p})$, which is absurd. Finally, among the $\xi(\varphi, \beta, \mathbf{p})$'s, there is only one containing the matrix J (or, equivalently, for which $\xi(\mathbf{e}_n) = J$), that is, $\xi(\varphi, \beta, \mathbf{0})$.

Proof of Proposition 4.2(iii). Assume that $I_{\beta\varphi}$ is nonsingular. By the second part of Theorem 2.5, $\xi(\varphi, \beta, (\varphi - \varepsilon)I_{\beta\varphi}^{-1}e_1)$ is a symmetric 1-algebra containing $T_{\varepsilon,\varphi}^{\beta,\beta}$. By the first part of Theorem 2.5, there is no other symmetric 1-algebra containing $T_{\varepsilon,\varphi}^{\beta,\beta}$. As regards the identity (4.5), notice that $\tau_{\varepsilon,\varphi}^{\beta,\beta} \subset \{A \in M_n(\mathbb{C}) : \mathcal{C}_A(T_{\varepsilon,\varphi}^{\beta,\beta}) = 0\}$. Conversely, let A be a matrix commuting with $T_{\varepsilon,\varphi}^{\beta,\beta}$ and consider the space $\tau_{\varepsilon',\varphi}^{\beta,\beta} = \xi(\varphi, \beta, (\varphi - \varepsilon')I_{\beta\varphi}^{-1}e_1)$, $\varepsilon' \neq \varepsilon$. Then apply Theorem 3.2 for $\mathbb{L} = \tau_{\varepsilon,\varphi}^{\beta,\beta}$ and $\mathbb{L}' = \tau_{\varepsilon',\varphi}^{\beta,\beta}$ to the matrix A to obtain $(\varepsilon - \varepsilon')A = (\varepsilon - \varepsilon')\tau_{\varepsilon,\varphi}^{\beta,\beta}(A^T\mathbf{e}_1)$. □

Now two interesting classes of matrix algebras $\mathcal{S}$ and $\mathcal{R}$, both corresponding to case (ii) in Proposition 4.2, are investigated. These algebras are also exploited to state, as special instances of formula (3.1), new efficient decompositions of a generic centrosymmetric matrix A (Theorem 4.3). Notice that the algebra $\mathcal{H}$, studied in [5] and related to the *Hartley transform*, is a particular element of $\mathcal{S}$.

The class $\boldsymbol{\mathcal{S}}$. Let $\varphi = \varepsilon = 0$ and $\beta = 1$ in (4.1)–(4.2). As $\Omega_0(e_1+e_{n-1}) = \tau(e_1+e_{n-1}) = I+J$ is singular, by Proposition 4.2(ii) there are infinite symmetric 1-algebras containing the matrix $T^{1,1}_{0,0}$, i.e., the spaces $\xi(0,1,\mathbf{p}^{\rm SK})$, where $\mathbf{p}^{\rm SK}$ is an arbitrary skewsymmetric vector ($\hat{\mathbf{p}}^{\rm SK} = -\mathbf{p}^{\rm SK}$). These spaces are denoted by $\mathcal{S}(\cdot;\mathbf{p}^{\rm SK})$ and can be represented as

$$(4.6)\ \mathcal{S}(\cdot;\mathbf{p}^{\rm SK}) = \xi(0,1,\mathbf{p}^{\rm SK}) = \{A \in M_n(\mathbb{C}) : \mathcal{C}_A(T^{1,1}_{0,0}) = \mathcal{C}_A(\mathcal{S}(\mathbf{e}_n;\mathbf{p}^{\rm SK})) = 0\}.$$

Each algebra $\mathcal{S}(\cdot;\mathbf{p}^{\rm SK})$ contains the algebra $C^{\rm S}$ of all $n\times n$ symmetric circulant matrices; therefore, by the identity $\{A : \mathcal{C}_A(T^{1,1}_{0,0}) = 0\} = C + JC$ (found in [9]), $\mathcal{S}(\cdot;\mathbf{p}^{\rm SK})$ must be equal to $C^{\rm S} + J\tilde{C}$ for some subset $\tilde{C}$ (depending upon $\mathbf{p}^{\rm SK}$) of the space C of circulant matrices.

Algebra η. If $\mathbf{p}^{\rm SK} = \mathbf{0}$ we have the space

$$(4.7)\quad \eta = \mathcal{S}(\cdot;\mathbf{0}) = \xi(0,1,\mathbf{0}) = \tau^{1,1}_{0,0} = \{A \in M_n(\mathbb{C}) : \mathcal{C}_A(T^{1,1}_{0,0}) = \mathcal{C}_A(J) = 0\}.$$

Notice that $\eta = C^{\rm S} + JC^{\rm S}$; in fact $C^{\rm S} + JC^{\rm S} \subset \eta$ and

$$\begin{aligned}\dim(C^{\rm S} + JC^{\rm S}) &= \dim C^{\rm S} + \dim JC^{\rm S} - \dim C^{\rm S} \cap JC^{\rm S} = 2\dim C^{\rm S} - \dim C^{\rm S} \cap JC^{\rm S}\\ &= \begin{cases} 2\left(\frac{n}{2}+1\right) - 2 & \text{if } n \text{ is even},\\ 2\left(\frac{n+1}{2}\right) - 1 & \text{if } n \text{ is odd},\end{cases}\end{aligned}$$

that is, $\dim(C^{\rm S} + JC^{\rm S}) = n$.

Algebra $\mathcal{H}$. If $\mathbf{p}^{\rm SK} = \frac{1}{2}(e_2 - e_{n-2})$, we have the space

$$(4.8)\qquad \mathcal{H} = \mathcal{S}\left(\cdot;\frac{1}{2}(e_2 - e_{n-2})\right) = \xi\left(0,1,\frac{1}{2}(e_2 - e_{n-2})\right).$$

