Rigidity at the Boundary for Holomorphic Self-Maps of the Unit Disk

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Abstract - We prove a rigidity theorem which generalizes a result due to D. Burns and G. Krantz (see [3]) for holomorphic self-maps in the unit disk of the complex plane. Essentially, we found that some conditions on the (boundary) Schwarzian derivative of a holomorphic self-map at specific points of the boundary of the disk may be sufficient to conclude that the map is a completely determined rational map.

1. The aim of this paper is to investigate how rigid is the set of holomorphic self-maps in the unit disk Δ of the complex plane $\mathbb{C}$ after imposing some conditions on the boundary Schwarzian derivative. It is infact well known that the Schwarzian derivative of a holomorphic self-map f carries a global information on f: it vanishes identically if and only if f is a Moebius transformation.

The idea of considering the boundary Schwarzian derivative of a map f naturally arises as soon as one tries to generalize the rigidity result established in [3] in the following sense: consider a holomorphic self-map f of Δ such that, in a neighbourhood of 1 in Δ, its expansion is

$$f(z) = z + \frac{1}{2}f''(1)(z-1)^2 + \frac{1}{6}f'''(1)(z-1)^3 + o(z-1)^3;$$

then, after some calculations and by applying techniques similar to the ones used in [3], it is easily seen that if $\mathrm{Re}(f''(1)) = 0$ and $\mathrm{Re}(\mathcal{S}f(1)) = 0$, f is nothing but the parabolic automorphism given by

$$f(z) = \frac{(2-ib)z + ib}{(2+ib) - ibz},$$

where $b = \mathrm{Im}(f''(1))$. It can be also written as

$$\frac{1+f(z)}{1-f(z)} = \frac{1+z}{1-z} + ib. \tag{1}$$

Observe that, in the particular case when $f''(1) = f'''(1) = 0$, one finds out just the result proved in [3], that is $f \equiv \mathrm{Id}_\Delta$.

Keywords: holomorphic maps; boundary behaviour; Schwarzian derivative; finite Blaschke products.

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Assume now that for a holomorphic self-map f of Δ the radial expansion in 1 is

$$f(z) = 1 + \beta_1(z-1) + \frac{1}{2}f''(1)(z-1)^2 + \frac{1}{6}f'''(1)(z-1)^3 + o(z-1)^3;$$

where $\beta_1 = f'(1)$ is a positive real number. Set

$$a \stackrel{\text{def}}{=} \frac{\mathrm{Re}(f''(1)) - \beta_1(\beta_1 - 1)}{\beta_1^2},$$

then $\frac{1}{a+1}$ is, in some sense, the radius of curvature at 1 of the boundary of $f(\Delta)$. If $a \geq 0$, consider the disk internally tangent to $\partial\Delta$ in 1 of radius $\frac{1}{a+1}$ and assume that $f(\Delta)$ is just contained in this disk. This condition obviously holds when $\beta_1 = 1$ and $\mathrm{Re}(f''(1)) = 0$, that is when $a = 0$, because $f(\Delta) \subset \Delta$ (as in the above-mentioned case of parabolic automorphisms). This geometric property implies (see Theorem 2.6) that necessarily $\mathrm{Im}(\mathcal{S}f(1)) = 0$ (which is equivalent to saying that the point 1 is a vertex for the boundary of $f(\Delta)$).

Suppose furthermore that one knows that $\sigma_2, \ldots, \sigma_N$ are $N-1$ inverse images of 1 such that for each of them the module of the radial limit $\beta_k = |f'(\sigma_k)|$ is finite; then we proved (see Theorem 2.6) that $\mathrm{Re}(\mathcal{S}f(1))$ is less or equal to the non-positive real number

$$-6\beta_1 \sum_{k=2}^{N} \frac{1}{\beta_k} \cdot \frac{1}{|\sigma_k - 1|^2} \tag{2}$$

(if $N = 1$, this number is intended to be 0). More precisely, if $\mathrm{Re}(\mathcal{S}f(1))$ reaches the upper bound (2) then f is the rational map given by

$$\frac{1+f(z)}{1-f(z)} = \frac{1}{\beta_1} \cdot \frac{1+z}{1-z} + \sum_{k=2}^{N} \frac{1}{\beta_k} \cdot \frac{\sigma_k + z}{\sigma_k - z} + a + ib \tag{3}$$

for $z \in \Delta$, where $b = \mathrm{Im}\left(\frac{1+f(0)}{1-f(0)}\right)$.

Therefore, the original rigidity result contained in [3] has been successively extended from the identity, to the parabolic automorphisms (1), and finally to the more general family of rational maps (3).

