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On a set of matrix algebras related to discrete Hartley-type transforms

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Abstract

A set of fast real transforms including the well known Hartley transform is fully investigated. Mixed radix splitting properties of Hartley-type transforms are examined in detail. The matrix algebras diagonalized by the Hartley-type matrices are expressed in terms of circulant and (-1)-circulant matrices.

1. Introduction

The principal aim of this paper is to give a taxonomy for a set of fast Hartley-type (Ht) transforms and for the corresponding algebras of matrices diagonalized by Ht matrices. This taxonomy completes previous partial lists considered in [1,3,12,17,18,20,24–27] and is here proposed as the Hartley counterpart of the known classification of Jacobi transforms/algebras in [6,9,28,30,37].

One can reasonably expect that this set of Ht algebras has applications analogous to the Jacobi set. Some Ht algebras have been already used in displacement and Bezoutian theory, and in the preconditioning technique for conjugate gradient type methods [1,3,17,18,24–26]. Moreover, a class of new quasi-Newtonian methods for unconstrained minimization problems (recently introduced in [8,19,20]) consists in an iterative approximation of the Hessian involving algebras of matrices diagonalized

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by fast transforms. In fact, these methods have been initially implemented in terms of Ht transforms. Some of the quoted applications are further developed in [21]. Finally the Ht transforms could be visually attractive, as required in image processing [32].

Four Ht matrices, here denoted by H, K, K^{T} and G, have already appeared in literature [27]. By H we mean the matrix defining the well known discrete Hartley transform [10,11,36]. The algebra $\mathscr{H} = SDH$ of all matrices diagonalized by H is obtained in [3]. In [25,26] the matrices H, K, K^{T} and G are named H^{I}, H^{II}, H^{III} and H^{IV}, respectively, and are used to represent Toeplitz-plus-Hankel Bezoutians. The same matrices are pointed out in [12] as particular instances in a set of generalized discrete Fourier matrices. The skew-Hartley transforms defined by K or K^{T} appear in the context of displacement formulas [1,17,18]. The optimal preconditioners introduced in [24] are chosen in algebras of matrices defined in terms of H, K, K^{T} and G. The matrix algebras $\mathscr{K} = SDK$ and $\gamma = SDG$, together with other two Ht algebras η and μ and the corresponding Ht transforms, are studied in [1,17,18,24].

Here it is shown that H, K, K^{T} and G can be obtained by a recursive doubling technique applied, initially, to the Hartley matrix H (Section 2). In other words, the same radix-2 splitting formulas which lead to fast Ht transforms, are used to define the Ht matrices. The results of Section 2 are then generalized in Section 3 where, on the basis of Ht radix-n_2 splitting formulas for $U_{n_1 n_2}$, we obtain factorizations of U_n, corresponding to a suitable factorization of n, in terms of sparse orthogonal matrices. So numerically stable Ht algorithms can be conceived which are fast whenever n is a highly composed integer. Finally, in Section 4 the partial lists of Ht algebras/transforms present in literature are easily completed through slight modifications of the unitary matrices defining the fast transforms or of the explicit expression—in terms of circulant or (-1)-circulant matrices—of the related algebras. Thus eight Ht fast transforms together with their corresponding Ht algebras are described in a table at the end of the paper.

2. Radix-2 fast Hartley-type transforms

The basic Hartley-type (Ht) matrices, H_n, K_n, K_n^{T} and G_n, are orthogonal matrices U_n of order n whose generic entry (i, j) has the form

$$[U_n]_{ij} = \frac{1}{\sqrt{n}}\operatorname{cas}\frac{f(i)g(j)\pi}{n}, \quad 0 \leqslant i, j \leqslant n-1, \tag{2.1}$$

where cas denotes the function $\operatorname{cas} x = \cos x + \sin x$. By H_n we denote the matrix defining the well known discrete Hartley transform [10]. The other matrices, K_n, K_n^{T} and G_n, have appeared in more recent articles [1,12,18,20,24–27]; in [18] the transforms associated with K_n and K_n^{T} are called skew-Hartley transforms. In this section it is shown that the same Ht recursive doubling property, on which fast Ht algorithms are based, leads to the successive definition of H_n, K_n, K_n^{T} and G_n. Other four Ht matrices will be obtained in Section 4. Two of them are new.

First introduce the following notations. The even–odd or 2-stride (cf. [33, pp. 43–53] and [36, p. 12]) matrix of order $2n$ and the counteridentity of order n are permutation matrices defined, respectively, by

$$Q_{n,2}\mathbf{z} = [z_0 z_2 \cdots z_{2n-2} z_1 z_3 \cdots z_{2n-1}]^{\mathrm{T}}, \quad \mathbf{z} \in \mathbb{C}^{2n},$$
$$J\mathbf{z} = [z_{n-1} \cdots z_1 z_0]^{\mathrm{T}}, \quad \mathbf{z} \in \mathbb{C}^{n}.$$

The modified shift cyclic matrix of order n is defined by

$$P_\xi = \begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ \xi & & & & 0 \end{bmatrix}, \quad \xi \in \mathbb{C}.$$

Note that if $\mathbf{z} \in \mathbb{C}^n$, then

$$P_\xi \mathbf{z} = [z_1 \cdots z_{n-1} \xi z_0]^{\mathrm{T}}.$$

The diagonal matrix with z_h as (h, h) entry is

$$d(\mathbf{z}) = \mathrm{diag}(z_h, h = 0, \ldots, n-1).$$

Given $\varphi_h \in \mathbb{R}$, denote by $\mathbf{c}$ and $\mathbf{s}$ the vectors of $\mathbb{R}^n$ with entries $c_h = \cos\varphi_h$, $s_h = \sin\varphi_h$, $h = 0, \ldots, n-1$.

Also observe that the function cas verifies the identities

$$\mathrm{cas}(x+y) = \cos y \,\mathrm{cas}\, x + \sin y \,\mathrm{cas}(-x), \quad \mathrm{cas}(x+\pi) = -\mathrm{cas}\, x,$$
$$\mathrm{cas}(x+y) + \mathrm{cas}(x-y) = 2\cos y \,\mathrm{cas}\, x.$$

Let H_n be the $n \times n$ matrix

$$[H_n]_{ij} = \frac{1}{\sqrt{n}} \mathrm{cas} \frac{2ij\pi}{n}, \quad 0 \leqslant i, j \leqslant n-1. \tag{2.2}$$

The matrix H_n is symmetric and orthogonal, i.e. $H_n = H_n^{\mathrm{T}} = H_n^{-1}$. Moreover H_{2n} can be expressed in terms of H_n by the formula

$$H_{2n} = \frac{1}{\sqrt{2}} \begin{bmatrix} I & R_{\mathscr{K}} \\ I & -R_{\mathscr{K}} \end{bmatrix} \begin{bmatrix} H_n & \\ & H_n \end{bmatrix} Q_{n,2}, \tag{2.3}$$

where $I = I_n$ is the identity matrix and $R_{\mathscr{K}} = (R_{\mathscr{K}})_n$ is the cross shaped matrix $d(\mathbf{c}) + d(\mathbf{s})JP_1$ with $\varphi_h = h\pi/n, 0 \leqslant h \leqslant n-1$.

The matrix–vector product $H_n\mathbf{z}$, $\mathbf{z} \in \mathbb{C}^n$, is referred to as a *discrete Hartley transform* (*DHT*) of length n [10,11]. From (2.3) it follows that a *DHT* of length 2^m can be computed in $\mathrm{O}(2^m m)$ flops. In fact a repeated application of (2.3), $n = 2^{m-1}$, $2^{m-2}, \ldots, 1$, yields the decomposition of H_{2^m},

$$H_{2^m} = \frac{1}{\sqrt{2^m}} L_1 L_2 \cdots L_m Q, \tag{2.4}$$

where L_j is a block diagonal matrix whose diagonal blocks are

$$\begin{bmatrix} I_{2^{m-j}} & (R_{\mathscr{K}})_{2^{m-j}} \\ I_{2^{m-j}} & -(R_{\mathscr{K}})_{2^{m-j}} \end{bmatrix}$$

and Q is the bit reversing permutation matrix. This decomposition proves the desired result since each product $L_j\mathbf{z}$ requires $\mathrm{O}(2^m)$ flops. Notice that the coefficient of 2^m can be reduced by exploiting certain symmetries of the $R_{\mathscr{K}}$-matrices [10,13,31].