Notice that $\mathcal{H} = C^{\rm S} + JPC^{\rm SK}$, where P is the circulant matrix whose first row is $\mathbf{e}_2^T$ ($P = P_1 = C(\mathbf{e}_2)$) and $C^{\rm SK}$ is the set of all $n\times n$ skewsymmetric circulant matrices (a matrix A is skewsymmetric if $A^T = -A$). To prove this fact, first observe that $C^{\rm S} + JPC^{\rm SK}$ is commutative and that the matrices $T^{1,1}_{0,0}$ and

$$(4.9)\qquad \mathcal{H}(\mathbf{e}_n) = \mathcal{S}\left(\mathbf{e}_n;\frac{1}{2}(e_2 - e_{n-2})\right) = J + \frac{1}{2}\begin{pmatrix} 0 & \cdots\cdots & 0\\ \vdots & & \\ 0 & \tau(e_2 - e_{n-2}) & \end{pmatrix}$$

are elements of $C^{\rm S} + JPC^{\rm SK}$. The commutativity follows from the commutativity of the space C. Moreover, the matrices $\frac{1}{2}T^{1,1}_{0,0}$ and $JP\left(-\frac{1}{2}(P - P^T)\right)$ are elements of $C^{\rm S}$ and $JPC^{\rm SK}$, respectively, and their sum is the matrix in (4.9). Thus $C^{\rm S} + JPC^{\rm SK} \subset \mathcal{H}$. But

$$\begin{aligned}\dim(C^{\rm S} + JPC^{\rm SK}) &= \dim C^{\rm S} + \dim JPC^{\rm SK} - \dim C^{\rm S} \cap JPC^{\rm SK}\\ &= \begin{cases} \left(\frac{n}{2}+1\right) + \left(\frac{n}{2}-1\right) - 0 & \text{if } n \text{ is even},\\ \left(\frac{n+1}{2}\right) + \left(\frac{n-1}{2}\right) - 0 & \text{if } n \text{ is odd},\end{cases}\end{aligned}$$

that is, $\dim(C^{\rm S} + JPC^{\rm SK}) = n$, and the identity $\mathcal{H} = C^{\rm S} + JPC^{\rm SK}$ is proved. In [5] it is shown that the matrices of $\mathcal{H}$ are simultaneously diagonalized by a similarity transformation known as Hartley transform (see also Theorem 5.2 in the next section). A greater attention has been devoted to this particular *real* transform since Bracewell [11, 12] introduced the fast Hartley transform (FHT).

Observe that the proper inclusion $\mathcal{H} \supset C^{\mathrm{S}}$ is exploited in [5] to determine a new preconditioner of symmetric Toeplitz systems, competitive with the more usual circulant preconditioners (see also [13]). All algebras $\mathcal{S}(\cdot;\mathbf{p}^{\mathrm{SK}})$ include C^{S} and, besides $\mathcal{H}$, there may be other algebras $\mathcal{S}(\cdot;\mathbf{p}^{\mathrm{SK}})$ whose matrices are simultaneously diagonalized by a fast transform (this is, the case of $\eta = \mathcal{S}(\cdot;\mathbf{0})$; see Theorem 5.2). As it will be shown in a forthcoming paper, some of the algebras $\mathcal{S}(\cdot;\mathbf{p}^{\mathrm{SK}})$ (together with some other $\mathcal{R}(\cdot;\mathbf{p}^{\mathrm{S}})$ algebras described below) can lead to other efficient preconditioners of Toeplitz systems.

The class $\boldsymbol{\mathcal{R}}$. The choice $\varphi = \varepsilon = 0$, $\beta = -1$ leads to symmetric 1-algebras—containing $T_{0,0}^{-1,-1}$—naturally related to those of the class $\mathcal{S}$. These are the following:

$$\mathcal{R}(\cdot;\mathbf{p}^{\mathrm{S}}) = \xi(0,-1,\mathbf{p}^{\mathrm{S}}) = \left\{A \in M_n(\mathbb{C}) : \mathcal{C}_A\left(T_{0,0}^{-1,-1}\right) = \mathcal{C}_A(\mathcal{R}(\mathbf{e}_n;\mathbf{p}^{\mathrm{S}})) = 0\right\}, \tag{4.10}$$

where $\mathbf{p}^{\mathrm{S}}$ is an arbitrary symmetric vector ($\hat{\mathbf{p}}^{\mathrm{S}} = \mathbf{p}^{\mathrm{S}}$). Each algebra $\mathcal{R}(\cdot;\mathbf{p}^{\mathrm{S}})$ contains the algebra C_{-1}^{S} of all $n \times n$ symmetric (-1)-circulant matrices; therefore, by the identity $\{A : \mathcal{C}_A(T_{0,0}^{-1,-1}) = 0\} = C_{-1} + JC_{-1}$ (found in [9]), $\mathcal{R}(\cdot;\mathbf{p}^{\mathrm{S}}) = C_{-1}^{\mathrm{S}} + J\tilde{C}_{-1}$ for some subset $\tilde{C}_{-1}$ (depending on $\mathbf{p}^{\mathrm{S}}$) of the space C_{-1} of (-1)-circulant matrices.

Algebra μ. If $\mathbf{p}^{\mathrm{S}} = \mathbf{0}$, we have the space

$$\mu = \mathcal{R}(\cdot;\mathbf{0}) = \xi(0,-1,\mathbf{0}) = \tau_{0,0}^{-1,-1} = \{A \in M_n(\mathbb{C}) : \mathcal{C}_A(T_{0,0}^{-1,-1}) = \mathcal{C}_A(J) = 0\} \tag{4.11}$$

naturally related to η. Notice that $\mu = C_{-1}^{\mathrm{S}} + JC_{-1}^{\mathrm{S}}$; in fact $C_{-1}^{\mathrm{S}} + JC_{-1}^{\mathrm{S}} \subset \mu$ and

$$\begin{aligned}
\dim(C_{-1}^{\mathrm{S}} + JC_{-1}^{\mathrm{S}}) &= \dim C_{-1}^{\mathrm{S}} + \dim JC_{-1}^{\mathrm{S}} - \dim C_{-1}^{\mathrm{S}} \cap JC_{-1}^{\mathrm{S}} \\
&= 2\dim C_{-1}^{\mathrm{S}} - \dim C_{-1}^{\mathrm{S}} \cap JC_{-1}^{\mathrm{S}} \\
&= \begin{cases} 2\left(\dfrac{n}{2}\right) - 0 & \text{if } n \text{ is even,} \\ 2\left(\dfrac{n+1}{2}\right) - 1 & \text{if } n \text{ is odd,} \end{cases}
\end{aligned}$$

that is, $\dim(C_{-1}^{\mathrm{S}} + JC_{-1}^{\mathrm{S}}) = n$.

