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Geometria. — *On fixed points of holomorphic maps of simply connected proper domains in* $\boldsymbol{C}$. Nota di ROBERTO TAURASO, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — A criterion for the existence of fixed point of one-dimensional holomorphic maps is established.

KEY WORDS: Fixed point; Holomorphic map; Wolff point.

RIASSUNTO. — *Punti fissi di funzioni olomorfe.* Si stabilisce un criterio di esistenza di punto fisso per funzioni olomorfe di un dominio proprio, semplicemente connesso di $\boldsymbol{C}$.

Let D be a simply connected, proper domain in $\boldsymbol{C}$, and let f be a holomorphic map of D into D, different from the identity map. According to the Denjoy-Wolff theorem, unless F is an elliptic automorphism of D, the iterates $f^k = f \circ f \ldots \circ f$ of f converge as $k \to \infty$, for the topology of uniform convergence on compact sets, to a constant function, mapping D onto a point $c \in \overline{D}$ (the closure of D). If $c \in D$ then $f(c) = c$ and c is the unique fixed point of f. In the present *Note*, a sufficient condition for the existence of a fixed point $c \in D$ of f will be established, together with a localization of c.

After collecting some known facts in § 1, § 2 will be devoted to investigating the case of the open unit disc and § 3 to the general case.

1. Let $\Delta = \{z \in \boldsymbol{C}: |z| < 1\}$ be the open unit disk of $\boldsymbol{C}$. For $a \in \Delta$ the Möbius transformation

$$M_a(z) = \frac{z - a}{1 - \bar{a}z} \qquad \forall z \in \Delta$$

is a holomorphic automorphism of Δ, which can be extended continuously to a homeomorphism of $\overline{\Delta}$ onto itself. This extension will be denoted by the same symbol M_a.

On Δ we introduce the Poincaré distance $\rho(z, w) = \tanh^{-1} |M_w(z)|$, $\forall z, w \in \Delta$ and define the open ρ-ball of center $w \in \Delta$ and radius $R > 0$: $B_\rho(w, R) = \{z \in \Delta: \rho(z, w) < R\} \subset\subset \Delta$, and the horocycle of center $\tau \in \partial\Delta$ and radius $R > 0$: $E(\tau, R) = \{z \in \Delta: |\tau - z|^2/(1 - |z|^2) < R\} \subset \Delta$. Then $\overline{E(\tau, R)} \cap \partial\Delta = \{\tau\}$ and the open sets $B_\rho(w, R)$ and $E(\tau, R)$ are euclidean disks contained in Δ such that

$$\bigcup_{R>0} B_\rho(w, R) = \bigcup_{R>0} E(\tau, R) = \Delta .$$

For any $f \in \mathrm{Hol}(\Delta, \Delta)$, *i.e.* a holomorphic map f from Δ to Δ, let Fix f be the set of fixed points of f: $\mathrm{Fix}\, f \stackrel{d}{=} \{z \in \Delta: f(z) = z\}$. We collect here some known facts (cf. *e.g.* [1]):

(*) Nella seduta dell'8 gennaio 1994.

1) f is a contraction for the distance ρ

(1) $$\rho(f(z), f(w)) \leqslant \rho(z, w) \quad \forall z, w \in \Delta;$$

moreover, equality holds for some $z \neq w \in \Delta$ iff it holds for every z, $w \in \Delta$ iff $f \in \mathrm{Aut}(\Delta)$.

2) (*Julia's Lemma*). Let $\sigma \in \partial\Delta$ and

$$\liminf_{z \to \sigma} \frac{1 - |f(z)|}{1 - |z|} \stackrel{d}{=} \lambda_f(\sigma).$$

If $\lambda_f(\sigma) < \infty$ then there exists a unique $\tau \in \partial\Delta$ such that $f(E(\sigma, R)) \subset E(\tau, \lambda_f(\sigma) R)$, $\forall R > 0$; moreover

$$\lim_{r \to 1^-} f(r\sigma) = \tau \quad \text{and} \quad \lim_{r \to 1^-} |f'(r\sigma)| = \lambda_f(\sigma).$$

3) (*Wolff's Lemma*). If $\mathrm{Fix} f = \emptyset$ then there exists a unique point $\tau = \tau(f) \in \partial\Delta$, Wolff point of f, such that

(2) $$f(E(\tau, R)) \subset E(\tau, R) \quad \forall R > 0.$$

4) As a consequence of 1), if f has two different fixed points in Δ then f is the identity map in Δ.

