Fuchsian differential equations on Riemann surfaces with locally meromorphic solutions[1]

Francesca Tovena Dipartimento di Matematica, Università degli Studi di Roma Tor Vergata, Via della Ricerca Scientifica, 00133, Roma.

1 Introduction.

In the present paper, we study Appell's differential equations, namely linear differential equations on a compact Riemann surface X, with analytic coefficients and regular singularities, whose solutions are everywhere meromorphic (cf. [Po], ch. V; an equation with regular singularities is also called "fuchsian"). In his work [Ap], Appell studies the differential equations of the first order, that correspond to the study of meromorphic connections with logaritmic poles on line bundles (cf. example 2.15). Vitali ([Vi1], [Vi2]) extends the analysis to higher orders, specially to the second order.

In this paper, we study Appell's differential equations using cohomological tecniques, following the line of Deligne ([De]) and Gunning ([Gu1]). The different tecnique allows to extend Vitali's results.

For a fixed Appell differential equation E, the analytic continuation

[1] Conferenza tenuta il 22 dicembre 2003

The paper is in final form and no version of it will be published elsewhere

of a solution is still a solution. The choice of a fundamental set of solutions for the equation (in a point $x_0 \in X$ which is ordinary for E), yields a representation of the fundamental group $\pi_1(X, x_0)$ of X at x_0 with values in $GL(n, \mathbf{C})$, where n is the order of E, called the *monodromy representation* of E ([Po]); we remark that it is not necessary to remove the singularities of E. The bijection between the quotient of the group homomorphisms set Hom $(\pi_1(X, x_0), GL(n, \mathbf{C}))$, under the action of $GL(n, \mathbf{C})$ by conjugation and the cohomology set $H^1(X, GL(n, \mathbf{C}))$, associates a cohomology class to each Appell differential equation.

In section 2, one finds the formal definition of an *Appell differential equation of order n* (Def. 2.3). A differential equation is here identified with a basis of the space of his solutions in an ordinary point and its analytic continuations along curves on X, that is a meromorphic section E of the holomorphic vector bundle $\mathcal{O}(\chi^*)$ associated to the monodromy $\chi^* \in H^1(X, GL(n, \mathbf{C}))$. Some examples are given in order to justify the notions that are introduced and to interprete them in classical terms. We are interested in so-called Riemann-Hilbert problem, i.e. the study of the relationships between the monodromy and the singularities of the equation; in the present case, a singular point is classified by the data of the valuations of the solutions, which is a list of n different integers called the local *exponents*. Denoting by ν_P the minimal exponent in $P \in X$, one defines a divisor $D_E = \sum_P \nu_P$, called the *divisor* associated to E. One easily proves that any divisor linearly equivalent to D_E is the divisor of an Appell differential equation with the same monodromy (prop. 2.7). In prop. 2.10, b), we prove that $deg\, D_E \leq (n-1)(g-1)$.

But the divisor D_E is not the unique responsible for the singularities of E: a singularity occurs any time the list of the exponents is different by $\{0, 1, 2, \ldots, n-1\}$; even if $\nu_P = 0$, one can have a gap in the exponents (cf. example 2.4). Chosen a meromorphic section g with divisor $-D_E$ of the line bundle $\mathcal{O}(D_E)$, the product $\overline{E} = Eg$ gives a surjective morphism $\overline{E} : \mathcal{O}(\chi) \to \mathcal{L}_E$ of E, called the *reduced form* of E: this can be interpreted as a new differential equation such that the minimal exponent is always 0 and it is the analogue of the classical "first reduced form" (cf. [Po] p. 78). Studying the properties of $\overline{E}$, one defines a new divisor B, that associates to each point P the difference between the sum of the exponents in P of $\overline{E}$ and the sum $\frac{n(n-1)}{2}$ of the exponents in an ordinary point; any exponents of $\overline{E}$ is of the form $a - \nu_P$ for an exponent a of E in P. In

Prop. 2.10 a relation is given between deg B and deg D_E :

$$deg\, B = -n\, deg D_E + n\,(n-1)(g-1),$$

and this yields a generalization of Fuchs relation on the Riemann sphere between the exponents in Cor.2.11. A special attention is given to the Appell differential equations whose divisor is effective, so that all their singularities are apparent.

An Appell differential equation is interpreted, in prop.2.14, as a meromorphic connection on the jet bundle $J^{n-1}(\mathcal{L}_E)$ and, in prop.2.13, as a morphism $\Phi_E : J^n(\mathcal{L}_E) \to \mathcal{O}(B) \otimes \Omega^{\otimes n} \otimes \mathcal{L}_E$.

In section 3, we recall some results of the theory of Deligne [De], Gunning [Gu 1,2,3] and Telemann [Te 1,2] on the Appell differential equations with empty divisor B, called *cyclic*. This case has been extensively studied also by Biswas and Raina [BR], Hejhal [H] and Gallo, Kapovich and Marden [GKM], with different tecniques.

In the case of second order Appell differential equations, the ratio of two independent solutions is a local coordinate, outside a finite number of points (the support of B), and it changes upto a Moebius trasformation via analytic continuation along a loop. This remark is the basis of the work of Mandelbaum ([Ma 1,2,3]) and the corresponding classification of second order differential equations, given in section 4, Thm. 4.2. Any irreducible class $\chi^* \in H^1(X, GL(2, \mathbf{C}))$ is the monodromy of an Appell differential equation; more precisely, any meromorphic section of $\mathcal{O}(\chi^*)$ is a differential equation (Thm. 4.4 and Prop. 4.9).

Section 5 is devoted to a cohomological interpretation of special differential equations, that are Appell differential equations that are derivative of an effective differential equation. In theorem 5.5, the duality between $H^1(X, \mathbf{C}(\chi^*))$ and $H^1(X, \mathbf{C}(\chi))$ implies a relation between any pair of special differential equations of monodromy χ^* and $\overline{\chi}$ (the complex conjugated of χ). In theorem 5.7, it is given a characterization of a special differential equation by an orthogonality condition.

Many authors have studied related problems. We refer to [GKM] for a list of reference. We notice that, in [H] and [GKM], the authors are mainly interested to the ratio of the solutions of a second order differential equation and it is more convenient for them to represents a differential equations by using the second reduced form and Schwarzian derivative.

My aim to apply the present work to complex dynamics (in analogy to [C], [CMS]) motivates my interest in the study of the divisor D_E and its relation with B, and in the contents of section 5.

I warmly thank Francesco Baldassarri that introduced me into this subject, explaining me the contents of [Ba]. His letter to Prof. Vesentini inspired the present work. I also thank Emma Previato for very stimulating discussions and for her encouraging enthusiasm. Finally, I thank Salvatore Coen for his invitation and, specially, for his tireless and patient work.

2 Appell differential equations

Let X be a compact Riemann surface of genus g. We denote by $\mathcal{O}$ (resp., $\mathcal{M}$, $\mathcal{E}$) the sheaf of holomorphic (resp., meromorphic, C^∞) functions on X, by $\mathcal{O}_U$ the restriction of $\mathcal{O}$ to an open subset U of X, by Ω or $\mathcal{O}(\kappa)$ the canonical sheaf of holomorphic differentials on X, by $GL(n, \mathcal{O})$ the sheaf of holomorphic sections of the constant sheaf $GL(n, \mathbf{C})$ on X, by $\Gamma(U, \mathcal{F})$ the holomorphic section of a sheaf $\mathcal{F}$ over an open subset U of X, by $\mathcal{F}^*$ the dual of the holomorphic sheaf $\mathcal{F}$, by χ^* the dual of the cohomology class $\chi \in H^1(GL(n, \mathbf{C}))$.

We keep the notation of Gunning [Gu1] and Deligne [De]. A holomorphic vector bundle *with connection* on X is a couple $(\mathcal{V}, \nabla)$ of a holomorphic vector bundle $\mathcal{V}$ on X and a $\mathbf{C}$-linear morphism of sheaves

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega$$

verifying Leibnitz rule: $\nabla(fv) = df\, v + f\nabla v$, for every holomorphic local section f of $\mathcal{O}$ and v of $\mathcal{V}$, where d denotes the external derivative $d : \mathcal{O} \to \Omega$. The connection is called *meromorphic* if the codomain of ∇ is $\mathcal{V} \otimes \mathcal{M}^1$, where we denote by $\mathcal{M}^1$ the sheaf of meromorphic differentials; finally, the connection is called *meromorphic with logarithmic poles* if the codomain is $\mathcal{V} \otimes \Omega(D)$, where we denote by $\Omega(D)$ the sheaf of meromorphic differentials with logarithmic poles in a divisor D. The morphism ∇ is called the *covariant derivative* defined by the connection, and it induces a covariant derivative ∇^* on the dual bundle $\mathcal{V}^* = Hom(\mathcal{V}, \mathcal{O}_X)$ by the position $\nabla^*(\phi(v)) = d\phi(v) - \phi(\nabla v)$, for local sections ϕ of $\mathcal{V}^*$ and v of $\mathcal{V}$. A local section v of $\mathcal{V}$ is called *flat* if $\nabla v = 0$. If V is a complex local system on X and $\mathcal{V} \cong V \otimes \mathcal{O}_X$, there is a unique connection ∇ on $\mathcal{V}$, called *canonical connection*, such that the flat sections are the local sections of the subsheaf V; given local sections f and v of $\mathcal{O}_X$ and V, one has

$\nabla(vf) = df\, v$. Viceversa, given a vector bundle with a connection $(\mathcal{V}, \nabla)$, the subsheaf of the flat sections is a complex local system, i.e. there exists a cohomology class $\chi \in H^1(GL(n, \mathbf{C}))$ such that $\mathcal{V} \cong \mathbf{C}(\chi) \otimes \mathcal{O} = \mathcal{O}(\chi)$ and the connection ∇ corresponds to the canonical connection. The class χ is called a *flat representative* for $\mathcal{V}$. In the bundle $\mathcal{O}(\chi)$, we will always use the canonical connection.

