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Geometria. — *On fixed points of C^1 extensions of expanding maps in the unit disc.* Nota di Roberto Tauraso, presentata (*) dal Socio E. Vesentini.

Abstract. — Using a result due to M. Shub, a theorem about the existence of fixed points inside the unit disc for C^1 extensions of expanding maps defined on the boundary is established. An application to a special class of rational maps on the Riemann sphere and some considerations on ergodic properties of these maps are also made.

Key words: Fixed point; Expanding map; Blaschke product; Invariant measure.

Riassunto. — *Punti fissi di estensioni C^1 di funzioni espansive nel disco unitario.* Sulla base di un risultato di M. Shub, si dimostra un teorema riguardante la presenza di punti fissi all'interno del disco unitario per estensioni C^1 di funzioni espansive definite sul bordo. La *Nota* si conclude con un'applicazione ad una classe di funzioni razionali della sfera di Riemann e alcune considerazioni sulle proprietà ergodiche di tali funzioni.

1. Let $D^2 \stackrel{d}{=} \{z \in \mathbf{C}: |z| \leq 1\}$ be the closed unit disk of $\mathbf{C}$, $S^1 \stackrel{d}{=} \{z \in \mathbf{C}: |z| = 1\}$ its boundary in $\mathbf{C}$ and $\text{int}(D^2) \stackrel{d}{=} D^2 \setminus S^1$ its interior in $\mathbf{C}$.

Let X be a «nice» topological space (in our case X will be either D^2 or S^1) and $f: X \to X$ be a continuous map with a finite number of fixed points in X. It is possible to associate to each fixed point p of f an integer $i(X, f, p)$, called the «index», which describes the way in which the map, locally, «winds around» the point. If U is a non-empty open set of X and ∂U is its boundary in X, then we denote by $\mathcal{C}(X, U)$ the set of all continuous maps $f: X \to X$ with a finite number of fixed points in U and, if ∂U is not empty, none of them in ∂U. Then, the index of $f \in \mathcal{C}(X, U)$ on U, $i(X, f, U)$, is the sum of the indices of the fixed points of f which lie in U.

Its main properties are (see [3]):

1) Localization: if $f, g \in \mathcal{C}(X, U)$ and $f(x) = g(x)$ for all $x \in U$ then $i(X, f, U) = i(X, g, U)$.

2) Homotopy: if $H: X \times [0, 1] \to X$ is a homotopy and $f_t(\cdot) \stackrel{d}{=} H(\cdot, t) \in \mathcal{C}(X, U)$ for all $t \in [0, 1]$ then $i(X, f_0, U) = i(X, f_t, U)$ $\forall t \in [0, 1]$.

3) Additivity: let $f \in \mathcal{C}(X, U)$ and $U_1, \dots, U_s$ be a set of mutually disjoint open subsets of U such that $U \setminus \bigcup_{j=1}^{s} U_j$ does not contain any fixed point of f, then $f \in \mathcal{C}(X, U_j)$ for $j = 1, \dots, s$ and $i(X, f, U) = \sum_{j=1}^{s} i(X, f, U_j)$.

4) Normalization: if we denote by $L(X, f)$ the Lefschetz number of $f \in \mathcal{C}(X, X)$ in X (see [3]), then $i(X, f, X) = L(X, f)$.

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5) Commutativity: if $f, g: X \to X$ are continuous maps such that $g \circ f \in \mathcal{C}(X, U)$, then $f \circ g \in \mathcal{C}(X, g^{-1}(U))$ and $i(X, g \circ f, U) = i(X, f \circ g, g^{-1}(U))$.

Let $X = D^2$ and $f \in \mathcal{C}(D^2, U)$ with U open set of D^2 containing only one fixed point p, then, if we define for all $z \in \mathbf{C}$

$$F(z) \stackrel{d}{=} \begin{cases} f(z) & \text{if } z \in D^2 , \\ f(z/|z|) & \text{otherwise} , \end{cases}$$

the index $i(X, f, U) = i(X, f, p)$ is the local degree of the map Id-F restricted to an appropriately small open set about 0.