5) If f is not an elliptic automorphism then the sequence of iterates $\{f^k\}_N$ converges, uniformly on compact sets of Δ, to a point c of $\bar{\Delta}$. If $\mathrm{Fix} f \neq \emptyset$ then $c \in \Delta$ and $f(c) = c$; if $\mathrm{Fix} f = \emptyset$ then $c = \tau(f) \in \partial\Delta$, the Wolff point of f.

The next result was established by Goebel [6] in a more general context and will be useful in the following.

For α, $\beta \in \Delta$, let $K_\alpha^\beta \stackrel{d}{=} \{z \in \bar{\Delta} : |1 - \bar{\beta} z|^2/(1 - |\beta|^2) \leqslant |1 - \bar{\alpha} z|^2/(1 - |\alpha|^2)\}$, and let

(3) $$K \stackrel{d}{=} \bigcap_{\alpha \in \Delta} K_\alpha^{f(\alpha)}.$$

If $\mathrm{Fix} f \neq \emptyset$ then $K = \mathrm{Fix} f$, otherwise $K = \tau(f)$.

Now, we conclude this first part with some classical results on bounded holomorphic function theory (see [8, 5, 7]). Consider a family $\{\alpha_j\}_J$ of points in Δ (not necessarily all different), indexed by a set J of consecutive positive integers starting from 1. With $\# J$ we will mean the cardinality of the set J.

Set for $1 \leqslant n \leqslant \# J$

$$B_n(z) \stackrel{d}{=} \prod_{j=1}^{n} (-|\alpha_j|/\alpha_j)((z - \alpha_j)/(1 - \bar{\alpha}_j z)) \quad \forall z \in \Delta$$

with the convention that $|\alpha_j|/\alpha_j = 1$ when $\alpha_j = 0$. If the family $\{\alpha_j\}_J$ is such that $\sum_{j \in J} (1 - |\alpha_j|) < \infty$ then we can define the Blaschke product B associated to that family: if J is empty then $B(z) \stackrel{d}{=} 1$ for all $z \in \Delta$, if J is finite then B is B_n with $n = \# J$, while in the infinite case we set

$$B(z) \overset{d}{=} \lim_{n \to \infty} B_n(z) \quad \forall z \in \Delta .$$

REMARK. The definition of B is independent of the ordering of the elements α_j.
The principal properties of the Blaschke product are:

1) when $\# J = \infty$ then the partial products $B_n \to B$ uniformly on compact sets of Δ;

2) $B \in \mathrm{Hol}(\Delta, \Delta)$;

3) $|B(r\sigma)| \to 1$ when $r \to 1^-$ for a.e. $\sigma \in \partial\Delta$ with respect to the Lebesgue measure on $\partial\Delta$ (that is B is an inner map);

4) the zeros of B in Δ are exactly $\{\alpha_j\}_J$ and a zero in the family is repeated as many times as its multiplicity.

The map $f \in \mathrm{Hol}(\Delta, \Delta)$ has a factorization of the form

$$f(z) = B(z)g(z) \quad \forall z \in \Delta \tag{4}$$

where B is a Blaschke product with zeros the family $\{\alpha_j\}_J$ that are exactly the zeros of f with the same multiplicities and $g \in \mathrm{Hol}(\Delta, \Delta)$ is without zeros in Δ.

2. It is easy to verify that if $\sigma, \tau \in \partial\Delta$, $t_0 > 0$ and $f(E(\sigma, R)) \subset E(\tau, t_0 R)$ for all $R > 0$ then $0 < \lambda_f(\sigma) = \min\{t > 0 : f(E(\sigma, R)) \subset E(\tau, tR)\ \forall R > 0\} \leqslant t_0 < \infty$. For this reason $\lambda_f(\sigma)$ is called the boundary dilatation coefficient.