Now the equality (2.3) can be rewritten as

$$H_{2n} = \frac{1}{\sqrt{2}} \begin{bmatrix} H_n & R_{\mathscr{K}} H_n \\ H_n & -R_{\mathscr{K}} H_n \end{bmatrix} Q_{n,2}. \tag{2.5}$$

Moreover, $R_{\mathscr{K}} = (R_{\mathscr{K}})^{\mathrm{T}} = (R_{\mathscr{K}})^{-1}$. Thus the matrix

$$K_n = R_{\mathscr{K}} H_n \tag{2.6}$$

appearing in (2.5) is orthogonal. Since

$$\sqrt{n}[R_{\mathscr{K}} H_n]_{ij} = \cos\frac{i\pi}{n}\,\mathrm{cas}\frac{2ij\pi}{n} + \sin\frac{i\pi}{n}\,\mathrm{cas}\frac{2(n-i)j\pi}{n} = \mathrm{cas}\frac{i(2j+1)\pi}{n},$$

the generic entry of K_n^{T} is

$$\left[K_n^{\mathrm{T}}\right]_{ij} = \frac{1}{\sqrt{n}}\,\mathrm{cas}\,\frac{(2i+1)j\pi}{n}, \quad 0 \leqslant i, j \leqslant n-1. \tag{2.7}$$

The matrix K_{2n}^{T} can be expressed in terms of K_n^{T} by a formula analogous to (2.3) and (2.5). In fact we have

$$K_{2n}^{\mathrm{T}} = \frac{1}{\sqrt{2}} \begin{bmatrix} K_n^{\mathrm{T}} & R_\gamma K_n^{\mathrm{T}} \\ K_n^{\mathrm{T}} & -R_\gamma K_n^{\mathrm{T}} \end{bmatrix} Q_{n,2}, \tag{2.8}$$

where $R_\gamma = d(\mathbf{c}) + d(\mathbf{s})J$ with $\varphi_h = (2h+1)\pi/2n$, $0 \leqslant h \leqslant n-1$. Note that $R_\gamma = (R_\gamma)^{\mathrm{T}} = (R_\gamma)^{-1}$. So the matrix

$$G_n = R_\gamma K_n^{\mathrm{T}} \tag{2.9}$$

appearing in (2.8) is orthogonal. As

$$\begin{aligned} \sqrt{n}\left[R_\gamma K_n^{\mathrm{T}}\right]_{ij} &= \cos\frac{(2i+1)\pi}{2n}\,\mathrm{cas}\frac{(2i+1)j\pi}{n} \\ &\quad + \sin\frac{(2i+1)\pi}{2n}\,\mathrm{cas}\frac{(2(n-i-1)+1)j\pi}{n}, \end{aligned}$$

we have

$$[G_n]_{ij} = \frac{1}{\sqrt{n}}\,\mathrm{cas}\,\frac{(2i+1)(2j+1)\pi}{2n}, \quad 0 \leqslant i, j \leqslant n-1. \tag{2.10}$$

The matrix G_n is symmetric, like H_n. Moreover G_n is persymmetric, i.e. $[G_n]_{i,j} = [G_n]_{n-j-1,n-i-1}$. The following equality holds:

$$G_{2n} = \frac{1}{\sqrt{2}} \begin{bmatrix} R_+ G_n & R_- G_n \\ -R_- J G_n & R_+ J G_n \end{bmatrix} Q_{n,2}, \tag{2.11}$$

where $R_\pm = d(\mathbf{c}) \pm d(\mathbf{s})J$ with $\varphi_h = (2h+1)\pi/4n$, $0 \leqslant h \leqslant n-1$. Now $R_\pm R_\pm^{\mathrm{T}} = I \pm J$ and therefore no block in the 2×2 block matrix in (2.11) is orthogonal.

A decomposition analogous to (2.11) holds for the K matrix:

$$[K_n]_{ij} = \frac{1}{\sqrt{n}} \operatorname{cas} \frac{i(2j+1)\pi}{n}, \quad 0 \leqslant i, j \leqslant n-1. \tag{2.12}$$

More precisely, as a Corollary of Theorem 3.3, $U = K$ (see Section 3), we have

$$K_{2n} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{R}_+ K_n & \tilde{R}_- K_n \\ -\tilde{R}_- J P_{-1} K_n & \tilde{R}_+ J P_{-1} K_n \end{bmatrix} Q_{n,2}, \tag{2.13}$$

where $\tilde{R}_\pm = d(\mathbf{c}) \pm d(\mathbf{s}) J P_{-1}$ with $\varphi_h = h\pi/2n$, $0 \leqslant h \leqslant n-1$. Again, since the matrices $\tilde{R}_\pm$ are singular, no block in the 2×2 block matrix in (2.13) is orthogonal. So the above procedure cannot be further utilized to generate new orthogonal matrices.

Clearly (2.8), (2.11), and (2.13) yield decompositions of $K_{2^m}^{\mathrm{T}}$, G_{2^m} and K_{2^m} analogous to (2.4) and hence corresponding algorithms of complexity $\mathrm{O}(m2^m)$ for the computation of discrete K^{T}, G and K transforms (respectively $DK^{\mathrm{T}}T$, *DGT* and *DKT*) of length 2^m. The efficiency of these Cooley–Tukey algorithms can be improved by exploiting certain symmetries of the *R*-matrices. Moreover, the R_γ-symmetries are similar to the $R_{\mathscr{K}}$-symmetries. Therefore to each fast *DHT* algorithm known in the literature [10,13,31] corresponds a fast $DK^{\mathrm{T}}T$ algorithm with an essentially identical complexity [7].

Of course further algorithms computing the *DHT*, the *DKT*, the *DGT* and the $DK^{\mathrm{T}}T$ are derived by exploiting the decompositions of U_{2^m}, $U = H, K, G, K^{\mathrm{T}}$, obtained by transposing (2.5), (2.8), (2.11) and (2.13), respectively. In particular the algorithm for the *DKT* so derived appears to be more convenient with respect to the algorithm based on (2.13).

Notice that decompositions of the form (2.4) may suggest a quite unusual strategy of constructing fast transforms. In fact the "information content" of $U (U \in \{H, K^{\mathrm{T}}, G, K\})$ essentially consists in the matrices *R* (apart the block-diagonal structure of the factors *L*). Thus a suitable choice of *R* may generate new fast transforms under the only conditions that the blocks of *L* be unitary and have minimal complexity.

Formulas (2.5), (2.8), (2.11) and (2.13) are collected in the following:

Theorem 2.1
(i) *For* $U = H, K^{\mathrm{T}}$, *we have*

$$U_{2n} = \frac{1}{\sqrt{2}} \begin{bmatrix} I & X \\ I & -X \end{bmatrix} \begin{bmatrix} U_n & \\ & U_n \end{bmatrix} Q_{n,2}, \tag{2.14}$$

where $X = R_{\mathscr{K}}$ *for* $U = H$ *and* $X = R_\gamma$ *for* $U = K^{\mathrm{T}}$.

(ii) *For $U = K, G$, we have*

$$U_{2n} = \frac{1}{\sqrt{2}}\begin{bmatrix} X & Y \\ -YW & XW \end{bmatrix}\begin{bmatrix} U_n & \\ & U_n \end{bmatrix} Q_{n,2} \tag{2.15}$$

where $X = R_+, Y = R_-, W = J$ *for* $U = G$ *and* $X = \tilde{R}_+, Y = \tilde{R}_-, W = JP_{-1}$ *for* $U = K$.

The Ht radix-2 splitting property, expressed through the identities (2.14) and (2.15), was already known for $U = H, K^{\mathrm{T}}, G$ (see respectively [36, pp. 224–228], [18] and [24]). For $U = K$ it is a new result. In fact, in [27] are considered fast algorithms where U_n transforms of length n, $U \in \{K, K^{\mathrm{T}}, G\}$, are reduced to $U'_{n/2}$ transforms of length $n/2$ with $U' \neq U$. In Section 3 we prove decompositions of $U_{n_1 n_2}$, $U = H, K^{\mathrm{T}}, G, K$, where n_1, n_2 are arbitrary positive integers ($n_i \geqslant 2$). In this way fast algorithms computing $U_n\mathbf{z}$ can be derived for any highly composed integer n. For $U = K^{\mathrm{T}}, G, K$, the Ht radix-n_2 splitting property is a new result.

In the following the matrices I, J, P_1, P_{-1} of dimension $r \neq n$ are denoted by I_r, J_r, $(P_1)_r$, $(P_{-1})_r$, respectively.

3. Mixed-radix fast Hartley-type transforms

Let U_n be one of the $n \times n$ Ht matrices H_n, K_n, K_n^{T}, G_n of Section 2 and call the matrix–vector product $U_n\mathbf{z}$, $\mathbf{z} \in \mathbb{C}^n$, a discrete U transform (DUT) of length n. Then *a DUT of length $n_1 n_2$ can be computed by n_2 DUTs of length n_1*.