Moreover, since D^2 is simply connected, every continuous map $f: D^2 \to D^2$ is homotopic to the constant map identically zero and we have $L(D^2, f) = 1$ for all $f \in \mathcal{C}(D^2, D^2)$.

2. Choose a fixed C^1 map $\varphi: S^1 \to S^1$. If $p \in S^1$ is an isolated fixed point of φ we will say that φ is transversally fixed in p if the derivative of φ in p, $D_p \varphi$, is different from 1 (*i.e.* the multiplicity of the fixed point p is 1).

Let $E^1(\varphi)$ be the set of all smooth extensions of φ inside D^2: $E^1(\varphi) \stackrel{d}{=} \{ f: D^2 \to \to D^2 : f \in C^1(D^2) \text{ and } f|_{S^1} \equiv \varphi \}$.

If $f \in E^1(\varphi) \cap \mathcal{C}(D^2, D^2)$ then the following theorems hold (see [4, 5]):

THEOREM 2.1. *If φ is transversally fixed in $p \in S^1$ then either $i(D^2, f, p) = 0$ or $i(D^2, f, p) = i(S^1, \varphi, p)$ which is either 1 or -1.*

THEOREM 2.2. *If φ is transversally fixed in $p \in S^1$ and $i(D^2, f, p) = 0$ then, chosen a neighborhood V of p in D^2 containing no other fixed point of f, there exists a homotopy $H: D^2 \times [0, 1] \to D^2$ such that, if $f_t(\cdot) \stackrel{d}{=} H(\cdot, t) \in \mathcal{C}(X, U)$ for all $t \in [0, t]$, then: $f_0 \equiv f$ in D^2, $f_t \equiv f$ in $(D^2 \setminus V) \cup S^1$ for all $t \in [0, 1]$ and $f_1 \in E^1(\varphi)$ has one and only one fixed point q in $V \cap \operatorname{int}(D^2)$. Moreover, $i(D^2, f_1, q) = -i(S^1, \varphi, p)$ while $i(D^2, f_1, p) = i(S^1, \varphi, p)$.*

THEOREM 2.3. *If $i(D^2, f, \operatorname{int}(D^2)) = 0$, there exists a $\tilde{f} \in E^1(\varphi)$ that has no fixed points in $\operatorname{int}(D^2)$.*

3. We shall say that a C^1 map $\varphi: S^1 \to S^1$ is expanding on S^1 (see [12, 9]) if there exist real numbers $c > 0$ and $\lambda > 1$ such that $|D_x \varphi^k| \geq c\lambda^k \ \forall x \in S^1$ and $\forall k \in \mathbf{N}$, where $\varphi^k = \varphi \circ \ldots \circ \varphi$ is the k-th iterate of φ.

The most trivial example of expanding maps on S^1 are the «rotations» $\Phi_N(x) \stackrel{d}{=} x^N$ $\forall x \in S^1$, with N integer such that $|N| \geq 2$. We can easily note that $\Phi_N \in \mathcal{C}(S^1, S^1)$ and if $p \in S^1$ is a fixed point of Φ_N then, by Theorem 2.1 and the properties of the index stated in the first section,

$$i(S^1, \Phi_N, p) = \begin{cases} -1 & \text{if } N \geq 2 , \\ 1 & \text{if } N \leq -2 . \end{cases} \tag{1}$$

Moreover, Shub has proved (see [12, 9]) that these «rotations» allow us to classify by conjugation the smooth expanding maps on S^1. The crucial device we need is the topological degree, $\deg\varphi$, of a map $\varphi\colon S^1 \to S^1$, that is the number of windings around S^1 of the path $\varphi(e^{2\pi i t})$ with t from 0 to 1; a winding is counted positively if counterclockwise and negatively in the other case (see [7]).

THEOREM 3.1. If the C^1 map $\varphi\colon S^1 \to S^1$ is expanding on S^1, then there exists a homeomorphism h of S^1 such that $h \circ \varphi \circ h^{-1} \equiv \Phi_N$ on S^1 where $N = \deg \varphi$ with $|N| \geq 2$.