Hence, by Wolff's lemma, if f has not a fixed point in Δ then

$$\lambda_f(\tau(f)) \leqslant 1 . \tag{5}$$

The next proposition follows easily from some basic results due to Carathéodory (see [3, Sections 298-300] and cf. also [2]):

PROPOSITION 2.1. Let f, g and h be maps $\in \mathrm{Hol}(\Delta, \Delta)$, such that $f = gh$ in Δ (g and h are divisors of f) then

$$\lambda_f(\sigma) = \lambda_g(\sigma) + \lambda_h(\sigma) \quad \forall \sigma \in \partial\Delta . \tag{6}$$

Moreover let $\{f_n\}_N \subset \mathrm{Hol}(\Delta, \Delta)$, if f_n is divisor of f, *i.e.* $f = f_n g_n$ with $g_n \in \mathrm{Hol}(\Delta, \Delta)$, for every n and $f_n \to f$ uniformly on compact sets of Δ, then

$$\lambda_{f_n}(\sigma) \to \lambda_f(\sigma) \quad \forall \sigma \in \partial\Delta . \tag{7}$$

Now, since the following relation holds

$$1 - |M_a(z)|^2 = ((1 - |a|^2)(1 - |z|^2))/|1 - \bar{a}z|^2 \quad \forall z, w \in \bar{\Delta} , \tag{8}$$

it is easy to compute λ_f when f is a Blaschke product:

LEMMA 2.2. Let B be the Blaschke product associated to the family $\{\alpha_j\}_J$

then for all $\sigma \in \Delta$

$$\lambda_B(\sigma) = \sum_{j \in J} (1 - |\alpha_j|^2)/|\sigma - \alpha_j|^2 .$$

Proof. If the family $\{\alpha_j\}_J$ is empty then there is nothing to prove.

Assume that $\# J \geq n > 0$: we can write the partial product of order n, B_n as product of n Möbius transformations

$$B_n(z) = e^{i\theta_n} \prod_{j=1}^{n} M_{\alpha_j}(z) \quad \text{with} \quad e^{i\theta_n} = \prod_{j=1}^{n} (-|\alpha_j|/\alpha_j) \in \partial\Delta .$$

Hence (6) and (8) yield for $\sigma \in \partial\Delta$

$$\lambda_{B_n}(\sigma) = \sum_{j=1}^{n} \lambda_{M_{\alpha_j}}(\sigma) = \sum_{j=1}^{n} (1 - |\alpha_j|^2)/|\sigma - \alpha_j|^2 .$$

If $\# J = \infty$, since $B_n \to B$ uniformly on compact set of Δ, then by (7)

$$\lambda_B(\sigma) = \lim_{n \to \infty} \lambda_{B_n}(\sigma) = \sum_{j=1}^{\infty} (1 - |\alpha_j|^2)/|\sigma - \alpha_j|^2 . \qquad \square$$

For $\alpha, \beta \in \Delta$ the set K_α^β (defined in § 1) depends essentially on the distance function ρ. In fact by (8) it is easy to prove that $K_\alpha^\beta \cap \Delta = \{z \in \Delta \colon \rho(z, \beta) \leq \rho(z, \alpha)\}$. Namely, in the case when β and α are different, the part of Δ that contains β and is delimited by the non-euclidean bisector of the non-euclidean segment with extreme points α and β, while $K_\alpha^\alpha = \bar{\Delta}$:

