

1806. Proposed by C. Pohoata (Romania).

Let M be a point on the circumcircle of triangle ABC and lying on the arc BC that does not contain A . Let I be the incenter of ABC , and let E and F be the feet of the perpendiculars from I to lines MB and MC , respectively. Prove that the value of

$$\frac{IE + IF}{AM}$$

is independent of the position of M .

Solution proposed by G.R.A.20 Problem Solving Group, Roma, Italy.

Let α, β, γ and θ be respectively the angles $\widehat{CAB}, \widehat{ABC}, \widehat{BCA}$ and \widehat{BCM} and let a, b, c be the sides CB, AC, AB .

Since IB bisects β and $\widehat{CMB} = \pi - \alpha$ ($ABMC$ is cyclic), then $\widehat{IBE} = \beta/2 + \alpha - \theta$ and

$$IE = IB \sin(\beta/2 + \alpha - \theta) = r \frac{\sin(\beta/2 + \alpha - \theta)}{\sin(\beta/2)}.$$

Similarly

$$IF = IC \sin(\gamma/2 + \theta) = r \frac{\sin(\gamma/2 + \theta)}{\sin(\gamma/2)}.$$

Moreover,

$$\frac{CM}{\sin(\alpha - \theta)} = \frac{BM}{\sin(\theta)} = \frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)} = 2R$$

and therefore, since $ABMC$ is cyclic, we have that

$$AM = CM \cdot \frac{c}{a} + MB \cdot \frac{b}{a} = \frac{2R}{\sin(\alpha)} (\sin(\alpha - \theta) \sin(\gamma) + \sin(\theta) \sin(\beta)).$$

Hence

$$\frac{IE + IF}{AM} = \frac{r \sin(\alpha) \sin(\gamma) \sin(\beta)}{2R \sin(\beta/2) \sin(\gamma/2)} \cdot f(\theta)$$

where

$$f(\theta) = \frac{\sin(\beta/2 + \alpha - \theta) \sin(\gamma/2) + \sin(\gamma/2 + \theta) \sin(\beta/2)}{\sin(\alpha - \theta) \sin(\gamma) + \sin(\theta) \sin(\beta)}.$$

Reminding that $\alpha = \pi - \beta - \gamma$, after some tedious computations we verified that $f(\theta) = f(0)$ for $\theta \in [0, \alpha]$ that is $(IE + IF)/AM$ is independent of the position of M .