2. Let $\Delta \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk of $\mathbb{C}$ whose boundary is $\partial\Delta = \{z \in \mathbb{C} : |z| = 1\}$ and let $\mathrm{Hol}(\Delta, \Delta)$ be the set of all holomorphic maps from Δ into itself. If $H \stackrel{\text{def}}{=} \{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$ is the right half-plane of $\mathbb{C}$ then, for $\sigma \in \partial\Delta$, let $\varphi_\sigma(z) = \frac{\sigma+z}{\sigma-z}$ be the biholomorphism of Δ onto H with inverse $z = \varphi_\sigma^{-1}(w) = \sigma\frac{w-1}{w+1}$. Given $\sigma \in \partial\Delta$ and $R > 0$, the horocycle $E(\sigma, R)$ of center σ and (hyperbolic) radius R is the disk in Δ of (euclidean) radius $R/(R+1)$ tangent to $\partial\Delta$ in τ which is analytically defined as

$$E(\sigma, R) = \left\{ z \in \Delta : \frac{|\sigma - z|^2}{1 - |z|^2} < R \right\},$$

with the convention that $E(\sigma,+\infty)=\Delta$.

For $f\in\mathrm{Hol}(\Delta,\Delta)$ and $\sigma,\tau\in\partial\Delta$, we define $\beta_f(\sigma,\tau)$ the following positive real number

$$\beta_f(\sigma,\tau)\stackrel{\mathrm{def}}{=}\sup_{z\in\Delta}\left\{\frac{|\tau-f(z)|^2}{1-|f(z)|^2}\Bigg/\frac{|\sigma-z|^2}{1-|z|^2}\right\};$$

in other words, this means that for any $R>0$

$$f(E(\sigma,R))\subset E(\tau,\beta_f(\sigma,\tau)R). \tag{4}$$

For $f\in\mathrm{Hol}(\Delta,\mathbb{C})$, $l\in\widehat{\mathbb{C}}$ is the non-tangential limit of f at $\sigma\in\partial\Delta$ if $f(z)$ tends to l as z tends to σ in Δ within an angular sector of vertex σ and opening less than π. We summarize this definition by writing

$$\operatorname*{K-lim}_{z\to\sigma} f(z)=l.$$

Moreover, if $f(z)$ tends to l as z tends to σ in Δ along the radius $\{r\sigma\mid r\in\,]0,1[\}$ connecting 0 to σ, we say that f has radial limit l at σ and we write

$$\operatorname*{r-lim}_{z\to\sigma} f(z)=l.$$

Clearly if l is the non-tangential limit of f at σ, it coincides with the radial limit of f at σ.

We recall the Julia-Wolff-Carathéodory Theorem for $f\in\mathrm{Hol}(\Delta,\Delta)$:

Theorem 2.1 [1] *Let $f\in\mathrm{Hol}(\Delta,\Delta)$ and $\sigma,\tau\in\partial\Delta$. Then*

$$\operatorname*{K-lim}_{z\to\sigma}\frac{\tau-f(z)}{\sigma-z}=\tau\overline{\sigma}\beta_f(\sigma,\tau).$$

If $\beta_f(\sigma,\tau)$ is finite, then

$$\operatorname*{K-lim}_{z\to\sigma} f(z)=\tau \quad \textit{and} \quad \operatorname*{K-lim}_{z\to\sigma} f'(z)=\tau\overline{\sigma}\beta_f(\sigma,\tau).$$

The Schwarzian derivative of a map f at a point z is defined as

$$\mathcal{S}f(z)\stackrel{\mathrm{def}}{=}\frac{f'''(z)}{f'(z)}-\frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Suppose that $I\subset\mathbb{R}$ is an interval and $\gamma: I\to\mathbb{C}$ is a C^2 curve such that $\gamma'(t)\neq 0\ \forall t\in I$. Then (see, e.g., [5] or [6]) the (signed) euclidean curvature of γ is given by

$$k_\gamma(t)=\mathrm{Im}\left(\frac{\gamma''(t)}{|\gamma'(t)|\gamma'(t)}\right).$$

and, if γ is C^3, its variation is

$$\frac{\mathrm{d}k(t)}{\mathrm{d}t}=\frac{\mathrm{Im}(\mathcal{S}\gamma(t))}{|\gamma'(t)|}.$$

We recall that a point of a curve where the variation of the curvature vanishes is called a *vertex*.