Algebra $\mathcal{K}$. If $\mathbf{p}^{\mathrm{S}} = -\frac{1}{2}(e_2 + e_{n-2})$, we have the space

$$\mathcal{K} = \mathcal{R}\left(\cdot;-\frac{1}{2}(e_2 + e_{n-2})\right) = \xi\left(0,-1,-\frac{1}{2}(e_2 + e_{n-2})\right) \tag{4.12}$$

naturally related to $\mathcal{H}$. Notice that $\mathcal{K} = C_{-1}^{\mathrm{S}} + JP_{-1}C_{-1}^{\mathrm{SK}}$, where $P_{-1} = C_{-1}(\mathbf{e}_2)$ and C_{-1}^{SK} is the set of all $n \times n$ skewsymmetric (-1)-circulant matrices. In order to prove this fact, first show (by proceeding as for $\mathcal{H}$) the inclusion $C_{-1}^{\mathrm{S}} + JP_{-1}C_{-1}^{\mathrm{SK}} \subset \mathcal{K}$, and then use the identity

$$\begin{aligned}
\dim(C_{-1}^{\mathrm{S}} + JP_{-1}C_{-1}^{\mathrm{SK}}) &= \dim C_{-1}^{\mathrm{S}} + \dim JP_{-1}C_{-1}^{\mathrm{SK}} - \dim C_{-1}^{\mathrm{S}} \cap JP_{-1}C_{-1}^{\mathrm{SK}} \\
&= \begin{cases} \left(\dfrac{n}{2}\right) + \left(\dfrac{n}{2}\right) - 0 & \text{if } n \text{ is even,} \\ \left(\dfrac{n+1}{2}\right) + \left(\dfrac{n-1}{2}\right) - 0 & \text{if } n \text{ is odd,} \end{cases} = n.
\end{aligned}$$

The matrices of $\mathcal{K}$ are simultaneously diagonalized by a similarity transformation analogue to the Hartley transform (skew-Hartley transform). Also the algebra μ is associated with a fast discrete transform. (See Theorem 5.2 and the following remark.) In Theorem 4.3 the most significant displacement decompositions are stated in terms of the algebras η, μ, $\mathcal{H}$, and $\mathcal{K}$.

THEOREM 4.3. *If* $AT_{0,0}^{1,1} - T_{0,0}^{1,1}A = \sum_{m=1}^{\alpha} \mathbf{x}_m \mathbf{y}_m^T$*, then*

$$\begin{aligned} A\mathcal{S}(\mathbf{e}_n;\mathbf{p}^{\mathrm{SK}}) + \mathcal{R}(\mathbf{e}_n;\mathbf{p}^s)A = \frac{1}{2}\sum_{m=1}^{\alpha} \mathcal{R}(\mathbf{x}_m;\mathbf{p}^{\mathrm{S}})\mathcal{S}(\mathbf{y}_m;\mathbf{p}^{\mathrm{SK}}) \\ + \mathcal{S}(A^T\mathbf{e}_n;\mathbf{p}^{\mathrm{SK}}) + \mathcal{R}(\mathbf{e}_n;\mathbf{p}^s)\mathcal{S}(A^T\mathbf{e}_1;\mathbf{p}^{\mathrm{SK}}) \end{aligned} \tag{4.13}$$

and, in particular,

$$AJ + JA = \frac{1}{2}\sum_{m=1}^{\alpha} \mu(\mathbf{x}_m)\eta(\mathbf{y}_m) + \eta((AJ+JA)^T\mathbf{e}_1), \tag{4.14}$$

$$AJ + \mathcal{K}(\mathbf{e}_n)A = \frac{1}{2}\sum_{m=1}^{\alpha} \mathcal{K}(\mathbf{x}_m)\eta(\mathbf{y}_m) + \eta(A^T\mathbf{e}_n) + \mathcal{K}(\mathbf{e}_n)\eta(A^T\mathbf{e}_1), \tag{4.15}$$

$$A\mathcal{H}(\mathbf{e}_n) + JA = \frac{1}{2}\sum_{m=1}^{\alpha} \mu(\mathbf{x}_m)\mathcal{H}(\mathbf{y}_m) + \mathcal{H}(A^T\mathbf{e}_n) + J\mathcal{H}(A^T\mathbf{e}_1), \tag{4.16}$$

$$A\mathcal{H}(\mathbf{e}_n) + \mathcal{K}(\mathbf{e}_n)A = \frac{1}{2}\sum_{m=1}^{\alpha} \mathcal{K}(\mathbf{x}_m)\mathcal{H}(\mathbf{y}_m) + \mathcal{H}(A^T\mathbf{e}_n) + \mathcal{K}(\mathbf{e}_n)\mathcal{H}(A^T\mathbf{e}_1). \tag{4.17}$$

Proof. For (4.13) set $\varepsilon = \varphi = \varepsilon' = \varphi' = 0$, $\beta = 1$, $\beta' = -1$ in Theorem 3.2. The particular cases (4.14)–(4.17) correspond, respectively, to the choices $\mathbf{p}^{\mathrm{S}} = \mathbf{p}^{\mathrm{SK}} = \mathbf{0}$, $\mathbf{p}^{\mathrm{S}} = -\frac{1}{2}(e_2 + e_{n-2})$ and $\mathbf{p}^{\mathrm{SK}} = \mathbf{0}$, $\mathbf{p}^{\mathrm{S}} = \mathbf{0}$ and $\mathbf{p}^{\mathrm{SK}} = \frac{1}{2}(e_2 - e_{n-2})$, and $\mathbf{p}^{\mathrm{S}} = -\frac{1}{2}(e_2 + e_{n-2})$ and $\mathbf{p}^{\mathrm{SK}} = \frac{1}{2}(e_2 - e_{n-2})$. ∎

If the matrix A is *centrosymmetric* (i.e., $AJ = JA$) the formulas (4.14)–(4.16) give explicit representations of A in terms of the algebras μ, η, $\mathcal{H}$, and $\mathcal{K}$. In fact the matrices $2J$, $J + \mathcal{K}(\mathbf{e}_n)$, and $J + \mathcal{H}(\mathbf{e}_n)$ are invertible. (It can be shown that $\det(J + \mathcal{H}(\mathbf{e}_n)) = \det(J + \mathcal{K}(\mathbf{e}_n)) = (-1)^{(n-1)/2}2n$ if n is odd; $\det(J + \mathcal{H}(\mathbf{e}_n)) = (-1)^{n/2}n^2$, $\det(J + \mathcal{K}(\mathbf{e}_n)) = (-1)^{n/2}4$ if n is even.) Notice that by Proposition 4.2(i) a symmetric 1-algebra containing $T_{\varepsilon,0}^{\beta,\beta}$, where $\beta = 1$ or $\beta = -1$, may exist only if $\varepsilon = 0$. As a (nonobvious) consequence of this fact, Theorem 3.2 cannot yield effective representations of a *generic* matrix A including algebras $\mathcal{S}(\cdot;\mathbf{p}^{\mathrm{SK}})$ or $\mathcal{R}(\cdot;\mathbf{p}^{\mathrm{S}})$. However, Theorem 3.2 yields such generic formulas, also in terms of non-Hessenberg algebras, if we let both $\mathbb{L}$ and $\mathbb{L}'$ be matrix algebras of the type considered in Proposition 4.2(iii). An example is easily obtained by choosing $\varphi' = \varphi$, $\beta' = \beta$ (in Theorem 3.2) and then—in order to ensure the existence of symmetric 1-algebras $\mathbb{L} \supset T_{\varepsilon,\varphi}^{\beta,\beta}$ and $\mathbb{L}' \supset T_{\varepsilon',\varphi}^{\beta,\beta}$ for $\varepsilon \neq \varepsilon'$—by requiring $I_{\beta\varphi}$ in (4.2) to be nonsingular (see Proposition 4.2). For the sake of brevity we mention only some values of β and φ for which $I_{\beta\varphi}$ is nonsingular and $I_{\beta\varphi}^{-1}$ is known (in the sense of Lemma 4.1) for any value of n: φ arbitrary, $\beta = 0$ [10]; $\varphi = 0$, $\beta^2 \neq 1$; $\varphi = 2$, $\beta = 1$; $\varphi = -2$, $\beta = -1$.