Now, we can establish the main theorem of this *Note* that generalizes a similar result obtained in [4, 5] in the case of the «rotations»:

THEOREM 3.2. Let $\varphi\colon S^1 \to S^1$ be a C^1 map expanding on S^1 and $N = \deg \varphi$.

1) If $N \geq 2$, then every $f \in E^1(\varphi)$ has a fixed point in $\operatorname{int}(D^2)$.

2) If $N \leq -2$, there exists a map $f \in E^1(\varphi)$ that has no fixed point in $\operatorname{int}(D^2)$.

PROOF. By Shub's theorem there is a homeomorphism h that conjugates φ to the «rotation» Φ_N. By this conjugation, φ has the same number of fixed points of Φ_N, that is $|N| - \operatorname{sign}(N) \geq 1$. Hence the set $\operatorname{Fix} \varphi \overset{d}{=} \{x \in S^1 \colon \varphi(x) = x\}$ is not empty and finite. This means that $\varphi \in \mathcal{C}(S^1, S^1)$ and, if $p \in S^1$ is a fixed point of φ, then $h(p)$ is the corresponding fixed point of Φ_N and, by the commutativity of the index we have

$$(2) \quad i(S^1, \varphi, p) = i(S^1, h^{-1} \circ (h \circ \varphi), p) = i(S^1, (h \circ \varphi) \circ h^{-1}, h(p)) = i(S^1, \Phi_N, h(p)) .$$

To apply the theorems stated in the previous section we have to show that φ is transversally fixed in each fixed point $p \in S^1$. In fact: $D_p \varphi^k = D_{\varphi^{k-1}(p)} \varphi \cdot D_{\varphi^{k-2}(p)} \varphi \ldots D_p \varphi = (D_p \varphi)^k$, and, since φ is expanding on S^1, then for all $k \geq 1$ $|D_p \varphi|^k = |D_p \varphi^k| \geq c\lambda^k$; this implies that $|D_p \varphi| \geq \lambda > 1$.

Now we distinguish the two cases.

1) If $N \geq 2$, we assume that there exists a map $f \in E^1(\varphi)$ without any fixed point in $\operatorname{int}(D^2)$.

Since $f \in \mathcal{C}(D^2, D^2)$, by the normalization property we have the contradiction

$$1 = L(D^2, f) = \sum_{p \in \operatorname{Fix} \varphi} i(D^2, f, p) \leq 0 ,$$

because, by Theorem 2.1, (2) and (1), either $i(D^2, f, p) = 0$ or $i(D^2, f, p) = i(S^1, \Phi_N, h(p)) = -1$.

2) If $N \leq -2$, we extend φ inside D^2 in the following manner

$$f_0(z) = \begin{cases} |z|^2 \varphi(z/|z|) & \text{if } z \in D^2 \setminus \{0\} , \\ 0 & \text{if } z = 0 . \end{cases}$$

It is easy to see that $f_0 \in E^1(\varphi)$ and that 0 is the only fixed point of f_0 in $\operatorname{int}(D^2)$. Since $|z| > |f_0(z)|$ for $0 \neq z \in \operatorname{int}(D^2)$, then $i(D^2, f_0, 0) = 1$. Moreover, if $p \in \operatorname{Fix} \varphi$, by Theorem 2.1, (2) and (1), either $i(D^2, f_0, p) = 0$ or $i(D^2, f_0, p) = i(S^1, \Phi_N, h(p)) = 1$.