This result, which is the Ht analogous of the *DFT* radix-n_2 splitting property [36, p. 79], is an obvious consequence of the decomposition formula for $U_{n_1 n_2}$ obtained in Theorem 3.3. Clearly it leads to a fast algorithm for the computation of *DUT*s whose length n is a highly composed integer (see Corollary 3.4). If n is a big prime or the product of big primes, then the same tricks adopted in the *DFT* case may be applied to develop fast algorithms. This is surely true for $U = H$ [29].

Theorem 3.3 is proved in detail only for $U = G$. However, suggestions for proving the remaining cases are given.

The proof of Theorem 3.3 is based on two preliminary lemmas where the *DUT* of vectors obtained by *shifting* ($P_{\pm 1}^a\mathbf{z}$) or by *stretching* ($I^{n_2}\mathbf{z}$) the entries of a given vector $\mathbf{z}$ are written in terms of the *DUT* of $\mathbf{z}$ and of suitable “shift” or “stretch” cross matrices (see Lemmas 3.1 and 3.2).

Let $\mathbf{z} \in \mathbb{C}^n$, $\mathbf{z} = [z_0 z_1 \cdots z_{n-1}]^{\mathrm{T}}$, and let $\mathbf{z}$ also denote the (infinite) antisymmetric periodic extension of $\mathbf{z}$, i.e.

$$z_{j'} = \begin{cases} z_j, & j' = j + 2an, \\ -z_j, & j' = j + (2a+1)n, \end{cases} \qquad a \in \mathbb{Z}, \quad j = 0, \ldots, n-1.$$

Observe that $P_{-1}^a\mathbf{z} = [z_a z_{a+1} \cdots z_{a+n-1}]^{\mathrm{T}}$, $a \in \mathbb{Z}$. Set also $\chi_s = 2s + 1$.

Lemma 3.1 (Shift). *Let* $\mathbf{z} \in \mathbb{C}^n$ *and* $a \in \mathbb{Z}$.

(i) *For* $U = H, K$, *we have*

$$U_n P_1^a \mathbf{z} = X_a U_n \mathbf{z}, \quad X_a = \begin{cases} d(\mathbf{c}^a) - d(\mathbf{s}^a) J P_1, & U = H, \\ d(\mathbf{c}^a) + d(\mathbf{s}^a) J P_{-1}, & U = K, \end{cases}$$

with $\varphi_h^{(a)} = h2a\pi/n,\ 0 \leqslant h \leqslant n-1$.

(ii) *For* $U = K^{\mathrm{T}}, G$, *we have*

$$U_n P_{-1}^a \mathbf{z} = X_a U_n \mathbf{z}, \quad X_a = \begin{cases} d(\mathbf{c}^a) - d(\mathbf{s}^a) J, & U = K^{\mathrm{T}}, \\ d(\mathbf{c}^a) + d(\mathbf{s}^a) J, & U = G, \end{cases}$$

with $\varphi_h^{(a)} = (2h+1)a\pi/n,\ 0 \leqslant h \leqslant n-1$.

Proof. Since $[P_{-1}^a \mathbf{z}]_j = z_{a+j}$, we obtain

$$\begin{aligned} \sqrt{n}[G_n P_{-1}^a \mathbf{z}]_i &= \sum_{j'=a}^{n-1+a} z_{j'} \operatorname{cas}\left(\frac{\chi_i \chi_{j'} \pi}{2n} - \frac{2a\chi_i \pi}{2n}\right) \\ &= \cos\varphi_i^{(a)} \sum_{j'=a}^{n-1+a} z_{j'} \operatorname{cas} \frac{\chi_i \chi_{j'} \pi}{2n} \\ &\quad + \sin\varphi_i^{(a)} \sum_{j'=a}^{n-1+a} z_{j'} \operatorname{cas} \frac{\chi_{n-i-1} \chi_{j'} \pi}{2n}. \end{aligned}$$

The index change $j = j' - a$ yields the assertion:

$$\left[G_n P_{-1}^a \mathbf{z}\right]_i = \cos\varphi_i^{(a)} [G_n \mathbf{z}]_i + \sin\varphi_i^{(a)} [G_n \mathbf{z}]_{n-i-1}. \qquad \square$$

Notice that the proof of Lemma 3.1 for H and K differs from the proof for K^{T} and G by the sole fact that the vector $\mathbf{z} \in \mathbb{C}^n$ is extended symmetrically, i.e. $z_{j'} = z_j,\ j' = j + an, a \in \mathbb{Z}, 0 \leqslant j \leqslant n-1$. In this way, $P_1^a \mathbf{z} = [z_a z_{a+1} \cdots z_{a+n-1}]^{\mathrm{T}}$, $a \in \mathbb{Z}$.

Now let $\mathbf{z} \in \mathbb{C}^{n_1}$ and let I^{n_2} be the $n_1 n_2 \times n_1$ matrix

$$[I^{n_2}]_{ij} = \begin{cases} 1, & i = kn_2, j = k, k = 0, \ldots, n_1 - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $I^{n_2}\mathbf{z} \in \mathbb{C}^{n_1 n_2}$ and

$$[I^{n_2}\mathbf{z}]_{j'} = \begin{cases} z_j, & j' = n_2 j, j = 0, \ldots, n_1 - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.2 (Stretch). *Let* $\mathbf{z} \in \mathbb{C}^{n_1}$. *We have*

$$U_{n_1 n_2} I^{n_2} \mathbf{z} = \frac{1}{\sqrt{n_2}} M_{n_1 n_2} \begin{bmatrix} U_{n_1} \\ U_{n_1} \\ \vdots \\ U_{n_1} \end{bmatrix} \mathbf{z},$$

where

$$M_{n_1 n_2} = \begin{cases} I_{n_1 n_2}, & U = H, K^{\mathrm{T}}, \\ d(\mathbf{c}) + d(\mathbf{s}) J_{n_2} \otimes J_{n_1} (P_{-1})_{n_1}, & U = K, \\ d(\mathbf{c}) + d(\mathbf{s}) J_{n_1 n_2}, & U = G, \end{cases}$$

with

$$\varphi_{i,h} = \begin{cases} \dfrac{(h + i n_1)(n_2 - 1)\pi}{n_1 n_2} + i\pi, & U = K, \\ \dfrac{(2(h + i n_1) + 1)(n_2 - 1)\pi}{2 n_1 n_2} + i\pi, & U = G, \end{cases}$$
$$0 \leqslant h \leqslant n_1 - 1, \quad 0 \leqslant i \leqslant n_2 - 1, \tag{3.1}$$

taken in lexicographic order.

Proof. For $i' = h + i n_1$, $h = 0, \ldots, n_1 - 1$, $i = 0, \ldots, n_2 - 1$, we obtain

$$\begin{aligned} \sqrt{n_1 n_2}[G_{n_1 n_2} I^{n_2} \mathbf{z}]_{i'} &= \sum_{j=0}^{n_1 - 1} z_j \operatorname{cas} \frac{\chi_{h + i n_1} \chi_{n_2 j} \pi}{2 n_1 n_2} \\ &= \sum_{j=0}^{n_1 - 1} z_j \operatorname{cas} \left(\frac{\chi_h \chi_j \pi}{2 n_1} - \frac{\chi_{h + i n_1} (n_2 - 1)\pi}{2 n_1 n_2} - i\pi \right) \\ &= \cos \varphi_{i,h} \sum_{j=0}^{n_1 - 1} z_j \operatorname{cas} \frac{\chi_h \chi_j \pi}{2 n_1} \\ &\quad + \sin \varphi_{i,h} \sum_{j=0}^{n_1 - 1} z_j \operatorname{cas} \frac{\chi_{n_1 - h - 1} \chi_j \pi}{2 n_1} \\ &= \sqrt{n_1} \left(\cos \varphi_{i,h} [G_{n_1} \mathbf{z}]_h + \sin \varphi_{i,h} [G_{n_1} \mathbf{z}]_{n_1 - h - 1} \right). \qquad \square \end{aligned}$$

Notice that, by Lemma 3.2, for $U = H$ or $U = K^{\mathrm{T}}$ the vector $U_{n_1 n_2} I^{n_2} \mathbf{z}$, $\mathbf{z} \in \mathbb{C}^{n_1}$, is directly expressed in terms of $U_{n_1} \mathbf{z}$, i.e. the stretch cross matrix $M_{n_1 n_2}$ is the identity. This is the reason why the proof of Theorem 3.3 is much easier in the cases $U = H, K^{\mathrm{T}}$.