Formula (4.14) is exploited in section 5 to state a simple espression of the inverse of a centrosymmetric Toeplitz-plus-Hankel matrix $T + H$. This expression allows us to calculate $(T + H)^{-1}\mathbf{f}$, $\mathbf{f} \in \mathbb{C}^n$, by performing essentially 10 DFTs reducible to 8 in the case $H = 0$, $[T^{-1}]_{11} \neq 0$, matching both best limits known so far.

5. Toeplitz-plus-Hankel inversion formulas. Theorem 3.2, the results of the previous section, and the fact that the rank of $\mathcal{C}_{T^{\beta,\beta}_{\varepsilon,\varphi}}((T+H)^{-1})$ is 4 for all values of ε, φ, β (see [26] for the case $\varepsilon=\varphi=\beta=0$) yield new representations of the inverse of a Toeplitz-plus-Hankel matrix $T+H$ (or, more generally, of $(T+H)$-like matrices, that is, structured matrices A for which rank$\mathcal{C}_{T^{\beta,\beta}_{\varepsilon,\varphi}}(A)$ is small with respect to n). These are similar to other formulas found in [1, 6, 9, 10, 16, 17, 20, 23, 32], but they involve new n-dimensional matrix algebras different from HAs. The formulas so obtained can be used to solve a linear system $(T+H)\mathbf{x}=\mathbf{f}$, $\mathbf{f}=\mathbb{C}^n$, in $O(n\log n)$ arithmetic operations (via the computation of $(T+H)^{-1}\mathbf{f}$), provided the 8 vectors defining $\mathcal{C}_{T^{\beta,\beta}_{\varepsilon,\varphi}}((T+H)^{-1})$ are known. Here only the centrosymmetric case is considered in detail.

This approach (compared to a direct triangular factorization of $T+H$ [33, 27]) is significant especially in case a distinction is emphasized between a preprocessing stage—where only operations on elements of $T+H$ are performed—and a successive stage of complexity $O(n\log n)$, where the linear system $(T+H)\mathbf{x}=\mathbf{f}$, $\mathbf{f}\in\mathbb{C}^n$, is solved. This distinction is justified when many different linear systems $(T+H)\mathbf{x}=\mathbf{f}_i$ have to be solved. The same point of view is assumed by Gohberg and Olshevsky in [21, 22], where the complexity of the computation of $A\mathbf{f}$ with preprocessing on A is studied for different types of structured matrices A, including the case $A=T^{-1}$ for a generic Toeplitz T. (Some results on the complexity of the preprocessing stage are also given in [21, 22].) In particular, they show that the application of T^{-1} to the vector $\mathbf{f}$ can be accomplished with a cost of 6 DFTs of order n and thus generalize the analogous result obtained by Ammar and Gader in the Hermitian case [1]. We mention the fact that if T is symmetric, the above limit can be reduced to 11 DFTs of order $\frac{n}{2}$ by using a formula for T^{-1} involving circulant and (-1)-circulant matrices of order $\frac{n}{2}$ (see [15, 17]). Moreover, it is known [10, 16] that 6 discrete transforms are also enough to compute the product $(T+H)^{-1}\mathbf{f}$, where $T+H$ is a centrosymmetric Toeplitz-plus-Hankel matrix. This fact is also shown in the present paper by using a decomposition of $(T+H)^{-1}$ in terms of Hartley-type matrix algebras (see the remarks after Theorems 5.1 and 5.2).

Let T, $[T]_{ij}=t_{i-j}$, and H, $[H]_{ij}=h_{i+j-2}$, $i,j=1,\dots,n$, be, respectively, a symmetric Toeplitz and a persymmetric Hankel matrix with complex elements, and assume that $T+H$ is nonsingular. Then [26]

$$\begin{aligned}(T+H)^{-1}T^{\beta,\beta}_{\varphi,\varphi}-T^{\beta,\beta}_{\varphi,\varphi}(T+H)^{-1}&=(\mathbf{x}_1-\varphi\mathbf{e}_1-\beta\mathbf{e}_n)\mathbf{w}_1^T+(\hat{\mathbf{x}}_1-\varphi\mathbf{e}_n-\beta\mathbf{e}_1)\hat{\mathbf{w}}_1^T\\&\quad-\mathbf{w}_1(\mathbf{x}_1-\varphi\mathbf{e}_1-\beta\mathbf{e}_n)^T-\hat{\mathbf{w}}_1(\hat{\mathbf{x}}_1-\varphi\mathbf{e}_n-\beta\mathbf{e}_1)^T,\end{aligned}\tag{5.1}$$

where $\mathbf{w}_1$ and $\mathbf{x}_1$ are such that

$$(T+H)\mathbf{w}_1=\mathbf{e}_1\quad\text{and}\quad(T+H)\mathbf{x}_1=[t_1+h_{-1}\ t_2+h_0\cdots t_n+h_{n-2}]^T,h_{-1},t_n\in\mathbb{C}$$

(see also [16, 10]). Equality (5.1) for $\beta=0$, $\varphi=1$ and Theorem 3.2 for $\varepsilon=\varphi=1$, $\varepsilon'=\varphi'=-1$, $\beta=\beta'=0$ let us regain the decomposition of $(T+H)^{-1}$

$$2(T+H)^{-1}=\tau_{-1,-1}(\mathbf{x}_1+\mathbf{e}_1)\tau_{1,1}(\mathbf{w}_1)-\tau_{-1,-1}(\mathbf{w}_1)\tau_{1,1}(\mathbf{x}_1-\mathbf{e}_1)\tag{5.2}$$

found in [10]. Moreover, Theorem 3.2 (via Theorem 4.3) yields new decompositions of $(T+H)^{-1}$ in terms of the matrix algebras η, μ, $\mathcal{H}$, and $\mathcal{K}$ studied in section 4.