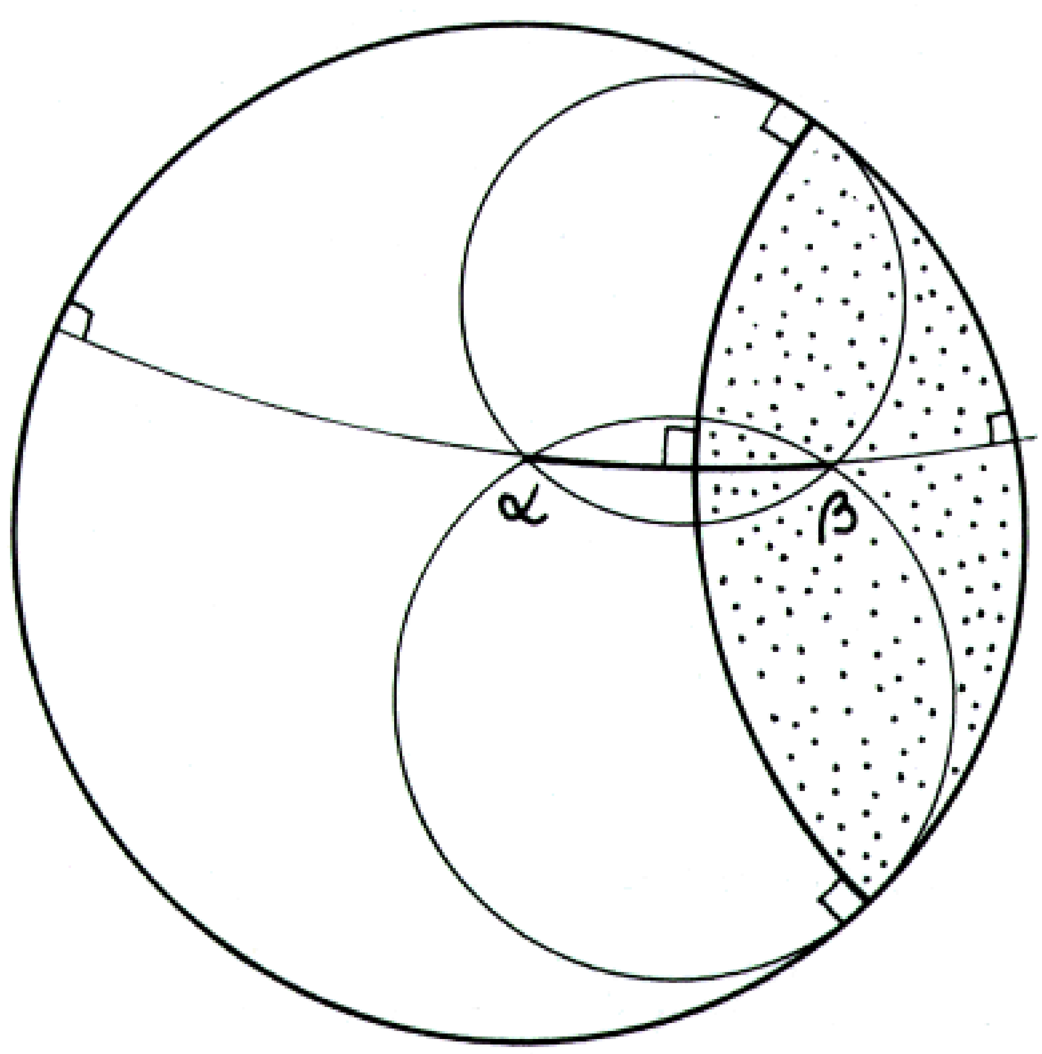

Fig. 1. – The set K_α^β is the dotted part of the picture.

3. If $D \subset \mathbf{C}$ is a domain we can define the Carathéodory pseudo-distance on D (see for example [4]) by $\rho_D(z, w) \stackrel{d}{=} \sup\{\rho(g(z), g(w)) \colon g \in \mathrm{Hol}(D, \Delta)\}$. This pseudo-distance is contracted by holomorphic maps, in the sense that if D_1 and D_2 are two domains of $\mathbf{C}$ and $F \in \mathrm{Hol}(D_1, D_2)$, then $\rho_{D_2}(F(z), F(w)) \leq \rho_{D_1}(z, w)$ $\forall z, w \in D_1$. Since

$\rho_\Delta = \rho$, Riemann's mapping theorem implies that if D is a proper simply connected domain of $\mathbf{C}$ and F is any biholomorphic map from D onto Δ then ρ_D is a distance in D and

$$\rho_D(z, w) = \rho(F(z), F(w)) = \tanh^{-1} |M_{F(z)}(F(w))| \quad \forall z, w \in D. \tag{9}$$