In Theorem 3.3 the discrete U transform $U_{n_1 n_2} \mathbf{z}$, $\mathbf{z} \in \mathbb{C}^{n_1 n_2}$, is reduced to n_2 *DUT*s of length n_1. The result is obtained as follows. First rewrite $\mathbf{z}$ as the sum of n_2 vectors, each obtained by shifting the entries of the n_2-stretched version of the vector $\mathbf{w}^j = [z_j z_{j+n_2} \cdots z_{j+(n_1-1)n_2}]^{\mathrm{T}}$. Then apply the shift/stretch Lemmas 3.1 and 3.2.

Theorem 3.3 (Ht radix-n_2 splitting property). *Let* $\mathbf{z} \in \mathbb{C}^{n_1 n_2}$ *and set, for* $0 \leqslant i, j \leqslant n_2 - 1$,

$$R_{i,j} = \begin{cases} d(\mathbf{c}) + d(\mathbf{s}) J_{n_1} (P_1)_{n_1}, & U = H, \\ d(\mathbf{c}) + d(\mathbf{s}) J_{n_1}, & U = K^{\mathrm{T}}, \\ d(\mathbf{c}) + d(\mathbf{s}) J_{n_1} (P_{-1})_{n_1}, & U = K, \\ d(\mathbf{c}) + d(\mathbf{s}) J_{n_1}, & U = G, \end{cases} \tag{3.2}$$

with the following corresponding values of φ_h*:*

$$\varphi_h = \varphi_{i,h}^{(j)} = \begin{cases} \dfrac{2j(in_1 + h)\pi}{n_1 n_2}, & U = H, \\ \dfrac{j(2(in_1 + h) + 1)\pi}{n_1 n_2}, & U = K^{\mathrm{T}}, \\ \dfrac{(h + in_1)((n_2 - 1) - 2j)\pi}{n_1 n_2} + i\pi, & U = K, \\ \dfrac{(2(h + in_1) + 1)((n_2 - 1) - 2j)\pi}{2n_1 n_2} + i\pi, & U = G, \end{cases} \quad 0 \leqslant h \leqslant n_1 - 1.$$

Then

$$U_{n_1 n_2} \mathbf{z} = \frac{1}{\sqrt{n_2}} \begin{bmatrix} R_{0,0} & R_{0,1} & \cdots & R_{0,n_2-1} \\ R_{1,0} & R_{1,1} & \cdots & R_{1,n_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n_2-1,0} & R_{n_2-1,1} & \cdots & R_{n_2-1,n_2-1} \end{bmatrix} \times \begin{bmatrix} U_{n_1} & & & \\ & U_{n_1} & & \\ & & \ddots & \\ & & & U_{n_1} \end{bmatrix} Q_{n_1,n_2} \mathbf{z},$$

where Q_{n_1,n_2} *is the* n_2*-stride* [33, pp. 43–53] *permutation matrix of order* $n_1 n_2$,

$$[Q_{n_1,n_2} \mathbf{z}]_{h+in_1} = z_{hn_2+i}. \tag{3.3}$$

Proof. Notice that $\mathbf{z} = \sum_{j=0}^{n_2-1} P_{-1}^{-j} I^{n_2} \mathbf{w}^j$. Therefore, by Lemmas 3.1 and 3.2 we have

$$G_{n_1 n_2} \mathbf{z} = \frac{1}{\sqrt{n_2}} \sum_{j=0}^{n_2-1} X_{-j} M_{n_1 n_2} \begin{bmatrix} G_{n_1} \\ G_{n_1} \\ \vdots \\ G_{n_1} \end{bmatrix} \mathbf{w}^j, \tag{3.4}$$

where

$$\begin{aligned} X_{-j} = {} & \mathrm{diag}\big(\mathrm{diag}(\cos\theta_{i,h}^{(j)}, h = 0, \ldots, n_1 - 1), i = 0, \ldots, n_2 - 1\big) \\ & - \mathrm{diag}\big(\mathrm{diag}(\sin\theta_{i,h}^{(j)}, h = 0, \ldots, n_1 - 1), i = 0, \ldots, n_2 - 1\big) J_{n_1 n_2} \end{aligned}$$

and

$$\begin{aligned} M_{n_1 n_2} = {} & \mathrm{diag}\big(\mathrm{diag}(\cos\varphi_{i,h}, h = 0, \ldots, n_1 - 1), i = 0, \ldots, n_2 - 1\big) \\ & + \mathrm{diag}\big(\mathrm{diag}(\sin\varphi_{i,h}, h = 0, \ldots, n_1 - 1), i = 0, \ldots, n_2 - 1\big) J_{n_1 n_2} \end{aligned}$$

with $\varphi_{i,h}$ as in (3.1) and $\theta_{i,h}^{(j)} = 2j\chi_{in_1+h}\pi/(2n_1n_2)$. Observe that $\varphi_{n_2-i-1,n_1-h-1} = -\varphi_{i,h} + 2(n_2-1)\pi$. Therefore

$$\cos\varphi_{n_2-i-1,n_1-h-1} = \cos\varphi_{i,h}, \quad \sin\varphi_{n_2-i-1,n_1-h-1} = -\sin\varphi_{i,h},$$

and, as a consequence, the following expression for $X_{-j}M_{n_1n_2}$ holds:

$$\begin{aligned} X_{-j}M_{n_1n_2} = {} & \operatorname{diag}\big(C_i^{(j)}, i = 0, \dots, n_2-1\big) \\ & + \operatorname{diag}\big(S_i^{(j)}, i = 0, \dots, n_2-1\big) J_{n_1n_2}, \end{aligned}$$

where

$$\begin{aligned} C_i^{(j)} &= \operatorname{diag}\big(\cos\big(\varphi_{i,h} - \theta_{i,h}^{(j)}\big), h = 0, \dots, n_1-1\big), \\ S_i^{(j)} &= \operatorname{diag}\big(\sin\big(\varphi_{i,h} - \theta_{i,h}^{(j)}\big), h = 0, \dots, n_1-1\big). \end{aligned}$$

Moreover, since $\varphi_{i,h} - \theta_{i,h}^{(j)} = \varphi_{i,h}^{(j)}$ we have that $C_i^{(j)} + S_i^{(j)}J_{n_1}$ is just the matrix $R_{i,j}$ defined in (3.2). Thus (3.4) becomes

$$G_{n_1n_2}\mathbf{z} = \frac{1}{\sqrt{n_2}} \sum_{j=0}^{n_2-1} \begin{bmatrix} R_{0,j}G_{n_1} \\ R_{1,j}G_{n_1} \\ \vdots \\ R_{n_2-1,j}G_{n_1} \end{bmatrix} \mathbf{w}^j$$

and the assertion is proved. □

A repeated application of Theorem 3.3 leads to a decomposition of U_n, $n = p_1p_2\cdots p_m$, which can be used to derive a fast algorithm computing the *DUT* for a generic highly composed integer n. This decomposition is stated in Corollary 3.4.

Corollary 3.4. *Let $n = p_1p_2\cdots p_m$. Set $n_{1,k} = p_{k+1}\cdots p_m$, $n_{2,k} = p_k$ and, for $0 \leqslant i, j \leqslant n_{2,k}-1$, let $R_{i,j}^{(k)}$ be obtained from $R_{i,j}$ in (3.2) by replacing n_1 and n_2 with $n_{1,k}$ and $n_{2,k}$. Then*

$$U_n = \frac{1}{\sqrt{n}} L_1L_2\cdots L_{m-1}S_mQ, \tag{3.5}$$

where L_k is a block diagonal matrix of order $p_1\cdots p_{k-1}$ whose diagonal blocks are all equal to

$$\hat{L}_k = \begin{bmatrix} R_{0,0}^{(k)} & R_{0,1}^{(k)} & \cdots & R_{0,n_{2,k}-1}^{(k)} \\ R_{1,0}^{(k)} & R_{1,1}^{(k)} & \cdots & R_{1,n_{2,k}-1}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n_{2,k}-1,0}^{(k)} & R_{n_{2,k}-1,1}^{(k)} & \cdots & R_{n_{2,k}-1,n_{2,k}-1}^{(k)} \end{bmatrix}$$

$(k = 1, \dots, m-1)$, S_m *is a block diagonal matrix of order* $p_1 p_2 \cdots p_{m-1}$ *whose diagonal blocks are all equal to* $\sqrt{p_m} U_{p_m}$, *and* Q *is the index-reversal* [36, p. 87] *permutation matrix*

$$Q = \prod_{k=m-1}^{1} \operatorname{diag}\left(Q_{p_{k+1}\cdots p_m, p_k}, j = 0, \dots, p_1 \cdots p_{k-1} - 1\right).$$

As a consequence of (3.5), if $\phi(\mathbf{x})$ denotes the number of complex multiplications required to compute the vector $\mathbf{x}$, then

$$\begin{aligned}\phi(\sqrt{n} U_n \mathbf{z}) &\leqslant 2n(p_1 + p_2 + \cdots + p_{m-1}) \\ &\quad + \phi(\sqrt{p_m} U_{p_m} \mathbf{y}) \prod_{k=1}^{m-1} p_k, \quad \mathbf{z} \in \mathbb{C}^n, \mathbf{y} \in \mathbb{C}^{p_m}.\end{aligned}$$

Notice, however, that this operation count does not take into account all possible symmetries of $R_{i,j}$. Moreover, since the matrices L_k defined in Corollary 3.4 are sparse orthogonal matrices, a good numerical stability of algorithms based on the decomposition (3.5) is assured. In some applications it is useful to have at disposal such algorithms, implementing directly the K, the K^{T} and the G transforms [21,25–27]. In particular, among the Ht displacement formulas for the inverse of a Toeplitz matrix T listed in [21], the one in terms of G and K, ignoring their relation to H, appears to be very efficient for the computation of $T^{-1}\mathbf{b}$.