Lemma 2.2 *Let $\tau \in \partial\Delta$; consider $\sigma_1, \cdots, \sigma_N$ distinct points in $\partial\Delta$, and $\beta_1, \cdots, \beta_N$ positive real numbers, then, for any $a + ib \in \overline{H}$, the map*

$$T(z) \stackrel{\text{def}}{=} \varphi_\tau^{-1}\Big(\sum_{k=1}^{N} \frac{1}{\beta_k}\varphi_{\sigma_k}(z) \ + \ a + ib\Big) \quad \forall z \in \Delta,$$

is a rational map of order N such that, for $j = 1, \cdots, N$,

$$\begin{aligned} T(\sigma_j) &= \tau, \ T'(\sigma_j) = \tau\overline{\sigma}_j\beta_j, \\ T''(\sigma_j) &= \tau\overline{\sigma}_j^2 \left[\beta_j(\beta_j - 1) + \beta_j^2 a + i\beta_j^2 \Big(b - \sum_{k=1,k\neq j}^{N} \frac{1}{\beta_k} \cdot \frac{2\mathrm{Im}(\sigma_k\overline{\sigma}_j)}{|\sigma_k - \sigma_j|^2}\Big) \right], \\ \mathrm{Re}(\sigma_j^2 \mathcal{S}T(\sigma_j)) &= -6\beta_j \sum_{k=1,k\neq j}^{N} \frac{1}{\beta_k} \cdot \frac{1}{|\sigma_k - \sigma_j|^2}. \end{aligned}$$

Moreover, T is a proper map such that $T(\Delta) = E(\tau, 1/a)$ and

$$\mathrm{Im}(z^2 \mathcal{S}T(z)) \equiv 0 \quad \forall z \in \partial\Delta.$$

In particular, T is a Blaschke product if and only if $a = 0$.

Proof. We can assume that $j = 1$ and $\sigma_1 = \tau = 1$, otherwise we may replace the map T with $\widetilde{T}(z) = \overline{\tau}T(\sigma_1 z))$.

Then, there is a neighbourhood $U \subset \mathbb{C} \setminus \{\sigma_2, \cdots, \sigma_N\}$ of the point 1 where T is holomorphic and

$$P_+(z)T(z) = P_-(z) \quad \text{for all } z \in U \tag{5}$$

with

$$P_\pm(z) = \frac{1}{\beta_1} + \frac{1 - z}{1 + z} \cdot (Q(z) \pm 1) \text{ and } Q(z) = \sum_{k=2}^{N} \frac{1}{\beta_k} \cdot \frac{\sigma_k + z}{\sigma_k - z} + a + ib.$$

It is easy to verify that

$$\begin{aligned} P_\pm(1) &= \frac{1}{\beta_1}, \\ P'_\pm(1) &= -\frac{1}{2}(Q(1) \pm 1), \\ P''_\pm(1) &= \frac{1}{2}(Q(1) \pm 1) - \sum_{k=2}^{N} \frac{1}{\beta_k} \cdot \frac{2\sigma_k}{(\sigma_k - 1)^2}, \\ P'''_\pm(1) &= -\frac{3}{4}(Q(1) \pm 1) + \sum_{k=2}^{N} \frac{1}{\beta_k} \cdot \left[\frac{3\sigma_k}{(\sigma_k - 1)^2} - \frac{6\sigma_k}{(\sigma_k - 1)^3} \right]. \end{aligned}$$

Therefore, deriving both sides of equation (5), we can find the following relations

$$
\begin{aligned}
T(1)=&\frac{P_-(1)}{P_+(1)}=1,\\
T'(1)=&\frac{P'_-(1)-P'_+(1)T(1)}{P_+(1)}=\beta_1,\\
T''(1)=&\frac{P''_-(1)-P''_+(1)T(1)-2P'_+(1)T'(1)}{P_+(1)}=(-1+(Q(1)+1)\beta_1)\beta_1=\\
=&\beta_1(\beta_1-1)+\beta_1^2a+i\beta_1^2(b-\sum_{k=2}^{N}\frac{1}{\beta_k}\cdot\frac{2\mathrm{Im}(\sigma_k)}{|\sigma_k-1|^2}),\\
\mathcal{S}T(1)=&\frac{P'''_-(1)-P'''_+(1)T(1)-3P''_+(1)T'(1)-3P'_+(1)T''(1)}{P_+(1)T'(1)}-\frac{3}{2}\left(\frac{T''(1)}{T'(1)}\right)^2=\\
=&\frac{3}{2}-\frac{3}{2}(Q(1)+1)\beta_1+3\beta_1\sum_{k=2}^{N}\frac{1}{\beta_k}\cdot\frac{2\sigma_k}{(\sigma_k-1)^2}+\\
&+\frac{3}{2}(Q(1)+1)(-1+(Q(1)+1)\beta_1)\beta_1-\frac{3}{2}(-1+(Q(1)+1)\beta_1)^2=\\
=&-6\beta_1\sum_{k=2}^{N}\frac{1}{\beta_k}\cdot\frac{1}{|\sigma_k-1|^2}.
\end{aligned}
$$

Finally T is holomorphic in a neighbourhood of $\overline{\Delta}$, and for $z\in\overline{\Delta}$ we have

$$\frac{1-|T(z)|^2}{|\tau-T(z)|^2}=\mathrm{Re}\left(\sum_{k=1}^{N}\frac{1}{\beta_k}\varphi_{\sigma_k}(z)\;+\;a+ib\right)\geq a,$$

and equality holds if and only if $z\in\partial\Delta\setminus\{\sigma_1,\cdots,\sigma_N\}$. This means that

$$T(\Delta)\subset E(\tau,1/a)\quad\text{and}\quad T(\partial\Delta)\subset\partial E(\tau,1/a).$$

But since T is an open map in $\overline{\Delta}$, and $\overline{\Delta}$ is compact and connected

$$T(\Delta)=E(\tau,1/a)\quad\text{and}\quad T(\partial\Delta)=\partial E(\tau,1/a),$$

so T is a proper map.