The derivative can be extended to meromorphic section of $\mathcal{V}$. If w is a holomorphic vector field on an open subset U of X, i.e., $w \in \Gamma(U, \mathcal{O}(\kappa^{-1}))$, one defines

$$\nabla_w(v) =< \nabla v, w > \in \Gamma(U, \mathcal{V}), \qquad \forall \text{ meromorphic section } v \text{ of } \mathcal{V} \text{ over } U.$$

The map ∇_w is called *covariant derivative along the vector field* w. One can iterate i times ∇_w and write $(\nabla_w \circ \cdots \circ \nabla_w) = \nabla_w^i$.

Let $(\mathcal{V}, \nabla)$ be a holomorphic vector bundle of rank n on X, with covariant derivative ∇. The definitions in the following generalize Deligne's definition of a "cyclic section", (cf. [De], 4.27, page 26):

Definition 2.1 *Let w be a holomorphic vector field over an open subspace U of X and assume that w is nowhere zero. A meromorphic section v of $\mathcal{V}$ over U is called* almost cyclic *if the local meromorphic sections $\{\nabla_w^i(v) | i = 1, \ldots, n\}$ generate over $\mathcal{O}_U$ a vector bundle of rank n. If, in particular, the sections $\{\nabla_w^i(v) | i = 1, \ldots, n\}$ generate $\mathcal{V}$ over U, the section v is called* cyclic *on U, and v is, actually, holomorphic.*

The given definition does not depend on the choice of the vector field w. A section v is almost cyclic (resp., cyclic) if and only if it is almost cyclic (resp., cyclic) the section $f\, v$ for every nowhere zero holomorphic function f.

Definition 2.2 *A global section of $\mathcal{V}$ is called* almost cyclic *(resp.,* cyclic*) if it is almost cyclic (resp., cyclic) over all open subspaces U of X.*

If $\mathcal{L}$ is a line bundle on X, a section of $\mathcal{V} \otimes \mathcal{L}$ is called almost cyclic *(resp.,* cyclic*) if it is almost cyclic (resp., cyclic) the corresponding section of $\mathcal{V}$ under all local isomorphism between $\mathcal{L}$ and $\mathcal{O}$. In particular, this definition applyes to a section of $\mathcal{V}^* \otimes \mathcal{L} \cong$ Hom$(\mathcal{V}, \mathcal{L})$, defining a* almost cyclic (or cyclic) morphism.

Here is the corresponding generalization of the definition of differential equation given by Deligne, (cf. [De], def. 4.3):

Definition 2.3 *An* Appell differential equation E of order n on X with monodromy χ^* *is an almost cyclic morphism* $E : \mathcal{O}(\chi) \to \mathcal{M}$*, i.e., an almost cyclic meromorphic section of* $\mathcal{O}(\chi^*)$*.*

Let $E : \mathcal{O}(\chi) \to \mathcal{M}$ be a fixed differential equation. We shall associate to E a line bundle $\mathcal{L}_E$ on X called the *image* of E, a set of local sections of $\mathcal{L}_E$ called *solutions* of E and a *divisor* D_E on X. In order to define this objects, we fix a coordinate covering $\mathcal{U} = \{U_\alpha, z_\alpha\}$ such that χ trivializes on $\mathcal{U}$ and a representative $(\chi_{\alpha\beta})$ of χ on $\mathcal{U}$. The morphism $E : \mathcal{O}(\chi) \to \mathcal{M}$ is described on U_α by a row vector $E_\alpha = (y_{1\alpha}, \ldots, y_{n\alpha})$ whose entries are meromorphic functions satisfying the relations $E_\alpha \chi_{\alpha\beta} = E_\beta$ on $U_\alpha \cap U_\beta \neq \emptyset$. The *divisor* D_E associated to E is given by

$$D_E = \sum_{p \in X} \nu_p(E) p \tag{2.1}$$

where $\nu_p(E)$ is the minimal order in p of the entries of the matrices representing E. In classical terms, D_E is the divisor on X obtained associating to $p \in X$ the so-called *minimal exponent* $\nu_p(E)$, i.e., the minimal valuation in p of the solutions of the differential equation in classical terms. The coefficient $\nu_p(E)$ is different from zero only in a finite set of points, which are *singularities* for E.

By refining the covering, one can assume that, for each α, only the point p_α given by $z_\alpha = 0$ can have a non-zero minimal exponent. Setting $\nu_\alpha = \nu_{p_\alpha}(E)$, the components of the n-tuple $\overline{E}_\alpha = (z_\alpha^{-\nu_\alpha} E_\alpha)$ are holomorphic funtions without common zeroes on U_α. The *image* of E is the subsheaf $\mathcal{L}_E$ of $\mathcal{M}$ generated on U_α by $z_\alpha^{\nu_\alpha}$, whose corresponding cohomology class $\xi \in H^1(X, \mathcal{O}^*)$ is represented by the cocycle defined by $\xi_{\alpha\beta} = z_\alpha^{-\nu_A} z_\beta^{\nu_B}$ on $U_\alpha \cap U_\beta \neq \emptyset$. One may identify $\mathcal{L}_E = \mathcal{O}(\xi)$; its sections satisfy $f_\alpha = \xi_{\alpha\beta} f_\beta$ on $U_\alpha \cap U_\beta \neq \emptyset$. We remark that

$$z_\alpha^{\nu_\alpha} \mathcal{O}_p = y_{1\alpha} \mathcal{O}_p + \ldots + y_{n\alpha} \mathcal{O}_p \qquad \forall p \in U_\alpha, \forall \alpha \tag{2.2}$$

and that $\overline{E}_\alpha \chi_{\alpha\beta} = \xi_{\alpha\beta} \overline{E}_\beta$ on $U_\alpha \cap U_\beta$, so that the matrices $\overline{E}_\alpha$ define an almost cyclic (surjective) morphism

$$\overline{E} : \mathcal{O}(\chi) \to \mathcal{L}_E.$$

We say that E is an *equation onto* $\mathcal{L}_E$ and that $\overline{E}$ is its *reduced form.* The *solutions* of E are the images, under $\overline{E}$ of the flat sections of $\mathcal{O}(\chi)$, and are local sections of $\mathcal{L}_E$.

We remark that the divisor D_E corresponds to $\mathcal{L}_E \cong \mathcal{O}(-D_E)$ in the identification between isomorphism classes of holomorphic line bundles on X and linear equivalence classes of divisors on X (in Gunning's notation) and, in particular, deg $D_E = -c_1(\mathcal{L}_E)$.

Example 2.4 On the Riemann sphere $X = \mathbf{P}^1$, one consider the standard coordinate covering given by $\{(U_0 = X \setminus \{P_\infty\}, x), (U_1 = X \setminus \{P_0\}, t = x^{-1})\}$. In classical terms, the differential equation given by:

$$\frac{d^2}{dx^2} f - \frac{1}{x}\frac{d}{dx} f = 0 \tag{2.3}$$

has solutions generated by 1 and x^2 and its singularities are P_0 (with exponents 0 and 2) and P_∞ (with exponents -2 and 0). The chosen solutions correspond to a pair $\{E_0 = (1, x^2), E_\infty = (1, t^{-2})\}$ that, according to definition 2.3, defines a differential equation $E : \mathcal{O} \oplus \mathcal{O} \to \mathcal{M}$ with monodromy $1 \in H^1(X, GL(2, \mathbf{C}))$ and divisor $D_E = -2P_\infty$. The associated line bundle $\mathcal{L}_E \cong \mathcal{O}_{\mathbf{P}^1}(2)$ is described by the cocycle $(\xi_{0\infty} = t^{-2})$ and the corresponding reduced form $\overline{E} : \mathcal{O} \oplus \mathcal{O} \to \mathcal{L}_E$ is given by $\{\overline{E}_0 = (1, x^2), \overline{E}_\infty = (t^2, 1)\}$. The solutions of E are the complex vector spaces generated, respectively, by 1 and x^2 on U_0 and by 1 and t^2 on U_∞. Notice that the analogue of the wronskian in given by a pair of holomorphic functions

$$w_0 = det\begin{pmatrix} \overline{E}_0 \\ \frac{d}{dx}\overline{E}_0 \end{pmatrix} = det\begin{pmatrix} 1 & x^2 \\ 0 & 2x \end{pmatrix} = 2x \qquad \text{on } U_0 \tag{2.4}$$