In Section 4 we study the *matrix algebras* $\mathscr{H} = SDH$, $\mathscr{K} = SDK$, $\delta = SDK^{\mathrm{T}}$, $\gamma = SDG$ where *SDU* denotes the n-dimensional space

$$SDU = \left\{U d(\mathbf{z}) U^* : \mathbf{z} \in \mathbb{C}^n\right\}$$

of all matrices diagonalized by the $n \times n$ unitary matrix U. We also study other four Ht transforms/matrix algebras of type $UI_{\mathscr{L}}/\mathscr{L} = SD(UI_{\mathscr{L}})$, $U = H, K, K^{\mathrm{T}}, G$, where $I_{\mathscr{L}}$ are suitable orthogonal $n \times n$ matrices with entries in the set $\{0, 1, \pm 1/\sqrt{2}\}$. Pairs of columns of U in (2.1) are combined so to obtain a matrix $UI_{\mathscr{L}} = [U_1|U_2]$, where the entries of U_1 have the form c_{ij} (s_{ij}), whereas the entries of U_2 have the form s_{ij} (c_{ij}) with

$$c_{ij} = \sqrt{\frac{2}{n}} \cos \frac{f(i)g(j)\pi}{n} \quad \text{and} \quad s_{ij} = \sqrt{\frac{2}{n}} \sin \frac{f(i)g(j)\pi}{n}.$$

Notice that both Ht algebras and the more widely known ξ-circulant algebras belong to the same class of *SDU* spaces and, as a consequence, both share the properties of *SDU* spaces (see Section 4).

4. Hartley-type matrix algebras

One of the possible representations of the Ht algebras considered in this section utilizes the well known ξ-circulant matrices. A ξ-circulant matrix is a Toeplitz matrix of the form

$$\mathscr{C}_\xi(\mathbf{z}) = \sum_{k=0}^{n-1} z_k (P_\xi)^k. \tag{4.1}$$

For the class $\mathscr{C}_\xi$ of ξ-circulant matrices, we have

$$\begin{aligned} &\mathscr{C}_\xi = SDF_\xi = \left\{ F_\xi d(\mathbf{z}) F_\xi^* : \mathbf{z} \in \mathbb{C}^n \right\}, \\ &[F_\xi]_{kj} = \tfrac{1}{\sqrt{n}} (\mathrm{e}^{-\mathbf{i}(\varphi + 2j\pi)/n})^k, \quad \mathbf{i} = \sqrt{-1}, \end{aligned} \tag{4.2}$$

whenever $\xi = \mathrm{e}^{-\mathbf{i}\varphi}$, $\varphi \in [0, 2\pi)$ (to prove (4.2) simply note that $P_\xi F_\xi = F_\xi D_{P_\xi}$ with D_{P_ξ} diagonal). Thus, if $|\xi| = 1$, then $\mathscr{C}_\xi$ belongs to the class of *SDU* spaces. As a consequence, $\mathscr{C}_\xi$ is closed under sum, product, inversion and conjugate transposition; moreover matrices from $\mathscr{C}_\xi$ commute. For $\xi = 1$, a ξ-circulant matrix is called, more simply, circulant. In a circulant matrix $\mathscr{C}(\mathbf{z})$ we have $[\mathscr{C}(\mathbf{z})]_{ij} = z_{j-i \bmod n}$, that is each row is derived from the row above by shifting right cyclically [15,16].

Properties first obtained for circulants were then observed to hold for any *SDU* space $\mathscr{L}$. In particular, a result which is fundamental to justify the introduction of the minimization $\mathscr{L}$*QN* methods in [8,19–21], was first derived for $\mathscr{L} = \mathscr{C}$ [34] and then extended to any *SDU* space [35]. It states that the eigenvalues of a hermitian matrix A are related to the eigenvalues of its *best least squares fit* $\mathscr{L}_A$ [24,34] by the following inequalities

$$\begin{aligned} &\nu_1(A) + \cdots + \nu_j(A) \leqslant \nu_1(\mathscr{L}_A) + \cdots + \nu_j(\mathscr{L}_A), \\ &\nu_{n-j+1}(\mathscr{L}_A) + \cdots + \nu_n(\mathscr{L}_A) \leqslant \nu_{n-j+1}(A) + \cdots + \nu_n(A), \end{aligned} \tag{4.3}$$

where $\nu_j(X)$ denote the eigenvalues of X in nondecreasing order. In [24] the important case $j = 1$ has been extended to spaces $\mathscr{L}$ more general than *SDU*: as a consequence, matrices from noncommutative group algebras could be exploited, in principle, in preconditioning techniques and in $\mathscr{L}$*QN* methods. We guess that an analogous extension could be obtained for all j.

Now consider the space

$$\tau = SDS, \quad [S]_{kj} = \sqrt{\frac{2}{n+1}} \sin \frac{(k+1)(j+1)\pi}{n+1}, \quad 0 \leqslant k, j \leqslant n-1. \tag{4.4}$$

One easily shows that $XS = SD_X$ where $X = P_0 + P_0^{\mathrm{T}}$ and $D_X = 2d(\mathbf{c})$, with $c_j = \cos((j+1)\pi/(n+1))$, $0 \leqslant j \leqslant n-1$, and therefore—since X is nonderogatory—τ is the set of all matrices of form

$$\tau(\mathbf{z}) = \sum_{k=0}^{n-1} z_k X_k, \tag{4.5}$$

where $X_0 = I$, $X_1 = X$, $X_{j+1} = X_j X - X_{j-1}$, $1 \leqslant j \leqslant n-1$.

The space τ is the prototype of a set of *eight Jacobi* matrix algebras $\mathscr{L} = SDU$, which are all generated by a symmetric tridiagonal matrix. These algebras and the corresponding real Jacobi transforms U have been extensively studied. In particular, it is known that the product $U\mathbf{z}$ can be performed in $\mathrm{O}(n \log n)$ flops, as in the case $U = F_\xi$. It is mainly by this property that the Jacobi algebras turn out to be competitive with the algebra $\mathscr{C}_\xi$ in several applications, such as preconditioning techniques of positive definite systems and matrix displacement formulas. For example, in several cases matrices from Jacobi algebras outperform ξ-circulants as preconditioners of Toeplitz linear systems [2,6,14,24,28,30]. Moreover, whereas one of the most efficient Gohberg-Semencul type representations of the inverse of a Toeplitz matrix involves circulant and (-1)-circulant matrices (see for example [21,22]), displacement formulas based on Jacobi algebras are more suitable to represent $T + H$-like matrices [1,4,5,9,22,23].