For $t\in\mathbb{R}$ set $\gamma(t)=T(e^{it})$. Since the curve $T(\partial\Delta)=\partial E(\tau,1/a)$ has constant curvature $1-\frac{1}{a}$, then for all $t\in\mathbb{R}$

$$0=\frac{\mathrm{d}k(t)}{\mathrm{d}t}=\frac{\mathrm{Im}(\mathcal{S}\gamma(t))}{|\gamma'(t)|}=-\frac{\mathrm{Im}((e^{it})^2\mathcal{S}T(e^{it}))}{|T'(e^{it})|}.$$

□

In the following theorem we prove an inequality which is a little bit more general than the analogous one obtained in [4]. Furthermore, this result for $N=1$ and $a=0$ is equivalent to (4).

Theorem 2.3 *Let $f : \Delta \to \Delta$ be a holomorphic map; take $\tau \in \partial\Delta$ and $\sigma_1, \cdots, \sigma_N$ distinct points in $\partial\Delta$ such that $0 < \beta_f(\sigma_k, \tau) < \infty$ for $k = 1, \cdots, N$. Then*

$$\frac{1-|f(z)|^2}{|\tau - f(z)|^2} \geq \sum_{k=1}^{N} \frac{1}{\beta_f(\sigma_k,\tau)} \cdot \frac{1-|z|^2}{|\sigma_k - z|^2} + a \qquad \forall z \in \Delta. \tag{6}$$

where

$$a = \inf_{z\in\Delta} \left\{ \frac{1-|f(z)|^2}{|\tau - f(z)|^2} \right\} \geq 0.$$

Moreover, equality in (6) holds at some $z_0 \in \Delta$ (and then at any $z \in \Delta$) if and only if f coincides with the following rational map:

$$T(z) \stackrel{\text{def}}{=} \varphi_\tau^{-1}\left(\sum_{k=1}^{N} \frac{1}{\beta_f(\sigma_k,\tau)} \cdot \varphi_{\sigma_k}(z) \; + \; a + ib\right) \quad \forall z \in \Delta \tag{7}$$

where $b = \mathrm{Im}(\varphi_\tau(f(0)))$.

Proof. Since, for $\sigma \in \Delta$, $\mathrm{Re}(\varphi_\sigma(z)) = \frac{1-|z|^2}{|\sigma - z|^2}$, the assertion is equivalent to proving that the function

$$F(z) \stackrel{\text{def}}{=} \varphi_\tau(f(z)) - \sum_{k=1}^{N} \frac{1}{\beta_f(\sigma_k,\tau)} \cdot \varphi_{\sigma_k}(z) - a$$

maps Δ into $\overline{H}$, that is $\mathrm{Re}(F(z)) \geq 0 \;\; \forall z \in \Delta$.

To do this, we define - for $n = 0, \cdots, N$ - the maps

$$\begin{cases} F_0(z) = \varphi_\tau(f(z)) - a \\ F_n(z) = F_{n-1}(z) - \frac{1}{\beta_f(\sigma_n,\tau)} \cdot \varphi_{\sigma_n}(z) \end{cases}.$$

and prove, by induction on $n \in \mathbb{N}$, that $\mathrm{Re}(F_n(z)) \geq 0$: the inequality (6) follows just for $n = N$. If $n = 0$ it follows immediately from the definition of a.

Assume that $\mathrm{Re}(F_n(z)) \geq 0$ for $0 \leq n < N$, then the map $f_n = \varphi_\tau^{-1} \circ F_n$ belongs to $\mathrm{Hol}(\Delta, \overline{\Delta})$ and

$$\begin{aligned} \frac{\tau - f_n(z)}{\sigma_{n+1} - z} &= \frac{1}{\tau - f(z)} \cdot \frac{2\tau}{F_n(z) + 1} \cdot \frac{\tau - f(z)}{\sigma_{n+1} - z} = \\ &= \frac{1}{\tau - f(z)} \cdot \frac{2\tau}{\frac{\tau + f(z)}{\tau - f(z)} - \sum_{k=1}^{n} \frac{1}{\beta_f(\sigma_k,\tau)} \cdot \frac{\sigma_k + z}{\sigma_k - z} - a + 1} \cdot \frac{\tau - f(z)}{\sigma_{n+1} - z}. \end{aligned}$$

