Norms and inequalities

1. The cone generated by vectors $a_1, a_2, \ldots, a_n$ is a set:

$$\operatorname{con}(a_1, a_2, \ldots, a_n) := \left\{ \sum_{i=1}^{n} \alpha_i a_i \colon \alpha_i \geqslant 0,\ i = 1, 2, \ldots, n \right\}.$$

Prove that a finitely generated cone is closed. Try to find a more simple proof then the one proposed on the seminar.

2. Prove that the set of all interior points and the closure of an arbitrary convex set are convex as well.

3. The affine hull of a set M is the intersection of all planes containing M (i. e. the affine hull of M is the minimal plane that contains M).

Prove that the affine hull of a set M coincides with the set of all affine combinations (i. e. linear combinations with real coefficients, for which the sum of the coefficients equals to 1) of finite systems of vectors from M.

4. Let V be a normed vector space and $B \subseteq V$ be a closed bounded convex set. Let also the origin be an interior point of B and if $x \in B$ then $\alpha x \in B$ for every α such that $|\alpha| \leqslant 1$. Prove, that the Minkowski functional

$$f(x) := \inf \left\{ \alpha > 0 \colon \frac{1}{\alpha} x \in B \right\}$$

defines a norm on V and B is the unity ball for f.

5. Let $a_1, \ldots, a_m \in \mathbb{R}^n$, $b_1, \ldots, b_m \in \mathbb{R}$. Prove, that a system of linear inequalities

$$a_1^{\mathrm{T}} x \leqslant b_1,\ \ldots,\ a_m^{\mathrm{T}} x \leqslant b_m$$

is inconsistent if and only if there exist coefficients $\alpha_1, \ldots, \alpha_m \geqslant 0$ such that the following equalities hold:

$$\alpha_1 a_1 + \ldots + \alpha_m a_m = 0, \quad \alpha_1 b_1 + \ldots + \alpha_m b_m = -1.$$

6. A matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ for which $a_{ij} \geqslant 0$ and $\sum\limits_{i=1}^{n} a_{ij} = \sum\limits_{j=1}^{n} a_{ij} = 1$ is called doubly stochastic.

Prove that any doubly stochastic matrix is a convex combination of some finite system of permutation matrices.

7. A point of a set M which is not an interior point for any line segment entirely contained in M is called a corner point of that set.

Find all corner points of the unity balls

$$B_p := \left\{ x \in \mathbb{R}^n \colon\ \|x\|_p \leqslant 1 \right\}, \quad 1 \leqslant p \leqslant \infty.$$

8. Let A and B be Hermitian matrices of order n and at least one of them be a positive definite one. Prove that all eigenvalues of matrix AB are real.

9. Prove that the Frobenius norm $\|\cdot\|_F$ and the spectral norm $\|\cdot\|_2$ are unitarily invariant.

10. Let $\sigma_1 \geqslant \ldots \geqslant \sigma_n$ be the singular values and $A = \sum\limits_{i=1}^{n} \sigma_i v_i u_i^*$ be a singular decomposition of matrix A. Prove the following equality for any unitarily invariant norm:

$$\min_{\operatorname{rank} B \leqslant k} \|A - B\| = \|A - A_k\|, \quad A_k = \sum_{i=1}^{k} \sigma_i v_i u_i^*.$$

11. Let $A, B \in \mathbb{C}^{n \times n}$. Prove that if at least one of the given matrices is normal then $\|AB - BA\|_F \leqslant \sqrt{2}\, \|A\|_F \|B\|_F$.

12. Let A be a Hermitian matrix of order n. Prove that for any subspace $V \subseteq \mathbb{C}^n$ the Rayleigh quotient $\dfrac{(Ax, x)}{(x, x)}$ attains its minimal and maximal values on V.

13. Let a matrix $A \in \mathbb{C}^{m \times n}$ be obtained from $B \in \mathbb{C}^{m \times (n-1)}$ by adding a column. Prove, that the singular values of A and B satisfy interlacing properties $\sigma_k(A) \geqslant \sigma_k(B) \geqslant \sigma_{k+1}(A)$ for $1 \leqslant k \leqslant n-1$.

14. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix, with the eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$. Let also $B \in \mathbb{C}^{n \times n}$ be a rank-1 Hermitian positive semi-definite matrix and the eigenvalues of matrix $A + B$ are $\mu_1 \geqslant \ldots \geqslant \mu_n$. Prove that $\lambda_1 + \|B\|_2 \geqslant \mu_1$ and $\mu_{k-1} \geqslant \lambda_{k-1} \geqslant \mu_k \geqslant \lambda_k$ for $2 \leqslant k \leqslant n$.

15. Let $A, B \in \mathbb{C}^{n \times n}$ be Hermitian matrices with eigenvalues $\lambda_1(A) \geqslant \ldots \geqslant \lambda_n(A)$, $\lambda_1(B) \geqslant \ldots \geqslant \lambda_n(B)$. Prove that

$$\sum_{i=1}^{n} [\lambda_i(A) - \lambda_i(B)]^2 \leqslant \|A - B\|_F^2.$$

16. For any vectors $x, y \in \mathbb{R}^n$ whose coordinates satisfy inequalities $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$ and $y_1 \geqslant y_2 \geqslant \ldots \geqslant y_n$ we say that x is majorised by y (denote that as $x \prec y$) if:

$$\sum_{i=1}^{k} x_i \leqslant \sum_{i=1}^{k} y_i, \ \ 1 \leqslant k \leqslant n-1; \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

Let A be a Hermitian matrix of order n. Let $\lambda_1 \geqslant \ldots \geqslant \lambda_n$ be the eigenvalues of A and $\alpha_1 \geqslant \ldots \geqslant \alpha_n$ be the main diagonal entries of A arranged in nonincreasing order. Prove that $[\alpha_1, \ldots, \alpha_n]^{\mathrm{T}} \prec [\lambda_1, \ldots, \lambda_n]^{\mathrm{T}}$.

17. For any vector norm $\|\cdot\|_\alpha$ on $\mathbb{C}^n$ the dual norm is a functional

$$\|y\|_\beta := \sup_{\|x\|_\alpha = 1} |y^* x| \,.$$

Prove that the dual norm to any norm $\|\cdot\|$ on $\mathbb{C}^n$ is a vector norm on the same space. Also, prove that the dual norm to the dual one coincides with the original norm. Find the dual norm to a Hölder norm $\|\cdot\|_p$ on $\mathbb{C}^n$ for $1 \leqslant p \leqslant \infty$.

18. Find the dual norm to the spectral norm $\|\cdot\|_2$ on $\mathbb{C}^{n\times n}$.

19. Let matrix $A \in \mathbb{C}^{n\times n}$ have the singular values $\sigma_1(A) \geqslant \ldots \geqslant \sigma_n(A)$. Prove that the functional $F_2(A) := \sigma_1(A) + \sigma_2(A)$ is a unitarily invariant matrix norm.

20. Let $A \in \mathbb{C}^{n\times n}$. The "$p$-th" Schatten norm is defined as

$$S_p(A) := \left[\sum_{i=1}^{n} \sigma_i^p(A)\right]^{1/p}, \quad 1 \leqslant p \leqslant \infty,$$

where $\sigma_1(A) \geqslant \ldots \geqslant \sigma_n(A)$ are the singular values of matrix A. Prove that the functional $S_p(\cdot)$ is a unitarily invariant matrix norm for $1 \leqslant p \leqslant \infty$. Find all values p for which $S_p(\cdot)$ is an operator norm.