$$w_\infty = det\begin{pmatrix} \overline{E}_\infty \\ \frac{d}{dt}\overline{E}_\infty \end{pmatrix} = det\begin{pmatrix} t^2 & 1 \\ 2t & 0 \end{pmatrix} = -2t \qquad \text{on } U_\infty \tag{2.5}$$

that define a holomorphic section $w = \{w_0, w_\infty\} \in H^0(X, \Omega \otimes \mathcal{L}_E^2)$ whose divisor is $B = P_0 \oplus P_\infty$. The divisor B is effective and it associates to each point the absolute value of the difference of the exponents, diminuished by 1. For this reason, B is called the *branching divisor* (cf. Def. 2.9). Since w is a holomorphic section, the degree of B must satisfy the relation:

$$deg\, B = deg\, (\Omega \otimes \mathcal{L}_E^2) = -2 + 4 = 2. \tag{2.6}$$

The relation (2.6) can be interpreted in two different ways. It is the Riemann-Hurwitz formula for the branched cover $x^2 = \frac{x^2}{1} : \mathbf{P}^1 \to \mathbf{P}^1$ given by the ratio of the entries of E (that are a fundamental set of solutions of the equations, in classical terms). Moreover, recalling that D_E is the divisor of the minimal exponents, one notices that the union of the supports of D_E and of B is the set of the singular points of the differential equation; then

$$deg\, B + 2deg\, D_E + \text{[number of singular points]} = \sum_{singular\, points} \text{[exponents]}$$

and the relation (2.6) is the Fuchs's relation between the exponents ([Po], p. 77) for a fuchsian differential equation on $\mathbf{P}^1$ (cf. Cor. 2.11 for a generalization):

$$\sum_{singular\, points} \text{[exponents]} = \text{[number of singular points]} - 2. \qquad (2.7)$$

Example 2.5 As in the previous example, one easily shows that the reduced form of the equation $\frac{d^2}{dx^2}f = 0$ is a morphism $\overline{E}_1 : \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1)$ with monodromy 1, divisor $D_{E_1} = -P_\infty$, empty branching divisor. On the other hand, the reduced form of the equation $x^2\frac{d^2}{dx^2}f - 2x\frac{d}{dx}f + 2 = 0$ is a morphism $\overline{E}_2 : \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1)$ onto the same line bundle and with the same monodromy and branching divisor as E_1, but with divisor $D_{E_2} = P_0 - 2\,P_\infty$ (which is linearly equivalent to D_{E_1}). $\triangle$

In the next proposition, we show that, for a fixed monodromy, the degree of the divisor of a differential equation is bounded from above. In Proposition 2.10 we will show that $degD_E \leq (n-1)(g-1)$, so there exists an upper bound that does not depend on the monodromy. We say that a line bundle ζ is a quotient of a vector bundle $\mathcal{O}(\Phi)$ if there is a surjective morphism of $\mathcal{O}_X$-modules $\mathcal{O}(\Phi) \to \zeta$.

Definition 2.6 *For all $\Phi \in H^1(X, \mathcal{GL}(n, \mathbf{C}))$, we set*

$$quot(\Phi) = min\ \{c(\zeta) : \text{ the line bundle } \zeta \text{ is a quotient of } \mathcal{O}(\Phi)\}.$$

Proposition 2.7 *a) For $\chi \in H^1(X, GL(n, \mathbf{C}))$, let $E : \mathcal{O}(\chi) \to \mathcal{M}$ be an Appell differential equation on X. Then $degD_E \leq -quot\,(\chi) < \infty$. In particular, there is a finite upper bound for the degree of all Appell differential equations with monodromy χ^*.*

b) If $\chi \in H^1(X, GL(n, \mathbf{C}))$ is the monodromy of an Appell differential equation E with divisor D_E, then χ is the monodromy of an Appell differential equation whose divisor is any divisor linearly equivalent to D_E.

Proof. a) In ([Gu1], page 80), Gunning defines $div(\Phi) = \max \{c(\zeta) : \zeta$ is a line bundle and $\zeta \subset \mathcal{O}(\Phi)\}$ and shows that $div(\Phi)$ is always a finite integer. Since $quot(\Phi) = c(det\Phi) - div(\Phi^*)$, one deduces that $quot(\Phi)$ is also finite. The proof of item b) is easy. $\triangle$

We now recall the notion of jet bundles in order to describe further properties of Appell's differential equations. We refer to [De] and [Bi] as a reference on jet bundles. Let Δ denote the diagonal divisor of $X \times X$, $p_i : X \times X \to X$ the projection onto the i^{th} factor and Δ_n the n^{th} order infinitesimal neighborhood of Δ in $X \times X$, defined by the non reduced divisor $(n+1)\Delta$ of $X \times X$.

For a holomorphic line bundle $\mathcal{L}$ on X and a positive integer k, the k-order jet bundle of $\mathcal{L}$, denoted by $J^k(\mathcal{L})$, is the direct image:

$$J^k(\mathcal{L}) = p_{1*}\left(\frac{p_2^*\mathcal{L}}{p_2^*\mathcal{V} \otimes \mathcal{O}_{X\times X}(-(k+1)\Delta)}\right) \tag{2.8}$$

with the $\mathcal{O}_X$-module structure induced by p_1. For every local section s of $\mathcal{L}$, p_2^*s defines a local section $d^k_{\mathcal{L}}(s)$ of $J^k(\mathcal{L})$. The $\mathbf{C}$-morphism of sheaves

$$d^k_{\mathcal{L}} : \mathcal{L} \to J^k(\mathcal{L}) \tag{2.9}$$

is the universal differential operator of order $\leq k$ and it has the following properties: $d^k_{\mathcal{L}}(t) = 1 \otimes t$, $d^k_{\mathcal{L}}(at) = (1 \otimes t)\, a = d^k_{\mathcal{L}}(t)\, a$ (here t (resp., a) is a local section of $\mathcal{L}$ (resp., $\mathcal{O}$)).

We recall that $J^k(\mathcal{L})$ is a holomorphic vector bundle of rank $k+1$, locally generated by the image of $\mathcal{L}$ under $d^k_{\mathcal{L}}$. If $\mathcal{L} \cong \mathcal{O}$, the choice of a local coordinate x on X gives a local isomorphism $J^k(\mathcal{O}) \cong \mathcal{O}^{\oplus k+1}$, under which the map $d^k_{\mathcal{O}}$ is given by:

$$\begin{array}{rcl} d^k_{\mathcal{O}} : \mathcal{O} & \to & J^k(\mathcal{O}) \cong \mathcal{O}^{\oplus k+1} \\ f & \mapsto & (f, \frac{d}{dx}f, \ldots, \frac{d^k}{dx^k}f). \end{array} \tag{2.10}$$

Moreover, one has a short exact sequence:

$$0 \to \Omega^{\otimes k} \otimes \mathcal{L} \to J^k(\mathcal{L}) \to J^{k-1}(\mathcal{L}) \to 0 \tag{2.11}$$

and

$$det\, J^k(\mathcal{L}) \cong \mathcal{L}^k \otimes \Omega^{\frac{k(k-1)}{2}}. \tag{2.12}$$

Fixed $\chi \in H^1(X, GL(n, \mathbf{C}))$ and $\lambda : \mathcal{O}(\chi) \to \mathcal{L}$ be a $\mathcal{O}$-morphism, the composition $d^k_{\mathcal{L}} \circ \lambda : \mathcal{O}(\chi) \to J^k(\mathcal{L})$ is a $\mathbf{C}$-morphism. Recalling that $Hom_{\mathbf{C}}(\mathbf{C}(\chi), \mathcal{N}) \cong Hom_{\mathcal{O}}(\mathcal{O}(\chi), \mathcal{N})$ for any $\mathcal{O}$-module $\mathcal{N}$, one defines a $\mathcal{O}$-morphism $\gamma^k_\lambda : \mathcal{O}(\chi) \to J^k(\mathcal{L})$ by the position $\gamma^k_\lambda(v) = d^k_{\mathcal{L}}(\lambda(v))$ for any flat local section v of $\mathcal{O}(\chi)$. The following diagram of $\mathcal{O}$-morphisms of $\mathcal{O}$-modules is commutative:

$$\begin{array}{ccc} \mathcal{O}(\chi) & \overset{\gamma^k_\lambda}{\rightarrow} & J^k(\mathcal{L}) \\ & \lambda \searrow & \uparrow d^k_{\mathcal{L}} \\ & & \mathcal{L} \end{array} \tag{2.13}$$

The following lemma is easy to prove by a local analysis:

Lemma 2.8 *Let $\lambda : \mathcal{O}(\chi) \to \mathcal{L}$ be a surjective $\mathcal{O}$-morphism.*
i) The morphism λ is almost cyclic if and only if $\gamma^{n-1}_\lambda : \mathcal{O}(\chi) \to J^{n-1}(\mathcal{L})$ induces a $\mathcal{O}$-isomorphism between $\mathcal{O}(\chi)$ and a rank n $\mathcal{O}$-module contained in $J^{n-1}(\mathcal{L})$.
ii) [[De], Lemma 4.9.2] The morphism λ is cyclic if and only if $\gamma^{n-1}_\lambda : \mathcal{O}(\chi) \to J^{n-1}(\mathcal{L})$ is a $\mathcal{O}$-isomorphism.