Passing to the K-limits as $z \to \sigma_{n+1}$ in both sides of the above equation, then $f(z) \to \tau$ and, by applying Theorem 2.1 to the map f, we obtain

$$\text{K-}\lim_{z\to\sigma_{n+1}} \frac{\tau - f_n(z)}{\sigma_{n+1} - z} = \tau\overline{\sigma}_{n+1}\beta_f(\sigma_{n+1},\tau) \neq 0. \tag{8}$$

Moreover $f_n(\Delta) \subset \Delta$, because, otherwise, by the maximum principle, $f_n \equiv \tau$ contradicting (8). Hence, once more from (8), by applying Theorem 2.1 to the map f_n, we have that $\beta_{f_n}(\sigma_{n+1}, \tau) = \beta_f(\sigma_{n+1}, \tau)$ and, from the definition of $\beta_{f_n}(\sigma_{n+1}, \tau)$ we can conclude that

$$\operatorname{Re}(F_n(z)) = \operatorname{Re}(\varphi_\tau(f_n(z))) = \frac{1 - |f_n(z)|^2}{|\tau - f_n(z)|^2} \geq \frac{1}{\beta_f(\sigma_{n+1}, \tau)} \cdot \frac{1 - |z|^2}{|\sigma_{n+1} - z|^2}$$

that is, $\operatorname{Re}(F_{n+1}(z)) \geq 0$.

Assume now that there exists a point $z_0 \in \Delta$ such that the equality in (6) holds, i.e. such that $\operatorname{Re}(F_N(z_0)) = 0$. Then $F_N(z_0) \in \overline{H}$ and by the maximum principle, $F_N \equiv ib$ for some $b \in \mathbb{R}$; since $\varphi_{\sigma_k}(0) = 1$ for $k = 1, \cdots, N$ then $b = \operatorname{Im}(\varphi_\tau(f(0))$, that is $f \equiv T$. $\square$

Theorem 2.4 *Let $f, g \in \operatorname{Hol}(\Delta, \Delta)$ and $\sigma, \tau \in \partial\Delta$ be such that*

$$\operatorname*{r-lim}_{z \to \sigma} \frac{f(z) - g(z)}{(z - \sigma)^3} = l \tag{9}$$

for some $l \in \mathbb{C}$, and

$$\frac{1 - |f(z)|^2}{|\tau - f(z)|^2} \geq \frac{1 - |g(z)|^2}{|\tau - g(z)|^2} \quad \forall z \in \Delta. \tag{10}$$

Then $f \equiv g$ if and only if $l = 0$. Moreover, $\overline{\tau}\sigma^3 l$ is a non-positive real number.

Proof. If $f \equiv g$ then obviously $l = 0$. Assume now that f and g are not indentically equal. We define the holomorphic map $h = -\varphi_\tau^{-1}(\varphi_\tau \circ f - \varphi_\tau \circ g)$, which, from (10), maps Δ into $\overline{\Delta}$. Then, by the maximum principle, if there is a point $z_0 \in \Delta$ such that $h(z_0) \in \partial\Delta$ then h is identically equal to a constant and, from (10), since f and g have the same radial limits at σ, this constant is $-\varphi_\tau^{-1}(0) = \tau$, therefore $f \equiv g$. This contradicts our assumption, so $h \in \operatorname{Hol}(\Delta, \Delta)$. Since $f - g$ never vanishes,

$$h = -\varphi_\tau^{-1}\left(2\tau \frac{f - g}{(\tau - f)(\tau - g)}\right) = \tau \frac{(\tau - f)(\tau - g) - 2\tau(f - g)}{(\tau - f)(\tau - g) + 2\tau(f - g)},$$

therefore for any $z \in \Delta$

$$\frac{\tau - h(z)}{\sigma - z} = \tau \frac{-4\tau \frac{f(z) - g(z)}{(z - \sigma)^3}}{\frac{\tau - f(z)}{\sigma - z} \frac{\tau - g(z)}{\sigma - z} + 2\tau(z - \sigma) \frac{f(z) - g(z)}{(z - \sigma)^3}}.$$

Passing to the radial limits as $z \to \sigma$ in both sides of the above equation, by (9) and by applying Theorem 2.1 to the maps f, g and h, we obtain that

$$\tau\overline{\sigma}\beta_h(\sigma, \tau) = \frac{-4\sigma^2 l}{\beta_f(\sigma, \tau)\beta_g(\sigma, \tau)}$$

which yields to

$$\overline{\tau}\sigma^3 l = -\frac{1}{4}\beta_h(\sigma,\tau)\beta_f(\sigma,\tau)\beta_g(\sigma,\tau) < 0.$$

□

The following corollary states the already recalled result proved by D. Burns and S. Krantz in [3]; observe, furthermore, that neither Herglotz representation nor Hopf's Lemma are required and that only the radial approach is considered.