The matrix algebra $\mathscr{H}$ diagonalized by the Hartley matrix H is defined in [3] by

$$\mathscr{H} = SDH, \quad [H]_{kj} = \frac{1}{\sqrt{n}} \operatorname{cas} \frac{2kj\pi}{n}, \quad 0 \leqslant k, j \leqslant n-1. \tag{4.6}$$

Whereas both $\mathscr{C}_\xi$ and τ are Hessenberg algebras [22] and then spanned by n polynomials in a single nonderogatory Hessenberg matrix, there is no Hessenberg matrix generating $\mathscr{H}$. In fact, tridiagonal matrices in $\mathscr{H}$ have necessarily some entry $(i, i+1)$ equal to zero, and therefore are derogatory. Notice that $\mathscr{H}$ includes the matrix $T^{1,1}_{0,0} = P_1 + P_1^{\mathrm{T}}$, but $T^{1,1}_{0,0}$ is not sufficient to define $\mathscr{H}$ since it is derogatory. In [18] it is shown that

$$\mathscr{H} = \left\{ A \in \mathbb{C}^{n\times n} : AT^{1,1}_{0,0} = T^{1,1}_{0,0}A \text{ and } A\mathscr{H}(\mathbf{e}_{n-1}) = \mathscr{H}(\mathbf{e}_{n-1})A \right\},$$

where $\mathscr{H}(\mathbf{e}_{n-1})$ is the matrix of $\mathscr{H}$ whose first row is $\mathbf{e}^{\mathrm{T}}_{n-1} = [0 \cdots 0\ 1]$. Similar characterizations hold for other four algebras named $\mathscr{K}$, η, μ [18] and γ [24] (for γ this is not true $\forall n$). Of course, any matrix $X = Hd(\mathbf{z})H$ with $z_i \neq z_j$, $i \neq j$, is such that $\mathscr{H} = \{A : AX = XA\}$, however it is not known a vector $\mathbf{z}$ for which X has (in some sense) a nice form.

In the following we define and study the set of *eight Hartley-type* (Ht) algebras $\mathscr{L} = SDU$, which includes the five algebras considered in [3,18,24]. We shall see that the matrices from Ht algebras are easily described in terms of circulant and (-1)-circulant matrices. Moreover, all the related Ht matrices U are defined in terms of the matrices H, K, K^{T}, G and thus (see Section 2) the matrix-vector products $U\mathbf{z}$ are all *fast* real transforms computable in $\mathrm{O}(n \log n)$ flops. As for the Fourier matrix F_ξ, these are really fast when n is just a power of 2. On the contrary, for U in the set of the Jacobi transforms the optimal value of n depends upon U [37].

One expects that this set of new algebras can have applications analogous to the Jacobi set. Some applications in Toeplitz systems preconditioning and in displacement or Bezoutian representations of Toeplitz-plus-Hankel-like matrices have been already studied in [1,3,18,21,24–26]. Moreover, these algebras $\mathscr{L}$ immediately lead to eight different new $\mathscr{L}QN$ minimization algorithms of complexity $\mathrm{O}(n \log n)$ per step [8,19–21]. Finally, some of the new transforms could be visually attractive, as required in image processing [32].

The Ht algebras can be defined in terms of symmetric and skew-symmetric (± 1)-circulant matrices

$$\mathscr{C}_{\pm 1}^{S} = \left\{ A \in \mathscr{C}_{\pm 1} : A = A^{\mathrm{T}} \right\}, \quad \mathscr{C}_{\pm 1}^{SK} = \left\{ A \in \mathscr{C}_{\pm 1} : A = -A^{\mathrm{T}} \right\}. \tag{4.7}$$

For $n = 3, 4$, the generic elements of $\mathscr{C}_{\pm 1}^{S}$ and $\mathscr{C}_{\pm 1}^{SK}$ are matrices of form:

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_1 \\ a_1 & a_0 & a_1 & a_2 \\ a_2 & a_1 & a_0 & a_1 \\ a_1 & a_2 & a_1 & a_0 \end{bmatrix} \in \mathscr{C}_1^S, \quad \begin{bmatrix} 0 & b_1 & 0 & -b_1 \\ -b_1 & 0 & b_1 & 0 \\ 0 & -b_1 & 0 & b_1 \\ b_1 & 0 & -b_1 & 0 \end{bmatrix} \in \mathscr{C}_1^{SK},$$

$$\begin{bmatrix} a_0 & a_1 & 0 & -a_1 \\ a_1 & a_0 & a_1 & 0 \\ 0 & a_1 & a_0 & a_1 \\ -a_1 & 0 & a_1 & a_0 \end{bmatrix} \in \mathscr{C}_{-1}^S, \quad \begin{bmatrix} 0 & b_1 & b_2 & b_1 \\ -b_1 & 0 & b_1 & b_2 \\ -b_2 & -b_1 & 0 & b_1 \\ -b_1 & -b_2 & -b_1 & 0 \end{bmatrix} \in \mathscr{C}_{-1}^{SK},$$

$$\begin{bmatrix} a_0 & a_1 & \pm a_1 \\ a_1 & a_0 & a_1 \\ \pm a_1 & a_1 & a_0 \end{bmatrix} \in \mathscr{C}_{\pm 1}^S, \quad \begin{bmatrix} 0 & b_1 & \mp b_1 \\ -b_1 & 0 & b_1 \\ \pm b_1 & -b_1 & 0 \end{bmatrix} \in \mathscr{C}_{\pm 1}^{SK}.$$

From the definition (4.7) one immediately realizes that $\mathscr{C}_{\pm 1}^{S}$ is an algebra, i.e. $A_1 A_2 \in \mathscr{C}_{\pm 1}^{S}$ whenever $A_1, A_2 \in \mathscr{C}_{\pm 1}^{S}$. On the contrary, $\mathscr{C}_{\pm 1}^{SK}$ is only a linear subspace of $\mathscr{C}_{\pm 1}$: for example

$$b_1 \neq 0 \;\Rightarrow\; \begin{bmatrix} 0 & b_1 & \mp b_1 \\ -b_1 & 0 & b_1 \\ \pm b_1 & -b_1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -2b_1^2 & \pm b_1^2 & b_1^2 \\ \pm b_1^2 & -2b_1^2 & \pm b_1^2 \\ b_1^2 & \pm b_1^2 & -2b_1^2 \end{bmatrix} \notin \mathscr{C}_{\pm 1}^{SK}.$$

Obvious basis for $\mathscr{C}_{\pm 1}^{S}$ and $\mathscr{C}_{\pm 1}^{SK}$ are, respectively, the sets $\{P_{\pm 1}^{j} + (P_{\pm 1}^{j})^{\mathrm{T}}\}$ and $\{P_{\pm 1}^{j} - (P_{\pm 1}^{j})^{\mathrm{T}}\}$. Since $\pm P_{\pm 1}^{n-1} = P_{\pm 1}^{\mathrm{T}} = P_{\pm 1}^{-1}$, this remark yields the following:

Proposition 4.1. *We have*

$$\begin{aligned} &\dim \mathscr{C}_1^S = \left\lceil \tfrac{n+1}{2} \right\rceil, \quad \dim \mathscr{C}_1^{SK} = \left\lfloor \tfrac{n-1}{2} \right\rfloor, \\ &\dim \mathscr{C}_{-1}^S = \left\lfloor \tfrac{n+1}{2} \right\rfloor, \quad \dim \mathscr{C}_{-1}^{SK} = \left\lceil \tfrac{n-1}{2} \right\rceil. \end{aligned} \tag{4.8}$$

Moreover, the set

$$
\begin{aligned}
&I, \quad P_{\pm1}^{j} \pm P_{\pm1}^{n-j} = P_{\pm1}^{j} + P_{\pm1}^{-j}\\
&\qquad = \begin{bmatrix}
 & & & 1 & & \pm1 & & \\
 & & & & \ddots & & \ddots & \\
 & & & & & & & \pm1\\
1 & & & & & & \ddots & \\
 & \ddots & & & & & & 1\\
\pm1 & & & & & & & \\
 & \ddots & & \ddots & & & & \\
 & & \pm1 & & 1 & & &
\end{bmatrix}, \quad j \geqslant 1,
\end{aligned}
$$

is a basis for $\mathscr{C}_{\pm1}^{S}$, *whereas the set*

$$
\begin{aligned}
&P_{\pm1}^{j} \mp P_{\pm1}^{n-j} = P_{\pm1}^{j} - P_{\pm1}^{-j}\\
&\qquad = \begin{bmatrix}
 & & & 1 & & \mp1 & & \\
 & & & & \ddots & & \ddots & \\
 & & & & & & & \mp1\\
-1 & & & & & & \ddots & \\
 & \ddots & & & & & & 1\\
\pm1 & & & & & & & \\
 & \ddots & & \ddots & & & & \\
 & & \pm1 & & -1 & & &
\end{bmatrix}, \quad j \geqslant 1,
\end{aligned}
$$

is a basis for $\mathscr{C}_{\pm1}^{SK}$. *Finally* $\mathscr{C}_{\pm1}^{S}$ *is the set of all polynomials in*

$$
T_{0,0}^{\pm1,\pm1} = P_{\pm1} + P_{\pm1}^{\mathrm{T}} = \begin{bmatrix}
0 & 1 & & & \pm1\\
1 & 0 & 1 & & \\
 & 1 & \ddots & \ddots & \\
 & & \ddots & 0 & 1\\
\pm1 & & & 1 & 0
\end{bmatrix}. \tag{4.9}
$$