Definition 2.9 *Let $E : \mathcal{O}(\chi) \to \mathcal{M}$ be a differential equation of order n onto $\mathcal{L}_E$ and $\overline{E} : \mathcal{O}(\chi) \to \mathcal{L}_E$ its reduced form. The $\mathcal{O}$-morphism*

$$det(\gamma^{n-1}_{\overline{E}}) : \mathcal{O}(det\, \chi) \to det\, J^{n-1}(\mathcal{L}_E) \cong \mathcal{L}^{\otimes n} \otimes \Omega^{\frac{n(n-1)}{2}} \tag{2.14}$$

defines a holomorphic section w_E of the bundle $\mathcal{L}^{\otimes n} \otimes \Omega^{\frac{n(n-1)}{2}} \otimes \mathcal{O}(det\, \chi^*)$. The section w_E is called wronskian *and its divisor, denoted by B, is an effective divisor called* branching divisor *satisfying:*

$$\mathcal{O}(B) \cong \mathcal{O}(det\, \chi^*) \otimes \mathcal{L}_E^{\otimes n} \otimes \Omega^{\frac{n(n-1)}{2}}. \tag{2.15}$$

The union of the supports of D_E and B is denoted by $Sing(E)$ and its elements are called the singularities *of E. A singularity P is called* apparent *if its coefficient in D_E is $\nu_P \geq 0$ (in classical terms, P is a singular point in which every solution is holomorphic).*

We remark that the given definition of branching divisor coincides with the definition given in ([GKM], pag. 680) for a second order differential equation.

The items a) and b) of the following proposition generalize, respectively, the statements of ([GKM], Cor. 11.3.1, p. 686) and ([Vi2], Lemma p. 14, [Ba]) for second order differential equations. The item c) is due to ([De], p.35, for $n = 2$), [Te1] and [BR] in the case $\mathcal{O}(det\, \chi^*)$=1.

Proposition 2.10 *Let $E : \mathcal{O}(\chi) \to \mathcal{L}_E$ be an Appell differential equation of order n onto $\mathcal{L}_E$. Denote by B its branching divisor. Then*

a) $deg\, B = n\ c(\mathcal{L}_E) + n(n-1)(g-1)$.

b) $deg\,(D_E) \leq (n-1)(g-1)$.

c) If $deg\,(D_E) = (n-1)(g-1)$, then the differential equation is cyclic and $\mathcal{L}_E^{-n} \cong \mathcal{O}(det\, \chi^*) \otimes \Omega^{\otimes \frac{n(n-1)}{2}}$.

The following corollary generalizes Fuchs's relation between the exponents for a fuchsian differential equation on $\mathbf{P}^1$ ([Po], p. 77):

Corollary 2.11 *Denote by m the cardinality of $Sing(E)$. Then*

$$\sum_{Sing(E)} [\text{exponents}] = n(n-1)(g-1) + \frac{n(n-1)}{2}\, m$$

Proof. One remarks that

$$deg\, B + n\, deg\, D_E + \frac{n(n-1)}{2}\, m = \sum_{Sing(E)} [\text{exponents}].$$

The thesis follows from Prop. 2.10, a). △

Example 2.12 We recall Example 2.4 in this context. Here the rank 3 vector bundle $J^2(\mathcal{L}_E)$ is given by the transition matrix (we denote by $'$ the derivative $\frac{d}{dt}$):

$$\begin{pmatrix} \xi_{0\infty} & 0 & 0 \\ \kappa_{0\infty}\xi'_{0\infty} & \kappa_{0\infty}\xi_{0\infty} & 0 \\ \kappa_{0\infty}\kappa'_{0\infty}\xi'_{0\infty} + \kappa^2_{0\infty}\xi''_{0\infty} & \kappa_{0\infty}\kappa'_{0\infty}\xi_{0\infty} + 2\kappa^2_{0\infty}\xi'_{0\infty} & \kappa^2_{0\infty}\xi_{0\infty} \end{pmatrix}$$

and the corresponding transition matrix for $J^1(\mathcal{L}_E)$ is obtained by deleting the last column and the last row.

The morphism $\gamma^1_{\overline{E}} : \mathcal{O} \oplus \mathcal{O} \to J^1(\mathcal{L}_E)$ is described on U_0 by the matrices

$$\begin{pmatrix} \overline{E}_0 \\ \frac{d}{dx}\overline{E}_0 \end{pmatrix} = \begin{pmatrix} 1 & x^2 \\ 0 & 2x \end{pmatrix} \text{ on } U_0 \tag{2.16}$$

$$\begin{pmatrix} \overline{E}_\infty \\ \frac{d}{dt}\overline{E}_\infty \end{pmatrix} = \begin{pmatrix} t^2 & 1 \\ 2t & 0 \end{pmatrix} \text{ on } U_\infty \tag{2.17}$$

In classical terms, the differential equation with a given system of solutions $\overline{E}_0$ is given by the condition:

$$det \begin{pmatrix} \overline{E}_0 & y \\ \frac{d}{dx}\overline{E}_0 & \frac{dy}{dx} \\ \frac{d^2}{dx^2}\overline{E}_0 & \frac{d^2y}{dx^2} \end{pmatrix} = det \begin{pmatrix} 1 & x^2 & y \\ 0 & 2x & \frac{dy}{dx} \\ 0 & 2 & \frac{d^2y}{dx^2} \end{pmatrix} = -2\frac{dy}{dx} + 2x\frac{d^2y}{dx^2} = 0$$

The pair $(0, -2, 2x)$ on U_0 and the corresponding $(0, 2, -2t)$ on U_∞ so obtained defines a morphism $\Phi_E : J^2(\mathcal{L}_E) \to \mathcal{M}$ onto the line bundle $\mathcal{O}(B) \otimes \Omega^{\otimes 2} \otimes \mathcal{L}_E$. Moreover, the composition of this map with the natural inclusion $\Omega^2 \otimes \mathcal{L}_E \to J^2(\mathcal{L}_E)$ is not identically zero and it is given by the wronskian. $\triangle$

The given example can be generalized, showing that an Appell differential equation is a differential operator according to the current definition. We recall that, by definition of differential equation of order n, the image of $\gamma^n_{\overline{E}}$ has rank n.

Proposition 2.13 *Let $E \in H^0(X, \mathcal{M}(\chi^*))$ be an Appell differential equation onto $\mathcal{L}_E$, with branching divisor B and wronskian w_E. Let $\overline{E} : \mathcal{O}(\chi) \to \mathcal{L}_E$ its reduced form and $\gamma^n_{\overline{E}}$ as defined in 2.13. Then there is a surjective $\mathcal{O}_X$-morphism*

$$\Phi_E : J^n(\mathcal{L}_E) \to \mathcal{O}(B) \otimes \Omega^{\otimes n} \otimes \mathcal{L}_E \tag{2.18}$$

such that $\Phi_E \circ \gamma^n_{\overline{E}} \equiv 0$ and the composition with the natural inclusion

$$\sigma : \Omega^{\otimes n} \otimes \mathcal{L}_E \to J^n(\mathcal{L}_E) \stackrel{\Phi_E}{\to} \mathcal{O}(B) \otimes \Omega^{\otimes n} \otimes \mathcal{L}_E \tag{2.19}$$

corresponds to the wronskian $w_E \in H^0(X, \mathcal{O}(B))$.

The morphism σ defined in the proposition is usually called *symbol*. We underline that the symbol of an Appell differential equation does not need to be an isomorphism: the special case in which σ is an isomorphism (i.e., B is empty) correspond to the differential equations studied by Deligne, and it will be described in the next section. When B is non empty, there may exists a morphism $\Phi : J^n(\mathcal{L}) \to \mathcal{O}(B) \otimes \Omega^{\otimes n} \otimes \mathcal{L}$ that do not arise from an Appell differential equation: a characterization can only be given by a local analysis that calculates the local exponents and ensures the absence of logarithmic singularities (cf. [Po], p. 69). In some cases this local analysis can be avoided, as in Remark 2.16 and in the case in which the symbol is an isomorphism (Prop. 3.1).

The next proposition shows that an Appell differential equation can be interpreted as a meromorphic connection. For differential equations of the first order, the two notions are equivalent, as it is recalled in the following example.