Since, by the normalization property,

$$\sum_{p \in \mathrm{Fix}\,\varphi} i(D^2, f_0, p) = L(D^2, f_0) - i(D^2, f_0, 0) = 1 - 1 = 0$$

then $i(D^2, f_0, p) = 0$ for all $p \in \mathrm{Fix}\,\varphi$. Choosing one of the fixed points $p \in S^1$ of φ then, by Theorem 2.2 there exists a map $f_1 \in E^1(\varphi)$ which coincides with f near 0, and therefore has the same index: $i(D^2, f_1, 0) = i(D^2, f_0, 0)$. Besides, f_1 has only another fixed point $q \in \mathrm{int}\, D^2$ and $i(D^2, f_1, q) = -i(D^2, f_1, p) = -i(S^1, \varphi, p) = -1$. Hence, $i(D^2, f_1, \mathrm{int}\, D^2) = i(D^2, f_1, 0) + i(D^2, f_1, q) = 0$ and, by Theorem 2.3, there exists $f_2 \in E^1(\varphi)$ without any fixed point in int (D^2). q.e.d.

Note that, in theorem, when $N \geqslant 2$, we can not weaken the hypothesis on the regularity of the extension f of φ because there exists a continuous extension which has no fixed point in int (D^2) (see [4]).

Besides, it follows directly from the definition of expanding map and the previous theorem that

Corollary 3.3. Let $\varphi: S^1 \to S^1$ be a C^1 map with $\deg \varphi = N \geqslant 2$. If $\min_{x \in S^1} |D_x \varphi| > 1$ then φ and $\bar{\varphi}$ are expanding on S^1 and since $\deg \bar{\varphi} = -N \leqslant -2$, every $f \in E^1(\varphi)$ has a fixed point in int (D^2), while there exists a map $f_0 \in E^1(\varphi)$ such that $\bar{f}_0$ has no fixed point in int (D^2).

4. Let $\hat{C} \stackrel{d}{=} C \cup \{\infty\}$ be the Riemann sphere and take a rational map $f: \hat{C} \to \hat{C}$ with degree $\geqslant 2$. The Fatou set of f, $\mathcal{F}$, that is the largest open set in $\hat{C}$ where the sequence of iterates $\{f^k\}$ is normal; let $\mathcal{J} \stackrel{d}{=} \hat{C} \setminus \mathcal{F}$ be the Julia set of f.

Classical properties of the Fatou set are that: $\mathcal{F}$ is completely invariant (*i.e.* $f(\mathcal{F}) = f^{-1}(\mathcal{F}) = \mathcal{F}$) and, if it is not empty, then it has one, two or else infinitely many open connected components.

Moreover (see for example [13, 2]), if f is expanding on $\mathcal{J}$, *i.e.*

$$\exists c > 0, \lambda > 1: \left| \frac{df^k}{dz}(x) \right| \geqslant c\lambda^k \qquad \forall x \in \mathcal{J} \quad \text{and} \quad \forall k \in \mathbf{N}$$

and if $\mathcal{F}$ has exactly two invariant components, then $\mathcal{J}$ is the common boundary of the components and is a Jordan curve. It can be either a circle in $\hat{C}$ or a highly irregular curve with tangents nowhere. In both cases, each component contains an attracting fixed point.

For example, a finite Blaschke product B (see for example [11]):

$$B(z) \stackrel{d}{=} e^{i\theta} \prod_{j=1}^{N} \left(\frac{z - a_j}{1 - \bar{a}_j z} \right) \qquad \forall z \in \hat{C}$$

where N is a positive integer, $\theta \in \mathbf{R}$ and $a_1, \dots, a_N \in \mathrm{int}\,(D^2)$, is a rational map with degree N such that $B(S^1) = S^1$, $B(\mathrm{int}\,(D^2)) = \mathrm{int}\,(D^2)$ and $B(\hat{C} \setminus D^2) = \hat{C} \setminus D^2$. Assume $N \geqslant 2$, then the Julia set of B can be either S^1 or a Cantor set contained in S^1. It is worth to note that, $B|_{S^1}$ is expanding on S^1 iff B has a fixed point in int (D^2) iff B has a fixed point in $\hat{C} \setminus D^2$. In this case the two fixed points in $\hat{C}\setminus S^1$ are attracting and symmetric with respect to S^1, while the Julia set of B is just S^1 and $\mathcal{F} = \text{int}(D^2) \cup \hat{C}\setminus D^2$.

Now we are ready to prove the following result.

