Problem 12105

Proposed by G. Brookfield (USA).

Let $r$ be a real number, and let $f(x) = x^3 + 2rx^2 + (r^2 - 1)x - 2r$. Suppose that $f$ has real roots $a$, $b$, and $c$. Prove $a, b, c \in [-1, 1]$ and

$$|\arcsin(a)| + |\arcsin(b)| + |\arcsin(c)| = \pi.$$

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Solution. If $x \in \{-1, 0, 1\}$ and $f(x) = 0$ then $r = 0$, $\{a, b, c\} = \{-1, 0, 1\}$ and the desired arcsin identity is valid.

We assume now that $x \not\in \{-1, 0, 1\}$ and therefore $r \neq 0$. If $f(x) = 0$ for some $r \in \mathbb{R} \setminus \{0\}$ then

$$f(x) = x^3 + 2rx^2 + (r^2 - 1)x - 2r = 0$$

is a quadratic equation with respect to $r$ and it follows that

$$\Delta = (x^2 - 1)^2 - x(x^3 - x) = 1 - x^2 \geq 0$$

which implies that $|x| \in (0, 1)$. It follows that the real roots $a, b, c$ are such that $|a|, |b|, |c| \in (0, 1)$.

Moreover,

$$(x^2 - a^2)(x^2 - b^2)(x^2 - c^2) = -f(x) f(-x) = x^6 - 2(r^2 + 1)x^4 + (r^4 + 6r^2 + 1)x^2 - 4r^2$$

and therefore

$$a^2 + b^2 + c^2 = 2(r^2 + 2), \quad a^2b^2 + b^2c^2 + c^2a^2 = (r^4 + 6r^2 + 1), \quad a^2b^2c^2 = 4r^2.$$

Hence

$$(a^2 + b^2 + c^2)^2 + 4a^2b^2c^2 - 4(a^2b^2 + b^2c^2 + c^2a^2) = (a^2 + b^2 + c^2)^2 + 4(a^2b^2 + b^2c^2 + c^2a^2) - 4(a^2b^2 + b^2c^2 + c^2a^2) = (2(r^2 + 1))^2 + 4 \cdot 4r^2 - 4 \cdot (r^4 + 6r^2 + 1) = 0 \quad \ast.$$

Finally we show that desired arcsin identity is implied by $\ast$.

Note that since $a + b + c = -2r = -abc$ and $abc = 2r$ it follows that $c = -(a + b)/(1 + ab)$ and for $0 < |a|, |b| < 1,$

$$\arcsin |a| + \arcsin |b| > \arcsin |c| = \arcsin \left| \frac{a + b}{1 + ab} \right|$$

because $h(t) = \arcsin(\tanh(t))$ is monotone and strictly subadditive in $(0, +\infty)$ ($h(0) = 0$ and $h$ is strictly concave in $[0, +\infty)$).

We may assume that $0 < |a| \leq |b| \leq |c| < 1$. Thus, by $\ast$,

$$(a^2 + b^2 + c^2)^2 + 4a^2b^2c^2 - 4(a^2b^2 + b^2c^2 + c^2a^2) = 0 \quad \Rightarrow \quad a^4 + b^4 + a^2b^2 + b^2c^2 + c^2a^2 = \frac{1}{2}(a^2b^2 + b^2c^2 + c^2a^2)$$

and we are done.