THEOREM 5.1.

$$(T+H)^{-1}=\frac{1}{2}\{\mu(\hat{\mathbf{x}}_1+\mathbf{e}_1)\eta(\mathbf{w}_1)-\mu(\mathbf{w}_1)\eta(\hat{\mathbf{x}}_1-\mathbf{e}_1)\},\tag{5.3}$$

$$(T+H)^{-1} = \frac{1}{2}(J+\mathcal{K}(\mathbf{e}_n))^{-1}\{[\mathcal{K}(\mathbf{x}_1+\mathbf{e}_n)+\mathcal{K}(\hat{\mathbf{x}}_1+\mathbf{e}_1)J]\eta(\mathbf{w}_1)$$

$$- [\mathcal{K}(\mathbf{w}_1)+\mathcal{K}(\hat{\mathbf{w}}_1)J]\eta(\mathbf{x}_1-\mathbf{e}_n)\}, \tag{5.4}$$

$$(T+H)^{-1} = \{\mu(\mathbf{x}_1+\mathbf{e}_n)[\mathcal{H}(\mathbf{w}_1)+J\mathcal{H}(\hat{\mathbf{w}}_1)]$$

$$- \mu(\mathbf{w}_1)[\mathcal{H}(\mathbf{x}_1-\mathbf{e}_n)+J\mathcal{H}(\hat{\mathbf{x}}_1-\mathbf{e}_1)]\}\frac{1}{2}(J+\mathcal{H}(\mathbf{e}_n))^{-1}. \tag{5.5}$$

Proof. Exploit (5.1) for $\varphi = 0$, $\beta = 1$ and formulas (4.14), (4.15), and (4.16) of Theorem 4.3, respectively. □

Formulas (5.2)–(5.5) can be used to compute $(T+H)^{-1}\mathbf{f}$ by means of a constant number of DFTs, Hartley-type transforms, trigonometric transforms, or mixed-type transforms all computable in $O(n \log n)$ arithmetic operations (see [5, 11, 10, 34], Theorem 5.2, and the following remark). In particular, formula (5.3) is competitive with the formulas found in [16, 10]. In fact, as an immediate consequence of Theorem 5.2, the matrix by vector product $(T+H)^{-1}\mathbf{f}$, $\mathbf{f} \in \mathbb{C}^n$, can be calculated by performing essentially 10 order n DFTs if $(T+H)^{-1}$ is replaced by its expression in (5.3) and if $\mathbf{x}_1$ and $\mathbf{w}_1$ are assumed as known. Moreover, we shall see that, for $H = 0$ and $w_{11} = [T^{-1}]_{11} \neq 0$, the number of DFTs can be reduced to 8. The limits 10 and 8 are identical to those obtained in [10] with (5.2); however, here the limit 8 is obtained without the further assumption that the entries of T are real, and the coefficient of n in the surplus of $O(n)$ operations is smaller. Recall that the limit 8 has been obtained for the first time by Ammar and Gader in [1]. Both in [1, 16, 10] and in (5.3) the number of discrete transforms is in any case 6 if the transforms of vectors not depending upon $\mathbf{f}$ are included in the preprocessing stage. Moreover, notice that Rost [32] obtains a simple representation for the "classical" Hankel Bezoutian (and therefore for H^{-1}) in terms of $\tau_{0,0}$ and $\tau_{0,1}$ matrices and refers to a future work concerning with the Toeplitz-plus-Hankel case and with the study of computational properties of these representations.

In the next theorem, $d(\mathbf{z})$, $\mathbf{z} \in \mathbb{C}^n$, denotes the $n \times n$ diagonal matrix whose (k,k) element is z_k, $k = 1,\dots,n$, and $\mathbf{i}$ is the imaginary unit. Moreover, if A is an $n \times n$ matrix with complex entries, then A^H denotes the transposed conjugate of A.

THEOREM 5.2. *Set* $\rho = \exp(-\mathbf{i}\pi/n)$, $\bar\rho = \rho^{-1}$, $\omega = \rho^2$, $[F]_{ij} = \frac{1}{\sqrt{n}}\omega^{(i-1)(j-1)}$, $i,j = 1,\dots,n$, $D_\rho = \operatorname{diag}(\rho^{i-1}, i = 1,\dots,n)$, *and* $D_\omega = D_\rho^2$. *Then, for all* $\mathbf{z} \in \mathbb{C}^n$,

$$\eta(\mathbf{z}) = M_\eta \Lambda(M_\eta^T\mathbf{z})M_\eta^H, \qquad \Lambda(M_\eta^T\mathbf{z}) = d(M_\eta^T\mathbf{z})d(M_\eta^T\mathbf{e}_1)^{-1}, \tag{5.6}$$

$$\mu(\mathbf{z}) = M_\mu \Lambda(M_\mu^T\mathbf{z})M_\mu^H, \qquad \Lambda(M_\mu^T\mathbf{z}) = d(M_\mu^T\mathbf{z})d(M_\mu^T\mathbf{e}_1)^{-1}, \tag{5.7}$$

where M_η *and* M_μ *are the unitary matrices:*

$$M_\eta = \frac{1}{\sqrt{2}}F\begin{pmatrix} \sqrt{2} & 0 & \cdots & & & & \cdots & 0\\ 0 & 1 & & & & & & -\omega\\ \vdots & & \ddots & & & & \cdot^{\cdot^{\cdot}} & 0\\ \vdots & & & 1 & 0 & -\omega^{\frac{n}{2}-1} & & \vdots\\ \vdots & & & 0 & \sqrt{2} & 0 & & \vdots\\ \vdots & & & \omega^{\frac{n}{2}+1} & 0 & 1 & & \vdots\\ \vdots & & \cdot^{\cdot^{\cdot}} & & & & \ddots & 0\\ 0 & \omega^{n-1} & 0 & \cdots & & \cdots & 0 & 1 \end{pmatrix}, \tag{5.8}$$