Proof. Only the last assertion needs a proof. Let $\mathscr{L}$ be the space of all polynomials in $T_{0,0}^{\pm1,\pm1}$. Clearly $\mathscr{L} \subset \mathscr{C}_{\pm1}^{S}$. Moreover, the dimension of $\mathscr{L}$ is the number of the distinct eigenvalues of $T_{0,0}^{\pm1,\pm1}$. Since the spectra of $T_{0,0}^{1,1}$ and $T_{0,0}^{-1,-1}$ are, respectively, $2\cos(2j\pi/n)$, $j = 0, \ldots, n-1$, and $2\cos((2j+1)\pi/n)$, $j = 0, \ldots, n-1$, we have $\dim\mathscr{L} = \dim\mathscr{C}_{\pm1}^{S}$. Therefore $\mathscr{L} = \mathscr{C}_{\pm1}^{S}$. $\square$

Now set

$$\mathscr{H} = \mathscr{C}_1^S + JP_1\mathscr{C}_1^{SK}, \quad \mathscr{K} = \mathscr{C}_{-1}^S + JP_{-1}\mathscr{C}_{-1}^{SK},$$
$$\eta = \mathscr{C}_1^S + J\mathscr{C}_1^S, \quad \mu = \mathscr{C}_{-1}^S + J\mathscr{C}_{-1}^S.$$

The algebras $\mathscr{H}$ and $\mathscr{K}$ are diagonalized, respectively, by the Hartley and by the skew-Hartley transforms, i.e.

$$\mathscr{H} = SDH, \quad \mathscr{K} = SDK.$$

The algebra $\mathscr{H}$ is introduced and shown to coincide with SDH in [3]. The algebras $\mathscr{K}$, η and μ are introduced in [1,18]: $\mathscr{K}$ as the (-1)-circulant version of $\mathscr{H}$; η and μ as new examples of matrix algebras—made up with matrices simultaneously symmetric and persymmetric—which can be involved in efficient displacement formulas. $\mathscr{H}$, $\mathscr{K}$, η, μ are particular examples in a class, characterized in [18], of symmetric (non Hessenberg) algebras including $T_{0,0}^{\pm 1,\pm 1}$.

For the algebras η and μ, we have

$$\eta = SDU_\eta, \quad \mu = SDU_\mu,$$

where

$$[U_\eta]_{kj} = \frac{1}{\sqrt{n}} \begin{cases} 1, & j = 0, \\ \operatorname{cas}\left(\frac{(2k+1)j\pi}{n} + \frac{\pi}{4}\right), & 1 \leqslant j \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor, \\ (-1)^k, & j = \frac{n}{2} \ (n \text{ even}), \\ \operatorname{cas}\left(\frac{(2k+1)j\pi}{n} - \frac{\pi}{4}\right), & \left\lceil \frac{n+1}{2} \right\rceil \leqslant j \leqslant n-1, \end{cases} \tag{4.10}$$

$$[U_\mu]_{kj} = \frac{1}{\sqrt{n}} \begin{cases} \operatorname{cas}\left(\frac{(2k+1)(2j+1)\pi}{2n} - \frac{\pi}{4}\right), & 0 \leqslant j \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor, \\ (-1)^k, & j = \frac{n-1}{2} \ (n \text{ odd}), \\ \operatorname{cas}\left(\frac{(2k+1)(2j+1)\pi}{2n} + \frac{\pi}{4}\right), & \left\lceil \frac{n}{2} \right\rceil \leqslant j \leqslant n-1, \end{cases} \tag{4.11}$$

$0 \leqslant k \leqslant n-1$ (see [24]). Notice that these are exactly the matrices obtained in [24] since $\operatorname{cas}(x + \frac{\pi}{4}) = \sqrt{2}\cos x$ and $\operatorname{cas}(x - \frac{\pi}{4}) = \sqrt{2}\sin x$. The use of the function cas and the equality $\operatorname{cas}(x + \frac{\pi}{4}) + \operatorname{cas}(x - \frac{\pi}{4}) = \sqrt{2}\operatorname{cas} x$ makes clearer the following relations among the transforms U_η, K, U_μ and G,

$$U_\eta = K^{\mathrm{T}} I_\eta, \tag{4.12}$$

$$U_\mu = G I_\mu, \tag{4.13}$$

where

$$I_\eta = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & & & & \\ & I_{\lfloor \frac{n-1}{2} \rfloor} & & J_{\lfloor \frac{n-1}{2} \rfloor} \\ & & \sqrt{2} & \\ & -J_{\lfloor \frac{n-1}{2} \rfloor} & & I_{\lfloor \frac{n-1}{2} \rfloor} \end{bmatrix},$$

$$I_\mu = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{\lfloor \frac{n}{2} \rfloor} & & -J_{\lfloor \frac{n}{2} \rfloor} \\ & \sqrt{2} & \\ J_{\lfloor \frac{n}{2} \rfloor} & & I_{\lfloor \frac{n}{2} \rfloor} \end{bmatrix} \tag{4.14}$$

(the presence of the central row and column including $\sqrt{2}$ depends on the oddness of n).

In [24] the algebra of all matrices diagonalized by G is called γ. More precisely,

$$\gamma = \mathscr{C}_{-1}^{S} + J\mathscr{C}_{-1}^{SK} = SDG.$$

Now the space

$$\delta = \mathscr{C}_{1}^{S} + J\mathscr{C}_{1}^{SK}$$

is naturally introduced as the circulant version of γ. The following result is obtained by using the shift Lemma for K (see Lemma 3.1).

Theorem 4.2. *The space $\delta = \mathscr{C}_{1}^{S} + J\mathscr{C}_{1}^{SK}$ is the algebra of all matrices diagonalized by K^{T}, that is $\delta = SDK^{\mathrm{T}}$.*

Proof. Since the dimension of δ is n,

$$\begin{aligned} \dim\big(\mathscr{C}_{1}^{S} + J\mathscr{C}_{1}^{SK}\big) &= \dim\big(\mathscr{C}_{1}^{S}\big) + \dim\big(J\mathscr{C}_{1}^{SK}\big) - \dim\big(\mathscr{C}_{1}^{S} \cap J\mathscr{C}_{1}^{SK}\big) \\ &= \left\lceil \frac{n+1}{2} \right\rceil + \left\lfloor \frac{n-1}{2} \right\rfloor - 0 = n, \end{aligned}$$

it is sufficient to prove that KAK^{T} is diagonal if $A \in \mathscr{C}_{1}^{S} \cup J\mathscr{C}_{1}^{SK}$. Let $A \in \mathscr{C}_{1}^{S}$. Then $A = p(T_{0,0}^{1,1})$ for some polynomial $p(x)$, and $KAK^{\mathrm{T}} = Kp(T_{0,0}^{1,1})K^{\mathrm{T}} = p(KT_{0,0}^{1,1}K^{\mathrm{T}})$. We claim that the matrix $KT_{0,0}^{1,1}K^{\mathrm{T}}$ is diagonal. To prove this claim it is sufficient to observe that

$$T_{0,0}^{1,1}K^{\mathrm{T}} = K^{\mathrm{T}}\mathrm{diag}\left(2\cos\frac{2k\pi}{n}, k = 0, \ldots, n-1\right).$$

In fact, one easily calculates

$$\begin{aligned} \sqrt{n}\big[T_{0,0}^{1,1}K^{\mathrm{T}}\big]_{ij} &= \operatorname{cas}\left(\frac{(2i+1)j\pi}{n} + \frac{2j\pi}{n}\right) + \operatorname{cas}\left(\frac{(2i+1)j\pi}{n} - \frac{2j\pi}{n}\right) \\ &= 2\cos\frac{2j\pi}{n}\operatorname{cas}\frac{(2i+1)j\pi}{n}, \quad i = 1, \ldots, n-2 \end{aligned}$$

(we left to the reader the cases $i=0$ and $i=n-1$). Now let $A \in J\mathscr{C}_1^{SK}$. As the matrices $E_j = P_1^j - P_1^{-j}$, $j=1,\ldots,\lfloor\frac{n-1}{2}\rfloor$, form a basis of the space $\mathscr{C}_1^{SK}$, let us show that KJE_jK^{T} is diagonal. We have

$$KE_jK^{\mathrm{T}} = KP_1^jK^{\mathrm{T}} - KP_1^{-j}K^{\mathrm{T}} = X_j - X_{-j} = 2d(\mathbf{s}^j)JP_{-1},$$

where X_j, X_{-j} are shift cross matrices and $s_k^j = \sin(2kj\pi/n)$, $k=0,\ldots,n-1$. Finally, the identity $KJ = -JP_{-1}K$ yields the assertion:

$$KJE_jK^{\mathrm{T}} = -JP_{-1}KE_jK^{\mathrm{T}} = -JP_{-1}2d(\mathbf{s}^j)JP_{-1} = 2d(\mathbf{s}^j). \qquad \square$$

Notice that K^{T} diagonalizes $\mathscr{C}_1^S$ whereas, as is shown in [18], K diagonalizes $\mathscr{C}_{-1}^S$.