PROPOSITION 4.1. Let $f: \hat{C} \to \hat{C}$ be a rational map with degree $N \geq 2$ and assume that $\mathcal{F}$ has exactly two invariant components, say V and W, f is expanding on $\mathcal{J}$ and $\mathcal{J}$ is a circle in $\hat{C}$ then if $\tilde{f}$ is a C^1 extension on $\bar{V}$ (on $\bar{W}$) of $f|_{\mathcal{J}}$ then $\tilde{f}$ has a fixed point in V (in W).

PROOF. Let f be a C^1 extension on $\bar{V}$ of $f|_{\mathcal{J}}$. Since $\mathcal{J}$ is a circle in $\hat{C}$ and $\mathcal{J}$ is the boundary of the domain V there exists a fractional linear map T such that $T(\mathcal{J}) = S^1$, $T(V) = \text{int}(D^2)$ and $T(a_V) = 0$ where a_V is the attracting fixed point of f in V. If we conjugate f by T then we obtain a Blaschke product $B = T \circ f \circ T^{-1}$ with degree $N \geq 2$ and a fixed point in 0. Now, let $\varphi \equiv B|_{S^1}$ then for all $x \in S^1$ and $k \geq 1$

$$|D_x \varphi^k| = \left| \frac{dB^k}{dz}(x) \right| \geq \lambda^k$$

where $\lambda \overset{d}{=} \min_{S^1} |dB/dz| > 1$ (see [1]). Since $\deg \varphi = N \geq 2$ and $T \circ \tilde{f} \circ T^{-1} \in E^1(\varphi)$, by Corollary 3.3, φ is expanding on S^1 and $T \circ \tilde{f} \circ T^{-1}$ has a fixed point in $\text{int}(D^2)$. This means that $\tilde{f}$ has a fixed point in V. q.e.d.

5. The expanding maps have some interesting ergodic properties that are summarized in the following result which is a particular case of a general theorem due to Walters (see [14]):

THEOREM 5.1. Let $\varphi: S^1 \to S^1$ be a C^2 map expanding on S^1. Then there exists an invariant probability measure μ for φ which is equivalent to the normalized Lebesgue measure σ. The following properties hold:

1) $\sigma \circ \varphi^{-k} \overset{*}{\rightharpoonup} \mu$ (where $\overset{*}{\rightharpoonup}$ denotes the convergence in the weak* topology).

2) φ is an exact endomorphism with respect to μ, that is: if $E \in \bigcap_{k=0}^{\infty} \varphi^{-k}(\mathcal{B})$ then $\mu(E)$ is either 0 or 1, where $\mathcal{B}$ is the σ-algebra of the borelian sets of S^1.

3) The entropy of φ with respect to μ is:

$$h_\mu(\varphi) = \int_{S^1} \log(|D_x \varphi|)\, d\mu(x).$$

Now, if φ is the restriction of a finite Blaschke product B, we can ask ourselves if there is any connection, in this special case, between the fixed point $a \in \text{int}(D^2)$ of B (see the previous section) and the invariant measure of the Theorem 5.1.

In fact, since the sequence of iterates $\{B^k(0)\}$ converges to the fixed point a (see [1, 6]), then $\sigma \circ \varphi^{-k} = \sigma_0 \circ B^{-k} = \sigma_{B^k(0)} \overset{*}{\rightharpoonup} \sigma_a = \mu$, where σ_y is the harmonic probability measure associated to $y \in \operatorname{int}(D^2)$ (see [10]),

$$\frac{d\sigma_y}{d\sigma}(x) = \frac{1-|y|^2}{|y-x|^2} \qquad \forall x \in S^1 .$$

Again by Theorem 5.1, the entropy formula is:

$$h_{\sigma_a}(\varphi) = \int_{S^1} \log\left(\left|\frac{dB}{dz}(x)\right|\right) d\sigma_a(x) .$$

Moreover, following [8] (see also [15]), by the conjugation property of the entropy and the variational principle (see also [15]) $0 < \log \lambda \leqslant h_{\sigma_a}(\varphi) \leqslant h(\varphi) = h(\Phi_N) = \log N < \infty$.

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