$$M_\mu = \frac{1}{\sqrt{2}} D_\rho F \begin{pmatrix} 1 & 0 & \cdots & & & & 0 & -\overline{\rho}^{n-1} \\ 0 & 1 & \ddots & & & & \iddots & 0 \\ \vdots & \ddots & \ddots & 0 & 0 & -\overline{\rho}^3 & & \vdots \\ \vdots & & 0 & 1 & -\overline{\rho} & 0 & & \vdots \\ \vdots & & 0 & -\overline{\rho}^{n-1} & 1 & 0 & & \vdots \\ & \iddots & & 0 & 0 & 1 & \ddots & \\ 0 & -\overline{\rho}^3 & & & & & \ddots & 0 \\ -\overline{\rho} & 0 & \cdots & & & & 0 & 1 \end{pmatrix} \tag{5.9}$$

for n even, and

$$M_\eta = \frac{1}{\sqrt{2}} F \begin{pmatrix} \sqrt{2} & 0 & \cdots & & & & & 0 \\ 0 & 1 & \ddots & & & & \iddots & -\omega \\ \vdots & \ddots & \ddots & & & & & 0 \\ \vdots & & \ddots & 0 & 0 & & \iddots & \vdots \\ \vdots & & 0 & 1 & -\omega^{\frac{n-1}{2}} & 0 & & \vdots \\ \vdots & & 0 & \omega^{\frac{n+1}{2}} & 1 & 0 & & \vdots \\ \vdots & & \iddots & 0 & 0 & & \ddots & \vdots \\ \vdots & \iddots & & & & & \ddots & 0 \\ 0 & \omega^{n-1} & 0 & \cdots & & & 0 & 1 \end{pmatrix}, \tag{5.10}$$

$$M_\mu = \frac{1}{\sqrt{2}} D_\rho F \begin{pmatrix} 1 & 0 & \cdots & & & 0 & -\omega^{\frac{n+1}{2}} \\ 0 & \ddots & & & & \iddots & 0 \\ \vdots & & 1 & 0 & -\omega^{n-1} & & \vdots \\ \vdots & & 0 & \sqrt{2} & 0 & & \vdots \\ \vdots & & \omega & 0 & 1 & & \vdots \\ 0 & \iddots & & & & \ddots & 0 \\ \omega^{\frac{n-1}{2}} & 0 & \cdots & & & 0 & 1 \end{pmatrix} \tag{5.11}$$

for n odd. Moreover, for all $\mathbf{z} \in \mathbb{C}^n$,

$$\mathcal{H}(\mathbf{z}) = \sqrt{n}\ H_+ d(H_+\mathbf{z}) H_+ = \sqrt{n}\ H_- d(H_-\mathbf{z}) H_-, \tag{5.12}$$

$$\mathcal{K}(\mathbf{z}) = \sqrt{n}\ K_+ d(K_+^T\mathbf{z}) K_+^T = \sqrt{n}\ K_- d(K_-^T\mathbf{z}) K_-^T, \tag{5.13}$$

where H_+, H_-, K_+, and K_- are the orthonormal matrices defined by

$$[H_\pm]_{ij} = (1/\sqrt{n}\,) \left(\cos \frac{2\pi(i-1)(j-1)}{n} \pm \sin \frac{2\pi(i-1)(j-1)}{n} \right), \tag{5.14}$$

$$[K_\pm]_{ij} = (1/\sqrt{n}\,) \left(\cos \frac{\pi(i-1)(2j-1)}{n} \pm \sin \frac{\pi(i-1)(2j-1)}{n} \right), \tag{5.15}$$

$i, j = 1, \ldots, n$.

Proof. The equalities (5.6) and (5.7) are shown only in the case n even ($n = 2m$). In the case n odd, the proof is similar. Notice that in order to find the matrices in (5.8), (5.10) and (5.9), (5.11), we had to look for a matrix diagonalizing J among the matrices diagonalizing $T_{0,0}^{1,1}$ and $T_{0,0}^{-1,-1}$, respectively.

Let us prove (5.6). Set $\mathbf{c}_i = \sqrt{n}\,F\mathbf{e}_i$, $i = 1,\dots,n$. By using the identities $\hat{\mathbf{c}}_i = \omega^{n-i+1}\mathbf{c}_{n-i+2}$, $i = 2,\dots,m$ (recall that, for a vector $\mathbf{z}$, $\hat{\mathbf{z}} = J\mathbf{z}$), one can easily show that

$$(5.16)\quad M_\eta = \frac{1}{\sqrt{2n}}\,[\sqrt{2}\mathbf{c}_1\ \ \mathbf{c}_2 + \hat{\mathbf{c}}_2 \cdots \mathbf{c}_m + \hat{\mathbf{c}}_m\ \ \sqrt{2}\mathbf{c}_{m+1}\ \ \mathbf{c}_{m+2} - \hat{\mathbf{c}}_{m+2} \cdots \mathbf{c}_n - \hat{\mathbf{c}}_n].$$

Moreover, as $T_{0,0}^{1,1} = P_1 + P_1^H$ and $P_1 = FD_\omega F^H$, $T_{0,0}^{1,1}F = F(D_\omega + D_\omega^H)$, i.e.,

$$(5.17)\quad T_{0,0}^{1,1}[\mathbf{c}_1\ \mathbf{c}_2 \cdots \mathbf{c}_n] = [\mathbf{c}_1\ \mathbf{c}_2 \cdots \mathbf{c}_n]\,\mathrm{diag}(2\cos\frac{2\pi(j-1)}{n},\quad j = 1,\dots,n).$$

By the centrosymmetry of $T_{0,0}^{1,1}$ (besides $\mathbf{c}_j$) also $\hat{\mathbf{c}}_j$ is an eigenvector of $T_{0,0}^{1,1}$ with associated eigenvalue $2\cos\frac{2\pi(j-1)}{n}$. This remark and equalities (5.17) and (5.16) allow us to say that

$$(5.18)\qquad T_{0,0}^{1,1}M_\eta = M_\eta\,\mathrm{diag}\left(2\cos\frac{2\pi(j-1)}{n},\ j = 1,\dots,n\right).$$

From (5.16) it also follows that

$$(5.19)\qquad \eta(\mathbf{e}_n)M_\eta = JM_\eta = M_\eta\begin{pmatrix} I & O \\ O & -I \end{pmatrix},$$

where the I in (5.19) is the $m \times m$ identity matrix ($\hat{\mathbf{c}}_1 = \mathbf{c}_1$, $\hat{\mathbf{c}}_{m+1} = -\mathbf{c}_{m+1}$). By exploiting, respectively, (5.18) and (5.19), we have that the matrix $M_\eta d(M_\eta^T\mathbf{z})$ $d(M_\eta^T\mathbf{e}_1)^{-1}M_\eta^H$ commutes with the matrices $T_{0,0}^{1,1}$ and J $\forall\mathbf{z} \in \mathbb{C}^n$. Moreover, as $M_\eta M_\eta^H = I$, its first row is $\mathbf{z}^T$, and therefore, by (4.7), we have $\eta(\mathbf{z}) = M_\eta d(M_\eta^T\mathbf{z})$ $d(M_\eta^T\mathbf{e}_1)^{-1}M_\eta^H$.