Corollary 2.5 *Let $f \in \mathrm{Hol}(\Delta,\Delta)$ and $\sigma \in \partial\Delta$ be such that*

$$\operatorname*{r-lim}_{z\to\sigma} \frac{f(z)-z}{(z-\sigma)^3} = 0. \tag{11}$$

Then $f \equiv \mathrm{Id}_\Delta$.

Proof. First we observe that (11) corresponds to condition (9) of Theorem 2.4 for $g = \mathrm{Id}_\Delta$. Moreover, we have

$$\frac{\sigma - f(z)}{\sigma - z} = 1 - \frac{f(z)-z}{(\sigma-z)^3}\cdot(\sigma-z)^2;$$

then, taking the radial limit for $z \to \sigma$, from (11) and by Theorem 2.1 it follows that $\beta_f(\sigma,\sigma) = 1$. Hence, from the definition of $\beta_f(\sigma,\sigma)$ also the condition (10) of Theorem 2.4 is certainly fulfilled for $g = \mathrm{Id}_\Delta$. Therefore, by the previous theorem, $f \equiv \mathrm{Id}_\Delta$.

□

Finally, the announced rigidity theorem.

Theorem 2.6 *Let $f : \Delta \to \Delta$ be a holomorphic map; take $\tau \in \partial\Delta$ and $\sigma_1, \cdots, \sigma_N$ distinct points in $\partial\Delta$ such that $0 < \beta_f(\sigma_k,\tau) < \infty$ for $k = 1, \cdots, N$. If, for some $j \in \{1,\ldots,N\}$, there exist complex numbers $f''(\sigma_j)$ and $f'''(\sigma_j)$ such that*

$$\operatorname*{r-lim}_{z\to\sigma_j} \frac{f(z) - \tau - f'(\sigma_j)(z-\sigma_j) - \frac{1}{2}f''(\sigma_j)(z-\sigma_j)^2 + \frac{1}{6}f'''(\sigma_j)(z-\sigma_j)^3}{(z-\sigma_j)^3} = 0, \tag{12}$$

where $f'(\sigma_j) = \tau\overline{\sigma_j}\beta_f(\sigma_j,\tau)$, and the following relation is satisfied

$$0 \le a \overset{\text{def}}{=} \frac{\mathrm{Re}(\sigma_j^2\overline{\tau}f''(\sigma_j)) - \beta_f(\sigma_j,\tau)(\beta_f(\sigma_j,\tau)-1)}{\beta_f(\sigma_j,\tau)^2} \le \inf_{z\in\Delta}\left\{\frac{1-|f(z)|^2}{|\tau - f(z)|^2}\right\}, \tag{13}$$

then

$$\mathrm{Im}(\sigma_j^2 \mathcal{S}f(\sigma_j)) = 0, \tag{14}$$

$$\mathrm{Re}(\sigma_j^2 \mathcal{S}f(\sigma_j)) \le -6\beta_f(\sigma_j,\tau)\sum_{k=1,k\neq j}^{N}\frac{1}{\beta_f(\sigma_k,\tau)}\cdot\frac{1}{|\sigma_k-\sigma_j|^2}. \tag{15}$$

Moreover, equality in (15) holds if and only if f is identically equal to the rational map

$$T(z) \stackrel{\text{def}}{=} \varphi_\tau^{-1}(\sum_{k=1}^{N} \frac{1}{\beta_f(\sigma_k,\tau)}\varphi_{\sigma_k}(z) \; + \; a + ib) \quad \forall z \in \Delta$$

where

$$b \stackrel{\text{def}}{=} \frac{\operatorname{Im}(\sigma_j^2 \overline{\tau} f''(\sigma_j))}{\beta_f(\sigma_j,\tau)^2} + \sum_{k=1, k\neq j}^{N} \frac{1}{\beta_f(\sigma_k,\tau)} \cdot \frac{2\operatorname{Im}(\sigma_k \overline{\sigma}_j)}{|\sigma_k - \sigma_j|^2}. \tag{16}$$

Proof. First of all, Theorem 2.3 and condition (13) imply that, for any $b \in \mathbb{R}$,

$$\frac{1-|f(z)|^2}{|\tau - f(z)|^2} \geq \sum_{k=1}^{N} \frac{1}{\beta_f(\sigma_k,\tau)} \cdot \frac{1-|z|^2}{|\sigma_k - z|^2} + a = \frac{1-|T(z)|^2}{|\tau - T(z)|^2}. \tag{17}$$

By Lemma 2.2, we already know that, for any $b \in \mathbb{R}$,

$$\tau = T(\sigma_j) \text{ and } f'(\sigma_j) = \tau \overline{\sigma}_j \beta_f(\sigma_j,\tau) = T'(\sigma_j). \tag{18}$$