Proposition 2.14 *An Appell differential equation $E : \mathcal{O}(\chi) \to \mathcal{M}$ of order n onto $\mathcal{L}_E$ defines a meromorphic connection on $J^{n-1}(\mathcal{L}_E)$.*

Proof. Let $\Psi \in H^1(X, \mathcal{GL}(n, \mathbf{C}))$ be the cohomology class such that $J^{n-1}(\mathcal{L}_E) \cong \mathcal{O}(\Psi)$. Fix a coordinate covering $\mathcal{U} = \{U_\alpha, z_\alpha\}$ such that χ and Ψ trivialize on $\mathcal{U}$, and choose representatives $(\chi_{\alpha\beta})$ of χ and $\Psi_{\alpha\beta}$ of Ψ on $\mathcal{U}$. The morphism $\gamma_{\overline{E}}^{n-1}$ is represented by $n \times n$ matrices Γ_α with meromorphic entries such that $\Gamma_\alpha \chi_{\alpha\beta} = \Psi_{\alpha\beta} \Gamma_\beta$ on $U_\alpha \cap U_\beta \neq \emptyset$. Every matrix Γ_α is invertible as matrix with meromorphic entries and the entries of the matrix $\Omega_\alpha = \Gamma_\alpha \; d\,(\Gamma_\alpha^{-1})$ are meromorphic differentials; the relation:

$$\Omega_\beta - \Psi_{\alpha\beta}^{-1} \, \Omega_\alpha \, \Psi_{\alpha\beta} \, = \Psi_{\alpha\beta}^{-1} \, d\Psi_{\alpha\beta} \tag{2.20}$$

shows that the matrices Ω_α define a meromorphic connection $J^{n-1}(\mathcal{L}_E) \to \Omega(B) \otimes J^{n-1}(\mathcal{L}_E)$, where we denote by $\Omega(B)$ the sheaf of meromorphic differentials on X whose poles are in B. △

Example 2.15 (e.g., [Vi1], p.65) A first order Appell differential equation is simply a non trivial meromorphic section E of a line bundle $\mathcal{O}(\chi^*)$ of degree zero. Such a section exists according to the fundamental existence theorem. One notice that $\frac{dE_\alpha}{E_\alpha} = \frac{dE_\beta}{E_\beta} = \omega$ defines a global meromorphic differential with integer residues. Notice that the sum of the residues of ω is $c(\mathcal{L}_E)$. In classical term, the general Appell differential equation of the first order with monodromy χ has the form:

$$dy = (\omega + \eta)\, y, \text{ with } \eta \in H^0(X, \Omega).$$

One easily sees that the classes of Appell differential equation of the first order on X form an homogeneous principal space with base $Pic(X)$ and group $H^0(X, \Omega)$.

Keeping the notation of the proposition, one has $\Gamma_\alpha = \overline{E}_\alpha, \Psi_{\alpha\beta} = \xi_{\alpha\beta}$ that is the cocycle of $\mathcal{L}_E$. The matrices Ω_α define a meromorphic connection on $\mathcal{L}_E$ with logarythmic poles. Viceversa, every connection of this type comes from an Appell differential equation.

Remark 2.16 *Monodromy holomorphically trivial ([Vi1],[Ba])*
Let $\chi \in H^1(X, SL(n, \mathbf{C}))$ and assume that there is an isomorphism $\mathcal{O}(\chi) \cong \mathcal{O}^n$ as holomorphic vector bundles, i.e. $i^*(\chi) = 1$ under the map $i^* : H^1(X, GL(n, \mathbf{C})) \to H^1(X, \mathcal{GL}(n, \mathbf{C}))$ induced in cohomology by the inclusion $\mathbf{C} \to \mathcal{O}$. Fix a coordinate covering $\mathcal{U} = \{U_\alpha, z_\alpha\}$ such that χ trivializes on $\mathcal{U}$, with a representative $(\chi_{\alpha\beta})$ of χ on $\mathcal{U}$. An isomorphism $E : \mathcal{O}(\chi) \to \mathcal{O}^n$ is described on U_α by a non-singular $n \times n$ matrix F_α with holomorphic entries satisfying the relations $F_\alpha \chi_{\alpha\beta} = F_\beta$ on $U_\alpha \cap U_\beta \neq \emptyset$, so that $dF_\alpha F_\alpha^{-1} = dF_\beta F_\beta^{-1} = \Lambda$ is a $n \times n$ matrix with entries in $H^0(X, \Omega)$ and zero trace. Each row of F_α defines a morphism $\mathcal{O}(\chi) \to \mathcal{O}$: if this morphism is almost cyclic, it defines an Appell differential equation. The flat representatives of $\mathcal{O} \oplus \mathcal{O}$ with determinant 1 are in one to one correspondence with the conjugacy classes of the matrices Λ as before. Viceversa, if $\chi \in H^1(X, SL(n, \mathbf{C}))$ is monodromy of n Appell differential equations such that the determinant of the morphism $\mathcal{O}(\chi) \to \mathcal{O}^n$ given by the direct sum is not identically zero (hence it is nowhere zero), then χ is holomorphically trivial (this is the "primo caso ridotto" in the paper [Vi1]). An analysis of the matrix Λ prove the following result for a second order differential equation, due to ([Vi1], p.60 and [Ba], p.7): *the monodromy $\chi \in H^1(X, SL(2, \mathbf{C}))$ admits two Appell differential equations such that the determinant of their direct sum is non zero if and only if the corresponding morphism $\Phi : J^2(\mathcal{O}) \to \Omega^{\otimes 3}$ is given by a matrix of the form (in the usual notation)*

$$(\varphi_\alpha \eta_\alpha^2, -d\eta_\alpha - 2\psi_\alpha \eta_\alpha, \eta_\alpha) \textit{ with } \varphi_\alpha, \eta_\alpha, \psi_\alpha \in H^0(X, \Omega).$$

In this case, χ is holomorphically trivial. △

Example 2.17 We recall the following example due to [Vi1],p.62. Let $g = 1$ and $\chi^* \in H^1(X, GL(2, \mathbf{C}))$ be the monodromy of a differential equation E onto a line bundle $\mathcal{L}_E$ such that D_E is effective (this is called an effective differential equation, in the next definition). Applying prop. 2.10, a), one sees that $c(\mathcal{L}_E) = 0$, hence $\mathcal{L}_E \cong \mathcal{O}$. Since the canonical bundle is trivial, the derivative ∇E is still a differential equation of monodromy χ^*. One concludes that χ is holomorphically trivial as before. Denoting by η the unique non zero holomorphic differential form on X, the explicit form becames:

$$(\nu\,\eta_\alpha^3, -\,d\eta_\alpha - 2\mu\,\eta_\alpha^2, \eta_\alpha) \qquad \text{for } \mu, \nu \in \mathbf{C}.$$

In [Vi1], p.10, it is shown that even in the case $g = 2$, each monodromy of a second order differential equation with effective divisor, is holomorphically trivial. △

The following definition correspond to the notion of a differential equation of "prima specie" (c.f. [Ap]):

Definition 2.18 *An Appell differential equation $E : \mathcal{O}(\chi) \to \mathcal{M}$ is called* effective *if the corresponding divisor D_E is effective, i. e. $\mathcal{L}_E \subset \mathcal{O}$ and E factorizes, as a $\mathcal{O}$-module morphism, through the inclusion $\mathcal{O} \to \mathcal{M}$:*

$$\begin{array}{ccc} E : \mathcal{O}(\chi) & \to & \mathcal{M} \\ \searrow & & \nearrow \\ & \mathcal{O} & \end{array} \qquad (2.21)$$

The following proposition follows from Prop. 2.10, a):

Proposition 2.19 *An Appell differential equation of order n onto $\mathcal{O}$ has at most $n(n-1)(g-1)$ apparent singularities.*

We refer to [O] for a related result on the existence of a differential equation (non necessary of Appell) with a given set of singularities and a bound on the number of the apparent singularities.

3 Cyclic differential equations

In this section we recall the work of [De], [BR], [Te1], [Te2] on cyclic differential equations, but we do not assume the determinant of the monodromy is 1. Let $(\mathcal{V}, \nabla)$ be a rank n vector bundle with connection and let $\chi \in H^1(X, GL(n, \mathbf{C}))$ be the flat representative defined by the connection. According to definition 2.3 and keeping the notation of the previous section, a differential equation $E : \mathcal{V} \cong \mathcal{O}(\chi) \to \mathcal{M}$ is cyclic if the corresponding reduced form $\overline{E} : \mathcal{V} \cong \mathcal{O}(\chi) \to \mathcal{L}_E$ is cyclic. In this case, the bundle $J^{n-1}(\mathcal{L}_E)$ is isomorphic to $\mathcal{O}(\chi)$, so it admits a connection; in particular, $\mathcal{L}_E^{\otimes -n} \cong \mathcal{O}(det\,\chi^*) \otimes \Omega^{\otimes \frac{n(n-1)}{2}}$ and $c_1(\mathcal{L}) = (n-1)(g-1)$ (cf. also Lemma 2.8, ii) and Prop. 2.10, c)).

Proposition 3.1 ([De], Prop. 4.9) *Let $\mathcal{L}$ be a fixed line bundle on X such that $c_1(\mathcal{L}) = (n-1)(g-1)$. There exists an equivalence between the two following categories, whose morphism are the isomorphisms:*
a) the category of line bundles $\mathcal{L}$ on X with a chosen holomorphic bundle morphism $P^n(\mathcal{L}) \to \Omega^{\otimes n} \otimes \mathcal{L}$ that induces the identity on the submodule $\Omega^{\otimes n} \otimes \mathcal{L}$ of $P^n(\mathcal{L})$;
b) the category of the cyclic differential equations of order n onto $\mathcal{L}$, i.e., the category of the triples $((\mathcal{V}, \nabla), \mathcal{L}, \lambda)$ given by a holomorphic fiber bundle $\mathcal{V}$ of rank n with a connection, a line bundle $\mathcal{L}$ and a surjective cyclic morphism: $\lambda : \mathcal{V} \to \mathcal{L}$.