So, it is possible to define, likewise the case of $D = \Delta$,

$$K_\alpha^\beta\{D, \rho_D) \stackrel{d}{=} \{z \in D: \rho_D(z, \beta) \leqslant \rho_D(z, \alpha)\} \quad \forall \alpha, \beta \in D. \tag{10}$$

Let D be a proper simply connected domain of $\mathbf{C}$ and $f \in \mathrm{Hol}(D, D)$. Assume that f is neither constant nor the identity map. Then, for $\zeta \in D, f^{-1}(\zeta)$ is a descrete subset of D. Fixing arbitrarly an ordering and repeating each element with its multiplicity, we construct from this set the family $\{\alpha_j\}_J$ of the counterimages of ζ. The following theorem yields a sufficient condition about the geometrical behaviour of the counterimages of ζ for the existence and uniquess of a fixed point of f in D.

THEOREM 3.1. If there exists $R \geqslant 0$ such that

$$\#\{j \in J: \alpha_j \in B_{\rho_D}(\zeta, R) \cup \{\zeta\}\} \stackrel{d}{=} C(\zeta, R) \geqslant (1 + \tanh R)/(1 - \tanh R) \tag{11}$$

then f has one fixed point in D. Furthermore, this fixed point belongs to the set $\bigcap_{j \in J} K_{\alpha_j}^\zeta(D, \rho_D)$.

PROOF. By (9) and (10), is is sufficient to prove the theorem in the case $D = \Delta$.

Uniqueness follows from § 1. Since the case $R = 0$ is trivial, assume that $R > 0$. The map f has a fixed point in Δ iff the same happens to $\tilde{f} = M_\zeta \circ f \circ M_\zeta^{-1}$. Moreover, by (4) $\tilde{f}$ can be written as $f = Bg$, where B is the Blaschke product associated to the family of the zeros of $\tilde{f}$, that is to $\{M_\zeta(\alpha_j)\}_J$. By the previous lemma, and by (6), for every $\sigma \in \partial\Delta$

$$\lambda_{\tilde{f}}(\sigma) \geqslant \lambda_B = \sum_{j \in J} (1 - |M_\zeta(\alpha_j)|^2)/|\sigma - M_\zeta(\alpha_j)|^2. \tag{12}$$

Since by the hypothesis there exist $C(\zeta, R)$ elements of the family $\{\alpha_j\}_J$ such that $\rho(\zeta, \alpha_j) < R$, that is $|M_\zeta(\alpha_j)| < \tanh R$, we have by (12) and (11)

$$\lambda_{\tilde{f}}(\sigma) \geqslant \sum_{j \in J} \frac{1 - |M_\zeta(\alpha_j)|}{1 + |M_\zeta(\alpha_j)|} > C(\zeta, R)\, \frac{1 - \tanh R}{1 + \tanh R} \geqslant 1.$$

By (5), this means that there does not exist the Wolff point of $\tilde{f}$. Hence f has a fixed point in Δ.

The second part of the theorem follows immediately from (3). □

For example, any map $f \in \mathrm{Hol}(\Delta, \Delta)$ that has at least three zeros or a zero with multiplicity $\geqslant 3$ in the set $\{z \in \Delta: |z| < 1/2\}$ satisfies the hypothesis and then has a fixed point in Δ.

Note that, if we want to construct a map $f = e^{i\varphi} B \in \mathrm{Hol}(\Delta, \Delta)$, with $\varphi \in \mathbf{R}$ and B a Blaschke product having a pre-assigned Wolff point $\tau \in \partial\Delta$, it is sufficient that the zeros of B go to τ «fast» and «tangentially».

A possible choice is the following: for every integer $j \geqslant 1$ take $\alpha_j \in \Delta \setminus \overline{E(\tau, 2^j)}$ such that $\lim_{j \to \infty} \alpha_j = \tau$. In fact

$$\lambda_f(\tau) = \lambda_B(\tau) = \sum_{j=1}^{\infty} (1 - |\alpha_j|^2)/|\tau - \alpha_j|^2 < \sum_{j=1}^{\infty} 2^{-j} = 1,$$

and by Wolff's lemma, we can take $e^{-i\varphi} = \lim_{r \to 1^-} B(r\tau) \in \partial\Delta$.

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