Let us prove (5.7). Set $\mathbf{c}_i = \sqrt{n}D_\rho F\mathbf{e}_i$, $i = 1,\dots,n$. The identities $\hat{\mathbf{c}}_i = -\bar{\rho}^{2i-1}\mathbf{c}_{n+1-i}$, $i = 1,\dots,m$, yield

$$(5.20)\qquad M_\mu = \frac{1}{\sqrt{2n}}\,[\mathbf{c}_1 + \hat{\mathbf{c}}_1 \cdots \mathbf{c}_m + \hat{\mathbf{c}}_m\ \ \mathbf{c}_{m+1} - \hat{\mathbf{c}}_{m+1} \cdots \mathbf{c}_n - \hat{\mathbf{c}}_n].$$

Moreover, as $T_{0,0}^{-1,-1} = P_{-1} + P_{-1}^H$ and $P_{-1} = D_\rho F\rho D_\omega F^H D_\rho^H$, we have $T_{0,0}^{-1,-1}D_\rho F = D_\rho F(\rho D_\omega + \bar{\rho}D_\omega^H)$, i.e.,

$$(5.21)\quad T_{0,0}^{-1,-1}[\mathbf{c}_1\ \mathbf{c}_2 \cdots \mathbf{c}_n] = [\mathbf{c}_1\ \mathbf{c}_2 \cdots \mathbf{c}_n]\,\mathrm{diag}\left(2\cos\frac{\pi(2j-1)}{n},\quad j = 1,\dots,n\right).$$

As in the case of (5.6), the equalities (5.21) and (5.20) yield

$$T_{0,0}^{-1,-1}M_\mu = M_\mu\,\mathrm{diag}\left(2\cos\frac{\pi(2j-1)}{n},\quad j = 1,\dots,n\right),$$

$$\mu(\mathbf{e}_n)M_\mu = JM_\mu = M_\mu\begin{pmatrix} I & O \\ O & -I \end{pmatrix},$$

where I is the $m \times m$ identity matrix. Thus the matrix $M_\mu d(M_\mu^T \mathbf{z}) d(M_\mu^T \mathbf{e}_1)^{-1} M_\mu^H$ commutes with the matrices $T_{0,0}^{-1,-1}$ and J $\forall \mathbf{z} \in \mathbb{C}^n$. Moreover, as $M_\mu M_\mu^H = I$, its first row is $\mathbf{z}^T$ and therefore, by (4.11), we have $\mu(\mathbf{z}) = M_\mu d(M_\mu^T \mathbf{z}) d(M_\mu^T \mathbf{e}_1)^{-1} M_\mu^H$.

Finally, let us prove (5.13). This proof is analogous to the proof of the first equality in (5.12), which is in [5]. Notice that $D_\rho F = \frac{1}{\sqrt{n}}\,(M - \mathbf{i}N)$, where $[M]_{ij} = \cos\frac{\pi(i-1)(2j-1)}{n}$ and $[N]_{ij} = \sin\frac{\pi(i-1)(2j-1)}{n}$, $i,j = 1,\dots,n$. Moreover, from the identities $(D_\rho F)^H D_\rho F = I$ and $(D_\rho F)^T D_\rho F = J$, we have

$$M^T M + N^T N = nI \quad \text{and} \quad M^T N + N^T M = 0,$$

respectively. Observe that $K_+ = \frac{1}{\sqrt{n}}\,(M+N)$ $\left[K_- = \frac{1}{\sqrt{n}}\,(M-N)\right]$. Thus, by the above equalities, $K_+^T K_+ = I$ $[K_-^T K_- = I]$. Moreover $M = MJ = -JP_{-1}M$ and $-N = NJ = -JP_{-1}N$; therefore, $K_+ J = -JP_{-1}K_+$ $[K_- J = -JP_{-1}K_-]$.

Let A be a generic (-1)-circulant matrix. We know that $(D_\rho F)^H A D_\rho F = D_A$, where D_A is a diagonal matrix and thus

$$\text{(5.22)} \qquad \mathrm{Re} D_A = \frac{1}{n}(M^T A M + N^T A N), \quad \mathrm{Im} D_A = \frac{1}{n}\,(N^T A M - M^T A N).$$

From (5.22) it follows that if A is a (-1)-circulant matrix, then

$$K_+^T A K_+ = \mathrm{Re} D_A - J \mathrm{Im} D_A \quad [K_-^T A K_- = \mathrm{Re} D_A + J\mathrm{Im} D_A].$$

Now let E be a generic element of $\mathcal{K} = C_{-1}^{\mathrm{S}} + JP_{-1}C_{-1}^{\mathrm{SK}}$ and assume that the entries of E are real, i.e., $E = E_{-1}^{\mathrm{S}} + JP_{-1}E_{-1}^{\mathrm{SK}}$, where E_{-1}^{S} is a real symmetric (-1)-circulant matrix and E_{-1}^{SK} is a real skewsymmetric (-1)-circulant matrix. Observe that the eigenvalues of E_{-1}^{S} is a real skewsymmetric (-1)-circulant matrix. Observe that the eigenvalues of E_{-1}^{S} and E_{-1}^{SK} are, respectively, real and purely imaginary. Thus

$$\begin{aligned} K_+^T E K_+ &= K_+^T E_{-1}^{\mathrm{S}} K_+ + K_+^T J P_{-1} E_{-1}^{\mathrm{SK}} K_+ = K_+^T E_{-1}^{\mathrm{S}} K_+ - J K_+^T E_{-1}^{\mathrm{SK}} K_+ \\ &= \mathrm{Re} D_{E_{-1}^{\mathrm{S}}} + \mathrm{Im} D_{E_{-1}^{\mathrm{SK}}} \\ &\left[K_-^T E K_- = \mathrm{Re} D_{E_{-1}^{\mathrm{S}}} - \mathrm{Im} D_{E_{-1}^{\mathrm{SK}}}\right]. \end{aligned}$$