A cyclic differential equation can be interpreted geometrically as follows: on a sufficiently small open U, the differential equation associates to U a curve in the projective space $\mathbf{P}^n(\mathbf{C})$ such that, in each point, such curve has a tangent and an osculating hyperplane well determinated. In particular, the ratio of af the entries of a second order cyclic differential equations gives an uniformization of the Riemann surface X:

Proposition 3.2 ([De], Prop. 5.8 for $n = 2$, [Te1], [Te2]) *Let $\mathcal{L}$ a line bundle on X such that $c_1(\mathcal{L}) = (n-1)(g-1)$, $n > 1$, and consider the set $\mathcal{S}$ of pairs given by a (n-1)-tuple $(\omega_2, \ldots, \omega_n)$ of global differential forms on X, with $\omega_i \in \Gamma(X, \Omega^{\otimes i})$, and a connection on $\mathcal{L}^{\otimes n} \otimes \Omega^{\otimes t}$, where $t = \frac{n(n-1)}{2}$. There exists a bijection between the set of cyclic differential equations of order n onto $\mathcal{L}$ and the set $\mathcal{S}$.*

Proof. The data of a connection on $\mathcal{L}^{\otimes n} \otimes \Omega^{\otimes t}$ can be interpreted as the choice of a flat representative for the determinant of the monodromy. We recall that, according to [Gu1] and [Te2], all the monodromies with determinant 1 of cyclic differential equations determine the same image under the composition of the natural maps: $H^1(X, SL(n, \mathbf{C})) \to H^1(X, PGL(n-1, \mathbf{C})) \to H^1(X, PGL(n-1, \mathcal{O}))$. The proof of the proposition reduces to the results of [Te1]. $\triangle$

According to Riemann-Roch theorem, the $(n-1)$-tuples $(\omega_2, \ldots, \omega_n)$ of global differential forms on X, with $\omega_i \in \Gamma(X, \Omega^{\otimes i})$ depends on $[(2g-2)(n-1+g)]$ parameters.

We refer to [BR] for a different characterization of cyclic differential equations.

4 Second order differential equations

In the case of Appell differentiall equation of the second order, on can apply Gunning's results on rank 2 vector bundles on the Riemann surface X ([Gu1]).

Let $\chi \in H^1(X, GL(2, \mathbf{C}))$ and let $E : \mathcal{O}(\chi) \to \mathcal{M}$ be an Appell differential equation with monodromy χ^*, onto a line bundle $\mathcal{L}_E$. We recall that, according to 2.10, one has deg $(D_E) \leq g-1$ and the wronskian is a section $w \in H^0(\mathcal{L}_E^2 \otimes \Omega \otimes \mathcal{O}(det\chi^*))$. The reduced form of the differential equation describes $\mathcal{O}(\chi)$ as an extension of $\mathcal{L}_E$, i.e., there exists a short exact sequence of vector bundles on X

$$0 \to \mathcal{L}_E^{-1} \otimes \mathcal{O}(\det\ \chi) \to \mathcal{O}(\chi) \xrightarrow{\overline{E}} \mathcal{L}_E \to 0. \tag{4.1}$$

Notice that the wroskian w is the Atiyah class of the extension (4.1), so the sequence does not split. We recall that the extensions of $\mathcal{L}_E$ by $\mathcal{L}_E^{-1} \otimes \mathcal{O}(\det\ \chi)$ are classified by the elements of $H^0(\mathcal{L}_E^2 \otimes \Omega \otimes \mathcal{O}(det\, \chi^*))$ upto isomorphism (cf., [Gu1], Th. 13 p. 72).

We want to give a classification of Appell differential equations with given divisor D_E and branching divisor B (i.e., with assigned singularities and exponents in each singularity) that generalizes the study of cyclic differential equations. We follow the line of [Gu1],[Gu2], [Gu3], [Ma1],[Ma2], [Ma3] (cf., also, [GKM]).

Definition 4.1 *Let $\{U_\alpha, z_\alpha\}$ be a coordinate covering of X. A* ramified projective covering of X *is the data $\{U_\alpha, h_\alpha, \Phi_{\alpha\beta}\}$, where $h_\alpha : U_\alpha \to P^1$ is a meromorphic function non locally constant $\forall\alpha$ and $\Phi_{\alpha\beta} = z_\alpha z_\beta^{-1}$ is a projective trasformation, on $U_\alpha \cap U_\beta \neq \emptyset$. A* ramified projective structure $\mathbf{H}$ *on X is an equivalence class of projective ramified coverings, where two coverings are equivalent if their union is a ramified projective covering.*

Theorem 4.2 *Let $\mathcal{L}$ be a line bundle on X with an injection $\mathcal{L} \to \mathcal{M}$. There exists a bijective correspondence between the sets:*
a) the set of the second order Appell differential equations onto $\mathcal{L}$ on X;
b) the set of the couples given by a ramified projective structure $\mathbf{H}$ on X with ramification divisor $B(\mathbf{H})$ such that deg $B(\mathbf{H}) = 2g-2+2c(\mathcal{L})$, and connection on $\Omega \otimes \mathcal{L}^{\otimes 2} \otimes \mathcal{H}$, where $\mathcal{H}$ is the line bundle on X having a section whose divisor is $-B(\mathbf{H})$.

Remark 4.3 The branching divisor $B = 0$ occurs precisely in the cyclic differential equations, and, in this case, the statement of the theorem is Deligne's prop. 3.2.

Let $\Phi \in H^1(X, GL(2, \mathbf{C}))$. We keep the notation of Def. 2.6. We recall that the holomorphic vector bundle $\mathcal{O}(\Phi)$ is called *stable* if $quot(\Phi) > 0$, *unstable* if $quot(\Phi) \leq 0$, *semistable* if $quot(\Phi) \geq 0$ (cf., e.g., [Gu1], p. 83). According to ([Gu1], Lemma 14, p. 81), one knows that $quot(\Phi) < g$.

Theorem 4.4 *Let $\chi \in H^1(X, GL(2, \mathbf{C}))$ such that quot $(\chi) \neq 0$ (recall that $g \geq$ quot $(\chi) \geq -g$). Let $\lambda : \mathcal{O}(\chi) \to \mathcal{L}$ be any surjective morphism where $\mathcal{L}$ is a line bundle on X with $c_1(\mathcal{L}) =$ quot (χ) (it exists by definition of quot (χ)). The following results hold:*

- *a) The morphism λ is a second order differential equation onto $\mathcal{L}$ with monodromy χ^*.*
- *b) Then each flat representative of $\mathcal{O}(\chi)$ is a monodromy for a differential equation onto $\mathcal{L}$.*
- *c) Any meromorphic section E of $\mathcal{O}(\chi^*)$ is an Appell differential equation with monodromy χ^*.*

Proof. Item a) follows from the fact that $\lambda : \mathcal{O}(\chi) \to \mathcal{L}$ is part of an extension as in (4.1), for $\mathcal{L} = \mathcal{L}_E$ and $\lambda = \overline{E}$. The sequence cannot split, since otherwise $c(\mathcal{L}) = 0$ according to Weil Theorem (cf., [Gu1], Th. 16, p. 110). The proof of item b) is similar. Item c) follows from the indecomposability of the holomorphic bundle $\mathcal{O}(\chi)$. $\triangle$

The following theorem study the divisor of minimal degree of a differential equation with a given monodromy, when the degree of the divisor is extremal:

Theorem 4.5 *Let $\chi \in H^1(X, GL(2, \mathbf{C}))$.*

a) Assume that quot $(\chi) < 0$, i.e. $\mathcal{O}(\chi)$ is unstable not semistable. There is a unique line bundle $\mathcal{L}$ that is a quotient of $\mathcal{O}(\chi)$ and satisfyes $c(\mathcal{L}) =$ quot (χ). Then the Appell differential equations with monodromy χ^* and associated divisor of degree $-$quot (χ) have linearly equivalent divisors.

b) Assume that quot $(\chi) = g > 0$, so that $\mathcal{O}(\chi)$ is stable. Then χ is the monodromy of an Appell differential equation onto each line bundle $\mathcal{L}$ with $c(\mathcal{L}) = g$.

Proof. Item a) follows directly from ([Gu1], Lemma 15, p.84), claiming that it is unique the line bundle of maximal degree contained in a rank two unstable non semistable vector bundle on X. Item b) follows from [Gu1], Lemma 16, p. 92. $\triangle$

Corollary 4.6 *For $g > 1$, let $\mathcal{L}$ a line bundle with $c(\mathcal{L}) = 1 - g$. Then the bundle $J^1(\mathcal{L})$ is unstable not semistable and the canonical projection $J^1(\mathcal{L}) \to \mathcal{L}$ is a cyclic differential equation with respect to each flat representative of $J^1(\mathcal{L})$. The monodromies for a differential equation onto $\mathcal{L}$ are then classified by the flat representatives of $J^1(\mathcal{L})$, and one can assigne to them the structure of an analytic variety isomorphic to $\mathbf{C}^{3g-3}$ (cf. [Gu3], Th.4 p. 57).*

The following proposition is a corollary of theorem 4.4 (cf. [Ma1], [GKM]):

Proposition 4.7 *Let $\chi \in H^1(X, GL(2,\mathbf{C}))$ such that quot $(\chi) = k \neq 0$ and consider the map:*

$$p : H^1(X, GL(2,\mathbf{C})) \to H^1(X, PGL(1,\mathbf{C})) \tag{4.2}$$

induced in cohomology by the standard projection $GL(2,\mathbf{C}) \to PGL(1,\mathbf{C})$. Then, the image $p(\chi)$ is the cohomology class for a ramified projective structure $\mathbf{H}$ on X, and $\mathbf{H} \in \mathbf{RPS}(2g-2+2k) = \{$ramified projective structures on X whose ramification divisor has degree $2g-2+2k\}$.