From the definitions of a in (13) and b in (16), and by Lemma 2.2, we also have

$$\operatorname{Re}(\sigma_j^2 \overline{\tau} f''(\sigma_j)) = \beta_f(\sigma_j,\tau) \cdot (\beta_f(\sigma_j,\tau) - 1) + \beta_f(\sigma_j,\tau)^2 a = \operatorname{Re}(\sigma_j^2 \overline{\tau} T''(\sigma_j));$$

$$\operatorname{Im}(\sigma_j^2 \overline{\tau} f''(\sigma_j)) = \beta_f(\sigma_j,\tau)^2 (b - \sum_{k=1, k\neq j}^{N} \frac{1}{\beta_f(\sigma_k,\tau)} \cdot \frac{2\operatorname{Im}(\sigma_k \overline{\sigma}_j)}{|\sigma_k - \sigma_j|^2}) = \operatorname{Im}(\sigma_j^2 \overline{\tau} T''(\sigma_j)),$$

so that

$$f''(\sigma_j) = T''(\sigma_j). \tag{19}$$

Since T is analytic in a neighbourhood of σ_j, the equations (18) and (19) together with the hypothesis (12) imply that

$$\operatorname*{r-lim}_{z\to\sigma_j} \frac{f(z) - T(z)}{(z-\sigma_j)^3} = \frac{1}{6}(f'''(\sigma_j) - T'''(\sigma_j)) \stackrel{\text{def}}{=} l. \tag{20}$$

From (17) and (20), and by applying Theorem 2.4 for $g = T$ we get that $\overline{\tau}\sigma_j^3 l$ is a non-positive real number. Therefore, from the definition of Schwarzian derivative, we have

$$\overline{\tau}\sigma_j^3 l = \frac{1}{6}\overline{\tau}\sigma_j^3 (f'''(\sigma_j) - T'''(\sigma_j)) = \frac{1}{6}\beta_f(\sigma_j,\tau)\sigma_j^2(\mathcal{S}f(\sigma_j) - \mathcal{S}T(\sigma_j)),$$

which, by Lemma 2.2, immediately yields to (14) and (15).

Furthermore, equality in (15) holds if and only if $l = 0$, and, again by Theorem 2.4, we can conclude that $f \equiv T$. □

Remark 2.7 It is worth making some comments on condition (13) of the previous theorem.

i) First we give a geometric interpretation of condition (13). For $r \in]0,1[$ and $t \in]-\pi,\pi]$ set $\gamma_r(t) = f(r\sigma_j e^{it})$. Since the K-limit $f'(\sigma_j) \neq 0$, there is a $r_0 \in]0,1[$ such that for any $r \in]r_0, 1[$ there exists a neighbourhood V_r of 0 in $]-\pi,\pi]$ such that $\gamma_r'(t) \neq 0$ for all $t \in V_r$. So we can compute the following limit

$$k_\tau \stackrel{\text{def}}{=} \lim_{r\to 1} k_{\gamma_r}(0) = \operatorname{Im}\left(\frac{-\sigma_j^2 f''(\sigma_j) - \sigma_j f'(\sigma_j)}{|f'(\sigma_j)| i\sigma_j f'(\sigma_j)}\right) = \frac{\operatorname{Re}(\overline{\tau}\sigma_j^2 f''(\sigma_j)) + \beta(\sigma_j,\tau)}{\beta(\sigma_j,\tau)^2}.$$

The real number k_τ can be regarded as the curvature at τ of the boundary of $f(\Delta)$. Hence, the corresponding hyperbolic curvature K_τ is easily calculated as

$$K_\tau \stackrel{\text{def}}{=} k_\tau - 1 = \frac{\operatorname{Re}(\sigma_j^2\overline{\tau} f''(\sigma_j)) - \beta_f(\sigma_j,\tau)(\beta_f(\sigma_j,\tau)-1)}{\beta_f(\sigma_j,\tau)^2} = a.$$

On the other hand, the non-negative real number

$$\inf_{z\in\Delta}\left\{\frac{1-|f(z)|^2}{|\tau - f(z)|^2}\right\}$$

is the hyperbolic curvature of the smallest horocycle centered at τ which contains $f(\Delta)$. Therefore the condition (13) says that the horocycle $E(\tau, \frac{1}{K_\tau})$, which, in some sense, better describes locally at τ the boundary of $f(\Delta)$, contains $f(\Delta)$.

ii) If we assume that the map $f \in \operatorname{Hol}(\Delta,\Delta)$ is also C^2 at the point $\sigma_j \in \partial\Delta$ then there is a neighbourhood U of 1 such that

$$0 \le \frac{\operatorname{Re}(\sigma_j^2\overline{\tau} f''(\sigma_j)) - \beta_f(\sigma_j,\tau)(\beta_f(\sigma_j,\tau)-1)}{\beta_f(\sigma_j,\tau)^2} = \inf_{z\in\Delta\cap U}\left\{\frac{1-|f(z)|^2}{|\tau - f(z)|^2}\right\}. \quad (21)$$