We have proved that $K_\pm^T E K_\pm = d(\mathbf{z}_E^\pm)$ for some $\mathbf{z}_E^\pm \in \mathbb{R}^n$. The thesis, in the real case, follows from the equalities $\mathbf{e}_1^T E K_\pm = \mathbf{e}_1^T K_\pm d(\mathbf{z}_E^\pm) = \mathbf{z}_E^{\pm T} d(K_\pm^T \mathbf{e}_1) = \frac{1}{\sqrt{n}}\ \mathbf{z}_E^{\pm T}$. For the complex case, simply observe that if $\mathbf{z} \in \mathbb{C}^n$, then $\mathbf{z} = \mathbf{z}_1 + \mathbf{i}\mathbf{z}_2$, where $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$, and that $\mathcal{K}(\mathbf{z}) = \mathcal{K}(\mathbf{z}_1 + \mathbf{i}\mathbf{z}_2) = \mathcal{K}(\mathbf{z}_1) + \mathbf{i}\mathcal{K}(\mathbf{z}_2)$. ∎

Remark. If n is an integer power of 2, then the *skew-Hartley* transform $\sqrt{n}\, K_\pm \mathbf{z}$ $(\sqrt{n}\, K_\pm^T \mathbf{z})$, $\mathbf{z} \in \mathbb{R}^n$, can be computed in at most $\frac{3}{2} n \log_2 n$ additions and $n \log_2 n$ multiplications of real numbers, i.e., with the same cost of the Hartley transform $\sqrt{n}\, H_\pm \mathbf{z}$. (For this last transform, see [5] and the references cited therein.) In fact, for $K_\pm^{(n)} = K_\pm$ we have

$$K_\pm^{(n)} = \frac{1}{\sqrt{2}} Q \begin{pmatrix} K_\pm^{(\frac{n}{2})} & K_\pm^{(\frac{n}{2})} \\ K_\pm^{(\frac{n}{2})} R_\pm & -K_\pm^{(\frac{n}{2})} R_\pm \end{pmatrix},$$

where $R_\pm = \mathrm{diag}(\cos\frac{(2j-1)\pi}{n}, j = 1,\dots,\frac{n}{2}) \pm J\,\mathrm{diag}(\sin\frac{(2j-1)\pi}{n}, j = 1,\dots,\frac{n}{2})$ and Q is the permutation matrix $Q\mathbf{e}_j = \mathbf{e}_{2j-1}$, $Q\mathbf{e}_{n-j+1} = \mathbf{e}_{n-2j+2}$, $j = 1,\dots,\frac{n}{2}$. (For

$H_{\pm}^{(n)} = H_{\pm}$ an analogous identity holds, where $R_{\pm} = \mathrm{diag}(\cos \frac{2\pi(j-1)}{n}, j = 1, \dots, \frac{n}{2}) \pm JP_{\beta}\,\mathrm{diag}(\sin \frac{2\pi(j-1)}{n}, j = 1, \dots, \frac{n}{2})$.

If $H = 0$ and $w_{11} = [T^{-1}]_{11} \neq 0$, then $\mathbf{x}_1 = -(1/w_{11})P_0\mathbf{w}_1$ [25] (see also [16]). By exploiting this fact and the identities (5.6) and (5.7) in Theorem 5.2, formula (5.3) becomes

$$
\begin{aligned}
T^{-1} &= \frac{1}{2w_{11}}\{\mu(\mathbf{w}_1)\eta(JP_1\mathbf{w}_1) - \mu(JP_{-1}\mathbf{w}_1)\eta(\mathbf{w}_1)\} \\
&= \frac{1}{2w_{11}} M_{\mu}\{\Lambda(M_{\mu}^T\mathbf{w}_1)M_{\mu}^H M_{\eta}\Lambda(M_{\eta}^T JP_1\mathbf{w}_1) \\
&\quad - \Lambda(M_{\mu}^T JP_{-1}\mathbf{w}_1)M_{\mu}^H M_{\eta}\Lambda(M_{\eta}^T\mathbf{w}_1)\}M_{\eta}^H.
\end{aligned} \tag{5.23}
$$

Observe that the vectors $\mathbf{z}$ in the four matrices $\Lambda(\mathbf{z})$ appearing in this last formula can be computed in $O(n)$ arithmetic operations once that $F\mathbf{w}_1$ and $FD_{\rho}\mathbf{w}_1$ are calculated (use the identities $F(JP_1)\mathbf{w}_1 = (JP_1)F\mathbf{w}_1$ and $FD_{\rho}(JP_{-1})\mathbf{w}_1 = -JFD_{\rho}\mathbf{w}_1$). Thus, if $\mathbf{w}_1$ is known, the vector $T^{-1}\mathbf{f}$, $\mathbf{f} \in \mathbb{C}^n$, can be computed by performing eight DFTs plus $\mathrm{O}(n)$ arithmetic operations.

In [1] Ammar and Gader obtain the same result by exploiting the representation in terms of circulant and (-1)-circulant matrices

$$
T^{-1} = \frac{1}{2w_{11}}\{C_{-1}(\mathbf{w}_1)C(\mathbf{w}_1)^T + C_{-1}(\mathbf{w}_1)^T C(\mathbf{w}_1)\}, \tag{5.24}
$$

which is a consequence of the following formula, holding for a *generic* nonsingular Toeplitz matrix $T = (t_{i-j})_{i,j=1}^n$,

$$
T^{-1} = \frac{1}{2}\{C_{-1}(\hat{\mathbf{w}}_n)C(\mathbf{e}_1 - \hat{\mathbf{x}}_1) + C_{-1}(\mathbf{e}_1 + \hat{\mathbf{x}}_1)C(\hat{\mathbf{w}}_n)\}, \tag{5.25}
$$

where $\mathbf{w}_n = T^{-1}\mathbf{e}_n$ and $T\mathbf{x}_1 = [t_1\ t_2 \cdots t_n]^T$, $t_n \in \mathbb{C}$ (see also [16]). Formulas of type (5.25), generalizing the Ammar–Gader formula (5.24), were first derived by Gohberg and Olshevsky in [20, 22]. Notice that, by using formula (5.25) or the analogous formulas in [20, 22], the product $T^{-1}\mathbf{f}$ for a generic T can be calculated with essentially 10 order n DFTs [21, 22], i.e., with the same amount of computation required to compute $(T+H)^{-1}\mathbf{f}$ for $T = T^T$ and $H = JHJ$ via (5.3). Both in (5.24), (5.25) and in (5.3), (5.23) the number of discrete transforms is 6 if the transforms of vectors not depending upon $\mathbf{f}$ are included in the preprocessing stage. Thus formulas (5.3) and (5.23) seem to be the analogues of the Ammar–Gader–Gohberg–Olshevsky-type formulas for the centrosymmetric Toeplitz-plus-Hankel case.

Acknowledgments. I wish to thank Professor Paolo Zellini for proposing the problem and for his comments and suggestions. I would also like to thank the referees for their useful comments.

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