Fixed the monodromy $\chi \in H^1(X, GL(2,\mathbf{C}))$ such that quot $(\chi) < 0$, it is possible to describe the differential equations with monodromy χ whose divisor has maximal degree.

Corollary 4.8 *If $k < 0$, for an arbitrary choice of $\xi \in H^(X, \mathbf{C}^*)$, there exists a bijection between the ramified projective structures in* $\mathbf{RPS}(2g - 2 + 2k)$ *and the cohomology classes $\chi \in H^1(X, GL(2, \mathbf{C}))$ such that det $\chi = \xi$ and quot $(\chi) = k$.*

In the case quot $(\chi) = 0$, one can formulate a result analogue to theorem 4.5 only if χ is associated to a class of irreducible representations of the fundamental group of X with value in $GL(2, \mathbf{C})$:

Proposition 4.9 *Let $\chi \in H^1(X, GL(2, \mathbf{C}))$ be a cohomology class representing a flat irreducible bundle with div $(\chi) = 0$. Let $\lambda : \mathcal{O}(\chi) \to \mathcal{L}$ be a surjective morphism onto a line bundle $\mathcal{L}$ with zero Chern class. Then λ is a differential equation onto $\mathcal{L}$ with monodromy χ. (cf. [Ma3], [GKM]).*

In its paper [Vi1], Vitali studies the effective Appell differential equations of second order. In particular, he treats the case of two equations with the same monodromy $\chi \in H^1(X, GL(2, \mathbf{C}))$.

Definition 4.10 *Two Appell's differential equations $E, F : \mathcal{O}(\chi) \to \mathcal{M}$ are* linearly independent *if they correspond to linearly independent meromorphic sections of $\mathcal{O}(\chi^*)$.*

Lemma 4.11 *Let $\chi^* \in H^1(X, GL(2, \mathbf{C}))$ be the monodromy of an effective differential equation on X. Then the number of linearly independent effective differential equations with monodromy χ^* is equal to $dim_{\mathbf{C}}\, H^0(X, \mathcal{O}(\chi^*))$.*

Proof. An holomorphic section of $\mathcal{O}(\chi^*)$ is a differential equation if and only if the extension corresponding to the surjection $\mathcal{O}(\chi^*) \to \mathcal{O}$ does not split (cf. 4.1). Using Serre's duality, this is an open condition with respect to at most g polynomial conditions. $\triangle$

We remark that the hypothesis in the proposition cannot be removed: a counterexample is given by $1 \in H^1(X, GL(2, \mathbf{C}))$. We notice, moreover, that the monodromy of an effective differential equation is unstable.

Corollary 4.12 *Let $\chi^* \in H^1(X, GL(2, \mathbf{C}))$ be the monodromy of an effective differential equation onto a line bundle $\mathcal{L}_E$ with $c(\mathcal{L}_E) = quot(\chi) < 0$. Then the number of linearly independent effective differential equations of monodromy χ^* is given by:*

$$H^0(X, \mathcal{L}_E^{-1}) = -c(\mathcal{L}_E) + 1 - g + H^0(X, \Omega \otimes \mathcal{L}_E) \leq g - 1.$$

Proof. According to [Gu1], Th. 23, one has $H^0(X, \mathcal{O}(\chi^*)) = H^0(X, \mathcal{L}_E^{-1})$, since χ is unstable non semistable. The thesis follows from Riemann-Roch Theorem, recalling that $-c(\mathcal{L}_E) \leq g-1$ by prop 2.10. $\triangle$

5 Integral differential equations

Definition 5.1 *An Appell differential equation $E : \mathcal{O}(\chi) \to \mathcal{L}_E$ of order n on X is* special *if $\mathcal{L}_E$ is special, i.e., there exist an injective morphism $\mathcal{L}_E \to \Omega$.*

We notice that, if E is special, then $2-2g \leq deg\, D_E \leq g-1$ and $deg\, B \leq 6g-6$.

Lemma 5.2 *[[Vi2], Thm. p.25]. Let χ^* be the monodromy of a special differential equation of the second order. The maximal number t of linearly independent differential equations with monodromy χ^* is given by $t = dim_{\mathbf{C}}\, H^0(X, \mathcal{O}(\chi^* \otimes \kappa)) = dim_{\mathbf{C}}\, H^1(X, \mathcal{O}(\chi))$. In particular, $2g-2 \leq t \leq 2g$ and the first inequality is strict if and only if $H^0(X, \mathcal{O}(\chi)) \neq 0$.*

Proof. The first statement can be proved as in lemma 4.11. According to the Riemann Roch theorem, one has $m = 2g-2+H^0(X, \mathcal{O}(\chi))$. If χ is irreducible, then $H^0(X, \mathcal{O}(\chi)) = 0$ by ([Gu1], Thm. 18, p.127). If χ is decomposable, then $\mathcal{O}(\chi)$ is the direct sum of two line bundle of degree 0, hence $H^0(X, \mathcal{O}(\chi)) \leq 2$. This concludes the proof. $\triangle$

Corollary 5.3 *The class $\chi^* \in$ is the monodromy of a special differential equation if the class χ is the monodromy of an effective differential equation.*

We notice that, if $g=1$ an Appell differential equation is special if and only if it is effective, so that there are 2 linearly independent equations according to Ex. 2.17.

The notion of "special" differential equation corresponds, in Vitali's notation, to a differential equation of " prima categoria la cui equazione integrale sia di prima specie".

Remark 5.4 Let $E : \mathcal{O}(\chi) \to \mathcal{L}_E$ be a special differential equation. Chosen an injection $\mathcal{L}_E \to \Omega$, one can identify E with a section $E \in$

$H^0(X, \mathcal{O}(\chi^* \otimes \kappa)) \subset H^0(X, \mathcal{M}(\chi^* \otimes \kappa))$. Consider the short exact sequence of sheaves on X:

$$0 \to \mathbf{C}(\chi^*) \to \mathcal{M}(\chi^*) \xrightarrow{d} d\mathcal{M}(\chi^*) \to 0 \tag{5.1}$$

where by d we denote the external derivative and by $d\mathcal{M}(\chi^*)$ the sheaf of germs of meromorphic differential forms on X which are sections of the vector bundle χ and zero residues everywhere. Recalling that $H^1(X, \mathcal{M}(\chi^*)) = 0$, the exact cohomology sequence associated to the previous sequence gives an isomorphism:

$$H^1(X, \mathbf{C}(\chi^*)) \cong \frac{H^0(X, d\mathcal{M}(\chi^*))}{dH^0(X, \mathcal{M}(\chi^*))} \tag{5.2}$$

The natural inclusion $\mathcal{O}(\chi^* \otimes \kappa) \to d\mathcal{M}(\chi^*)$ associates to the special differential equation E a cohomology class in $H^1(X, \mathbf{C}(\chi^*))$, i. e., an extension of the form:

$$0 \to \mathbf{C}(\chi^*) \to \mathbf{C}(\Psi) \to \mathbf{C} \to 0. \tag{5.3}$$

where $\Psi \in H^1(X, GL(n+1, \mathbf{C}))$. The class Ψ is the monodromy of an effective Appell differential equation $\int E$ whose derivative is given by E: we say that $\int E$ is the integral differential equation of E. We remark that this contruction depends on the chosen inclusion $\mathcal{L}_E \to \Omega$. $\triangle$

We notice that the construction of an integral equation can be performed in a bigger generality. The hypothesis we made are necessary to ensures that the integral equation is effective.

In ([Vi2], Art. 5), Vitali studies the relation between the periods of solutions of two second order differential equations with contragradient monodromies and finds a reciprocity law generalizing the reciprocity law for differential forms on X (of first, second and third type).

We suggest here a proof of Vitali reciprocity law via the duality between $H^1(X, \mathbf{C}(\chi^*))$ and $H^1(X, \mathbf{C}(\overline{\chi})) \cong H^1(X, \mathbf{C}(\chi))$, where the last isomorphism is given by conjugtion. The duality is express in two different ways, and the law is obtained by equating the two expressions.

Recall that $H^1(X, \mathbf{C}(\chi))$ is isomorphic to the cohomology of groups $H^1(\pi_1(X), \hat{\chi})$ of $\mathbf{C}^2$ considered as a $\pi_1(X)$-module via the representation $\hat{\chi}$ associated to χ ([Gu1], Thm.19, p.128). One fixes the choice of a set $\{\sigma_j, \tau_j\} \subset \pi_1(X, P_0)$ of representatives for a of a standard base of

$H^1(X, \mathbf{Z})$, i.e., σ_j, τ_j are 1-cycle in X that are disjoint out of the point P_0 and such that $D = X \setminus \{\sigma_j, \tau_j\}$ is simply connected.