Now the next step consists in defining a fast transform U_α, related to an algebra $\alpha = SDU_\alpha$, that has, *mutatis mutandis*, the same relation with H as U_η and U_μ have with K^{T} and G, respectively (see (4.12) and (4.13)):

$$[U_\alpha]_{kj} = \frac{1}{\sqrt{n}}\begin{cases} 1, & j=0,\\ \operatorname{cas}\left(\frac{2kj\pi}{n}+\frac{\pi}{4}\right), & 1 \leqslant j \leqslant \left\lfloor\frac{n-1}{2}\right\rfloor,\\ (-1)^k, & j=\frac{n}{2}\ (n \text{ even}),\\ \operatorname{cas}\left(\frac{2kj\pi}{n}-\frac{\pi}{4}\right), & \left\lceil\frac{n+1}{2}\right\rceil \leqslant j \leqslant n-1. \end{cases} \tag{4.15}$$

We have precisely

$$U_\alpha = HI_\alpha \tag{4.16}$$

where $I_\alpha = I_\eta^{\mathrm{T}}$. The relation between U_α and the space

$$\alpha = \mathscr{C}_1^S + JP_1\mathscr{C}_1^S$$

is defined by the following:

Theorem 4.3. *The space $\alpha = \mathscr{C}_1^S + JP_1\mathscr{C}_1^S$ is the algebra of all matrices diagonalized by U_α, that is $\alpha = SDU_\alpha$.*

Proof. Since the dimension of α is n,

$$\begin{aligned}\dim\left(\mathscr{C}_1^S + JP_1\mathscr{C}_1^S\right) &= \dim\left(\mathscr{C}_1^S\right) + \dim\left(JP_1\mathscr{C}_1^S\right) - \dim\left(\mathscr{C}_1^S \cap JP_1\mathscr{C}_1^S\right)\\ &= \begin{cases} 2\left(\frac{n}{2}+1\right)-2, & n \text{ even},\\ 2(\frac{n+1}{2})-1, & n \text{ odd},\end{cases}\end{aligned}$$

it is sufficient to prove that $U_\alpha^T AU_\alpha$ is diagonal if $A \in \mathscr{C}_1^S \cup JP_1\mathscr{C}_1^S$. If $A \in \mathscr{C}_1^S$, by exploiting the equalities $HT_{0,0}^{1,1}H = 2d(\mathbf{c})$, $c_h = \cos(2h\pi/n)$, $0 \leqslant h \leqslant n-1$ [3] and $d(\mathbf{c})I_\alpha = I_\alpha d(\mathbf{c})$, we obtain

$$U_\alpha^T A U_\alpha = I_\alpha^{\mathrm{T}} H p(T_{0,0}^{1,1}) H I_\alpha = I_\alpha^T p(2d(\mathbf{c})) I_\alpha = p(2d(\mathbf{c})).$$

If $A \in JP_1\mathscr{C}_1^S$, the identity $HJP_1 = JP_1H$ yields

$$\begin{aligned} U_\alpha^{\mathrm{T}} A U_\alpha &= I_\alpha^{\mathrm{T}} HJP_1 p(T_{0,0}^{1,1}) H I_\alpha = I_\alpha^{\mathrm{T}} JP_1 p(HT_{0,0}^{1,1}H) I_\alpha \\ &= I_\alpha^{\mathrm{T}} JP_1 I_\alpha p(2d(\mathbf{c})) = \begin{bmatrix} I_{\lceil \frac{n+1}{2} \rceil} & 0 \\ 0 & -I_{\lfloor \frac{n-1}{2} \rfloor} \end{bmatrix} p(2d(\mathbf{c})). \qquad \square \end{aligned}$$

Now the space

$$\beta = \mathscr{C}_{-1}^S + JP_{-1}\mathscr{C}_{-1}^S$$

is naturally introduced as the (-1)-circulant version of α. In the following theorem it is proved that the matrix U_β defined by

$$U_\beta = KI_\beta, \tag{4.17}$$

where $I_\beta = I_\mu^{\mathrm{T}}$, i.e.

$$[U_\beta]_{kj} = \frac{1}{\sqrt{n}} \begin{cases} \operatorname{cas}\left(\frac{k(2j+1)\pi}{n} - \frac{\pi}{4}\right), & 0 \leqslant j \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor, \\ (-1)^k, & j = \frac{n-1}{2}\ (n \text{ odd}), \\ \operatorname{cas}\left(\frac{k(2j+1)\pi}{n} + \frac{\pi}{4}\right), & \lceil \frac{n}{2} \rceil \leqslant j \leqslant n-1, \end{cases} \tag{4.18}$$

is just the matrix diagonalizing all matrices of β. Notice that the equality (4.17), involving the K transform, completes the list of equalities (4.12), (4.13) and (4.16) stating reciprocal links among the previous 8 Ht algebras/transforms.

Theorem 4.4. *The space $\beta = \mathscr{C}_{-1}^S + JP_{-1}\mathscr{C}_{-1}^S$ is the algebra of all matrices diagonalized by U_β, that is $\beta = SDU_\beta$.*

Proof. Proceed as in Theorem 4.3 by exploiting the equalities $K^{\mathrm{T}} T_{0,0}^{-1,-1} K = 2d(\mathbf{c})$, $c_h = \cos((2h+1)\pi/n)$, $h = 0, \ldots, n-1$ [18], $d(\mathbf{c})I_\beta = I_\beta d(\mathbf{c})$ and the fact that if $A \in \mathscr{C}_{-1}^S$ then A is a polynomial in $T_{0,0}^{-1,-1}$. $\square$

The Table 1, that is the analogous of the eight Jacobi algebras in [28], resumes the eight Hartley algebras $\mathscr{H}$, $\mathscr{K}$, μ, η, γ, δ, α, β. Here the Hartley algebra $\mathscr{H}$ is the prototype, as all transforms K, U_μ, U_η, G, K^{T}, U_α, U_β can be reduced to the only transform H. However, suitable procedures for each transform may be preferable to a systematic reduction to the Hartley transform. A good evidence of this fact is given by the possible applications of the *Ht* fast transforms in the theory of displacement formulas [21] and of Bezoutian representations [25,26].

Table 1

The eight Hartley-type algebras: $\mathscr{H}$, $\mathscr{K}$, α, β, δ, γ, η, μ	
(+1)-circulant type	(−1)-circulant type
$\mathscr{H} = SDH = \mathscr{C}_1^S + JP_1\mathscr{C}_1^{SK}$	$\mathscr{K} = SDK = \mathscr{C}_{-1}^S + JP_{-1}\mathscr{C}_{-1}^{SK}$
$\alpha = SD(HI_\alpha) = \mathscr{C}_1^S + JP_1\mathscr{C}_1^S$	$\beta = SD(KI_\beta) = \mathscr{C}_{-1}^S + JP_{-1}\mathscr{C}_{-1}^S$
$\delta = SDK^{\mathrm{T}} = \mathscr{C}_1^S + J\mathscr{C}_1^{SK}$	$\gamma = SDG = \mathscr{C}_{-1}^S + J\mathscr{C}_{-1}^{SK}$
$\eta = SD(K^{\mathrm{T}}I_\eta) = \mathscr{C}_1^S + J\mathscr{C}_1^S$	$\mu = SD(GI_\mu) = \mathscr{C}_{-1}^S + J\mathscr{C}_{-1}^S$
The basic Hartley-type transforms: H, K^{T}, K, G	
$[H]_{ij} = \frac{1}{\sqrt{n}}\operatorname{cas}\frac{2ij\pi}{n}$, $[K]_{ij} = \frac{1}{\sqrt{n}}\operatorname{cas}\frac{i(2j+1)\pi}{n}$, $[G]_{ij} = \frac{1}{\sqrt{n}}\operatorname{cas}\frac{(2i+1)(2j+1)\pi}{2n}$	

$K = R_{\mathscr{K}}H,\ G = R_\gamma K^{\mathrm{T}},\ I_\alpha = I_\eta^{\mathrm{T}},\ I_\beta = I_\mu^{\mathrm{T}};$
$R_{\mathscr{K}}$ and R_γ are defined in Section 2. I_η and I_μ are defined in (4.14).

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