Infact, taking $\tau = \sigma_j = 1$, the expansion of f at the point 1 is

$$f(z) = 1 + \beta(z-1) + \frac{1}{2}f''(1)(z-1)^2 + \gamma(z) \ \text{ with } \ \lim_{z\to 1}\frac{\gamma(z)}{(z-1)^2} = 0$$

where $\beta = \beta_f(\sigma_j,\tau)$. After conjugating the map f in the half-plane H by putting $\Phi = \varphi_1 \circ f \circ \varphi_1^{-1}$, we have

$$\frac{2}{1+\Phi(w)} = \beta\cdot\frac{2}{1+w} - f''(1)\cdot\frac{2}{(1+w)^2} - \gamma(\varphi_1^{-1}(w)),$$

and, setting $\Gamma(w) = \frac{1}{2}\gamma(\varphi_1^{-1}(w))(1+w)^2$, which tends to zero as $w \to \infty$, the previous equation can be written for $w = x+iy \in H$ as

$$\begin{aligned}\Phi(w) &= \frac{w^2 + (2-\beta)w + 1 - \beta + f''(1) + \Gamma(w)}{\beta w + \beta - f''(1) - \Gamma(w)} = \\ &= \frac{x^2 - y^2 + (2-\beta)x + 1 - \beta + i(2xy + (2-\beta)y) + f''(1) + \Gamma(w)}{\beta x + \beta + i\beta y - f''(1) - \Gamma(w)}.\end{aligned}$$

Hence, after some little manipulations, we obtain that

$$\operatorname{Re}(\Phi(w)) = \frac{x^3 + y^2(x + 1 + \frac{1}{\beta}\operatorname{Re}(f''(1)) - \beta)}{\beta(x^2 + y^2)} + O(w), \tag{22}$$

where $O(w)$ tends to zero as $w \to \infty$.

Since $f(\Delta) \subset \Delta$, $\operatorname{Re}(w) > 0$ for any $w \in H$. Assume that $\{z_n\}_{n\in\mathbb{N}}$ is a sequence in Δ which tends to 1 tangentially in (22), and set $w_n = x_n + iy_n = \varphi_1(z_n)$. Then $|y_n| \to \infty$, x_n is positive and bounded and

$$\liminf_{n\to+\infty} \operatorname{Re}(\Phi(w_n)) = \frac{1}{\beta}(\liminf_{n\to+\infty} x_n + 1 + \frac{1}{\beta}\operatorname{Re}(f''(1)) - \beta) \geq 0.$$

Therefore, when we impose that $x_n \to 0^+$, we get

$$\lim_{n\to+\infty} \operatorname{Re}(\Phi(w_n)) = \frac{\operatorname{Re}(f''(1)) - \beta(\beta - 1)}{\beta^2} \geq 0. \tag{23}$$

and we have just proved the inequality in (21).

Moreover, if z approaches 1 non-tangentially, that is if $x \to \infty$, then, by (22), $\operatorname{Re}(\Phi(w)) \to \infty$ and we can conclude that there exists a neighbourhood U of 1 where

$$\frac{\operatorname{Re}(f''(1)) - \beta(\beta - 1)}{\beta^2} \leq \operatorname{Re}(\Phi(w)) = \frac{1 - |f(z)|^2}{|\tau - f(z)|^2} \quad \forall z \in \Delta \cap U.$$

So, remembering (23), also the equality in (21) is proved.

Condition (21) is strictly weaker than condition (13) because of the following example:

$$f(z) = z + \frac{1}{2}(z-1)^2 + \frac{1}{4}(z-1)^3 \quad \forall z \in \Delta.$$

Infact, one can prove that $f \in \operatorname{Hol}(\Delta, \Delta)$ and that f is C^2 at 1; moreover $f(1) = f'(1) = f''(1) = 1$ and then $a = 1$, whereas $\inf_{z\in\Delta}\left\{\frac{1-|f(z)|^2}{|1-f(z)|^2}\right\} = 0$ since f has another fixed point on the boundary $\partial\Delta$, namely -1. Nevertheless one has that $\mathcal{S}f(1) = 0$.

iii) Notice that in [2], always assuming for f the regularity C^2 at σ_j, a particular case of the inequality (21), precisely when $\beta_f(\sigma_j, \tau) = 1$, appears at p. 52. Moreover if $\operatorname{Re}(\sigma_j^2\overline{\tau}f''(\sigma_j)) = 0$, at p. 67, it is proved that

$$\operatorname{Im}(\sigma_j^2\mathcal{S}f(\sigma_j)) = 0 \text{ and } \operatorname{Re}(\sigma_j^2\mathcal{S}f(\sigma_j)) \leq 0$$

which are certainly implied by (14) and (15).

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