Theorem 5.5 *Let E be a special differential equation on X of monodromy $\chi^* \in H^1(X, GL(n, \mathbf{C}))$ and let L be a special differential equation of monodromy $\overline{\chi}$. Denote by $A \in H^1(X, \mathbf{C}(\chi^*))$ (resp., $C \in H^1(X, \mathbf{C}(\overline{\chi}))$) the corresponding cohomology class associated to E as in the previous remark (resp., C). Moreover, denote by A_γ (resp., C_γ) a 1-cocycle representing A (resp., B) in the cohomology of groups.*

Then

$$0 = \sum_{j=1}^{g} [\overline{C}^t_\tau\, \hat{\chi}(\tau_j)^{-1} A_\sigma - \overline{C}^t_\sigma\, \hat{\chi}(\tau_j)^{-1} A_\tau] +$$

$$+ \sum_{j=1}^{g} [\overline{C}^t_\tau\, \hat{\chi}(\sigma_j \tau_j)^{-1} (A_\lambda - \hat{\chi}(\sigma_j)^{-1} A_\lambda)] -$$

$$- \sum_{j=1}^{g} [\overline{C}^t_\sigma\, \hat{\chi}(\tau_j \sigma_j)^{-1} (A_\lambda - \hat{\chi}(\tau_j)^{-1} A_\lambda)] =$$

$$= \sum_{j=1}^{g} [\overline{C}^t_{\sigma_j^{-1}} A_{\tau_j} - \overline{C}^t_{\tau_j^{-1}} A_{\sigma_j} - \overline{C}^t_{(\sigma_j\tau_j)^{-1}} A_{[\sigma_j, \tau_j]} + \overline{C}_{[\sigma_j, \tau_j]} A_{\lambda_{j-1}}]$$

where $\lambda_j = [\sigma_1, \tau_1] \ldots [\sigma_j, \tau_j] \in \pi_1(X)$ and $[\sigma, \tau] = [\sigma\tau\sigma^{-1}\tau^{-1}]$.

Proof. One applies ([Gu1], Thm. 21, p. 153 and p.156) to express the value of $< A, C >$ in the duality between $H^1(X, \mathbf{C}(\chi^*))$ and $H^1(X, \mathbf{C}(\chi))$. On the other hand, since E and L are special, then $< A, C >= 0$ according to ([Gu1], Thm. 25, p. 176); in fact, let $\psi \in H^0(X, d\mathcal{M}(\chi^*))$ (resp., $\varphi \in H^0(X, d\mathcal{M}(\overline{\chi}))$) be a representative of E (resp., L) via the isomorphism (5.2); one can consider $H^0(X, d\mathcal{M}(\chi^*))$ and $H^0(X, d\mathcal{M}(\overline{\chi}))$ as a subset of the cohomology $H^0(\tilde{X}, \mathcal{M} \oplus \mathcal{M})$ on the universal cover $\tilde{X}$ of X. Locally on X, one denotes by $\int \varphi$ a vector of n meromorphic functions whose derivative is φ. Then $< A, C >= 2\pi i\, \mathcal{R}(\psi^t \int \varphi) = 0$, where by $\mathcal{R}$ we denote the total residue. △

Remark 5.6 The theorem can be extended to the case of Appell differential equations that can be interpreted as elements of $H^0(X, d\mathcal{M}(\chi^*))$ (and $H^0(X, d\mathcal{M}(\overline{\chi}))$, using the equality $< A, C >= 2\pi i\, \mathcal{R}(\psi^t \int \varphi)$.

In ([Vi2], Art.3,4), Vitali characterizes special second order differential equations via an orthogonality condition. The long exact sequence of cohomology associated to the short exact sequence of sheaves:

$$0 \to \mathbf{C}(\chi^*) \to \mathcal{O}(\chi^*) \xrightarrow{d} \mathcal{O}(\kappa \otimes \chi^*) \to 0, \tag{5.4}$$

where d denotes the external derivative, gives an inclusion $\delta : H^0(X, \mathcal{O}(\kappa \otimes \chi)/dH^0(X, \mathcal{O}(\chi^*)) \to H^1(X, \mathbf{C}(\chi^*))$.

To each element $A \in H^1(X, \mathbf{C}(\chi^*))$, one associates a linear functional $\rho(A) : H^0(X, \mathcal{O}(\kappa \otimes \chi)/dH^0(X, \mathcal{O}(\chi)) \to \mathbf{C}$, defined as follows. Consider a coordinate cover U_α of X (sufficiently fine) and choose C^∞-sections $F_\alpha \in H^0(U_\alpha, \mathcal{E}(\chi^*))$ whose coboundary is a representative for A. Then, one sets

$$\rho(A)(\varphi) = \int_X \varphi^t \wedge dF_\alpha$$

for each $\varphi \in H^0(X, \mathcal{O}(\kappa \otimes \chi)/dH^0(X, \mathcal{O}(\chi))$.

According to Serre duality, an element A is in the image of δ if and only if $\rho(A) \equiv 0$ ([Gu1], Thm.22, p.159). In particular, one has the following:

Theorem 5.7 *Let E be a special differential equation on X of monodromy $\chi^* \in H^1(X, GL(n, \mathbf{C}))$ and let $A \in H^1(X, \mathbf{C}(\chi^*))$ be the class associated to E under the isomorphism (5.2). Then $\rho(A) \equiv 0$.*

The condition in the theorem is also sufficient for A to come from a special differential equation. Notice that testing $\rho(A)$ on a basis of $H^0(X, \mathcal{O}(\kappa \otimes \chi)/dH^0(X, \mathcal{O}(\chi))$ gives a list of (at most $2g$) independents conditions.

References

[Ap] P. Appell, Sur les integrales de fonctions a multiplicateurs et leur application au developpement des fonctions abeliennes en series trigonometriques, Acta mathematica **13** (1890), 1–174.

[Ba] F. Baldassarri, letter to Prof. Vesentini, may 25, 1983.

[Bi] I. Biswas, *Coupled connections on a compact Riemann surface*, J. Math. Pures Appl. **182** (2003), 1–42.

[BR] I. Biswas, A.K. Raina, *Projective Structures on a Riemann surface, II*, Internat. Math. Res. Notices **13** (1999), 687–716.

[C] C.Camacho, *Dicritical singularities of holomorphic vector fields*, Contemporary Mathematics **269** (2001), 39–46.

[CMS] C.Camacho, H.Movasati, P.Sad, *Fibered neighborhoods of curves in surfaces*, Preprint of the IMPA (2001).

[De] P. Deligne, Equations Différentielles à Points Singuliers Réguliers, Springer Lecture Notes **163**, (1970).

[GKM] D. Gallo, M. Kapovich, A. Marden, *The monodromy groups of Schwarzian equations on closed Riemann surfaces*, Annals of Mathematics **151** (2000), 625–704.

[Gu1] R.C. Gunning, Lectures on Vector Bundles over Riemann Surfaces, Princeton University Press (1967).

[Gu2] R.C. Gunning, *Special coordinate coverings of Riemann Surfaces*, Math. Ann. **170** (1967), 67–86.

[Gu3] R.C. Gunning, *Analytic structures on the space of flat vector bundles over a compact Riemann surface*, Several complex variables II, Mariland, Springer Lecture Notes **185** (1970).

[Gu4] R.C. Gunning, *The divisor order of vector bundles of rank two on a Riemann surface*, Bulletin of the Institute of Mathematics Academia Sinica **6** (1978), no. 2, part I, 295–302.

[H] D. A. Hejhal,*Monodromy groups and linearly polymorphic functions*, Acta Mathematica **135** (1975), no. 1, 1–55.

[O] M. Ohtsuki,*On the number of Apparent Singularities of a Linear Differential Equation*, Tokyo J. Math. **5** (1982), no. 1, 23–29.

[Po] E.G.C. Poole, Introduction to the theory of the linear differential equations, Dover Publications, New York, 1960.

[Ma1] R. Mandelbaum, *Branched Structures on Riemann Surfaces*, Trans. Amer. Math. Soc. **163** (1972), 261–275.

[Ma2] R. Mandelbaum, *Branched Structures and Affine and Projective Bundles on Riemann Surfaces*, Trans. Amer. Math. Soc. **183** (1973), 37–58.

[Ma3] R. Mandelbaum, *Unstable Bundles and Branched Structures on Riemann Surfaces*, Math. Ann. **214** (1975), 49–59.

[Te1] C. Teleman , *Sur les structures homographiques d'une surface de Riemann*, Comment. Math. Helv. **33** (1959), 206–211.

[Te2] C. Teleman , *Sur les structures fibrées osculatrices d'une surface de Riemann*, Comment. Math. Helv. **34** (1960), 175–184.

[Vi1] G. Vitali, *Sopra le equazioni differenziali lineari omogenee a coefficienti algebrici*, Rend. Circ. Matem. Palermo **XVI** (1902), 57–69.

[Vi2] G. Vitali, *Sopra le equazioni differenziali lineari omogenee a coefficienti algebrici*, Annali della R. Scuola Normale Superiore di Pisa **IX** (1903), 1–57.

Ricevuto il 24 febbraio 2004.