Incomplete factorization preconditioners and their updates with applications (Exercises)

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Exercise 1

With reference to the first lesson prove that the following implication for the definition of M-matrices holds,

1. Definition 1 and Definition 2 are equivalent [**hint**: Perron-Frobenius Theorem can be useful]
2. Definition 2 implies Definition 5,
3. Definition 5 implies Definition 2,
4. Definition 2 implies Definition 3.

Exercise 2

Given a non–singular matrix $A \in \mathbb{R}^{n\times n}$ and non zero vector $\mathbf{v} \in \mathbb{R}^n$ consider the Krylov subspace

$$\mathcal{K}_m(A,\mathbf{v}) \triangleq \operatorname{span}\left\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \ldots, A^{m-1}\mathbf{v}\right\},$$

and prove that:

1. $\mathcal{K}_m(A,\mathbf{v})$ is the subspace of all vectors of $\mathbb{R}^n$ that can be written as $\mathbf{x} = p(A)\mathbf{v}$ where p is a polynomial of degree $\delta(p) \leq m-1$,
2. If μ is the degree of the minimal polynomial of $\mathbf{v}$ with respect to A, i.e., the non–zero polynomial p of lowest degree such that $p(A)\mathbf{v} = 0$. Then $\mu \leq n$ and $\mathcal{K}_\mu(A,\mathbf{v})$ is invariant under A and $\mathcal{K}_m(A,\mathbf{v}) = \mathcal{K}_\mu(A,\mathbf{v})\ \forall\, m \geq \mu$.
3. the dimension of $\mathcal{K}_m(A,\mathbf{v})$ is m if and only if the degree μ of the minimal polynomial of $\mathbf{v}$ with respect to A is greater or equal than m, therefore,

$$\dim \mathcal{K}_m(A,\mathbf{v}) = \min\{m, \mu\}$$

Exercise 3

Consider the matrix

$$A = \begin{pmatrix} 5 & -1 & -1 & \dots & -1 \\ 2 & 5 & 0 & \dots & 0 \\ 2 & 0 & 5 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 2 & 0 & \dots & 0 & 5 \end{pmatrix}_{n \times n}$$

1. How many entries are non–zero in the (*complete*) LU factorization of A?
2. Find a permutation matrix P such that $\mathrm{nnz}(A) = \mathrm{nnz}(LU)$, where L and U comes from the LU factorization of PAP^T.

Exercise 4

To show that small or clustered eigenvalues do not always give a good indication about the conditioning number of a matrix provide an example of a matrix A such that the eigenvalues of A are all equal to 1 and for which $k_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$ is arbitrarily large.

Exercise 5

Given a non–singular matrix $A \in \mathbb{R}^{n\times n}$ and an arbitrarily chosen matrix $B \in \mathbb{R}^{n\times n}$ show that,

$$\exists\, \alpha \,:\, \det(A_\varepsilon) \neq 0, \text{ for } A_\varepsilon = A + \varepsilon B,\, \forall \varepsilon < \alpha$$

Exercise 6

Consider the $n \times n$ tridiagonal matrix of the form:

$$T(\alpha) = \begin{bmatrix} \alpha & -1 & & \\ -1 & \alpha & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & \alpha \end{bmatrix}_{n\times n}, \; \alpha \in \mathbb{R},$$

1. Compute the eigenvalues of $T(\alpha)$ [**hint**: write $T(\alpha) = \alpha I + B \dots$]
2. For what values of α does $T(\alpha)$ become positive definite?
3. Use Theorem 3 of Lesson 2 to devise a bound for the element of $T(\alpha)^{-1}$.

Exercise 7

Given the sparse approximate LU factorization of a matrix $A \in \mathbb{R}^{n\times n}$ write down the factor L^{-1} of the Van Duin preconditioner (see Lesson 2).

Exercise 8

Prove that if A, E are two arbitrary matrix in $\mathbb{R}^{n\times n}$, and λ is an eigenvalue of $A+E$. Then either λ is an eigenvalue of A or

$$1 \leq \|(\lambda I - A)^{-1}E\| \leq \|(\lambda I - A)^{-1}\|\|E\|.$$