The College Mathematics Journal, November 2008

887. *Proposed by J.L. Diaz-Barrero (Spain).*

Evaluate

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{\sqrt{n/k^3}}.$$

Solution proposed by G.R.A.20 Problem Solving Group, Roma, Italy.

We first transform the product in a sum by applying the logarithm

$$\log\left(\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{\sqrt{n/k^3}}\right)=\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{-3/2}\log\left(1+\frac{k}{n}\right).$$

Since such sum is actually a Riemann sum, its limit is the following integral

$$\begin{aligned}\int_0^1 x^{-3/2}\log(1+x)\,dx &= -2[x^{-1/2}\log(1+x)]_0^1+2\int_0^1\frac{x^{-1/2}}{1+x}\,dx\\ &= -2\log 2+2\int_0^1\frac{2}{1+t^2}\,dt\\ &= -\log 4+4[\arctan t)]_0^1=-\log 4+\pi\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{\sqrt{n/k^3}}=\exp(-\log 4+\pi)=\frac{e^{\pi}}{4}.$$

□