Problem 12084

Proposed by G. Stoica (Canada).

Let \( \{a_n\}_{n \geq 1} \) be a sequence of nonnegative numbers. Prove that \( \frac{1}{n} \sum_{k=1}^{n} a_k \) is unbounded if and only if there exists a decreasing sequence \( \{b_n\}_{n \geq 1} \) such that \( \lim_{n \to \infty} b_n = 0 \), \( \sum_{n=1}^{\infty} b_n \) is finite, and \( \sum_{n=1}^{\infty} a_n b_n \) is infinite. Is the word “decreasing” essential?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let \( A_n = \sum_{k=1}^{n} a_k \), then

\[
\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} A_n b_n - \sum_{n=1}^{N} A_{n-1} b_n = \sum_{n=1}^{N-1} A_n \cdot (b_n - b_{n+1}) + A_N b_N. \tag{1}
\]

Proof of \( \Leftarrow \). Assume that \( \frac{1}{n} \sum_{k=1}^{n} a_k \geq 1 \) is bounded, i.e. \( 0 \leq A_n \leq M n \) for some \( M \in \mathbb{R} \). Let \( \{b_n\}_{n \geq 1} \) be any decreasing nonnegative sequence, then by (1),

\[
\sum_{n=1}^{N} a_n b_n \leq M \left( \sum_{n=1}^{N-1} n (b_n - b_{n+1}) + N b_N \right) = M \sum_{n=1}^{N} b_n
\]

So if \( \sum_{n=1}^{\infty} b_n \) is finite then \( \sum_{n=1}^{\infty} a_n b_n \) is finite too and we have a contradiction.

Proof of \( \Rightarrow \). Assume that \( \frac{1}{n} \sum_{k=1}^{n} a_k \geq 1 \) is unbounded. Then there is a strictly increasing sequence \( \{n_k\}_{k \geq 1} \) of positive integers such that \( A_{n_k} \geq kn_k \). Let

\[
b_i := \sum_{k=j}^{\infty} \frac{1}{n_k k^2} \quad \text{for } i = n_{j-1} + 1, \ldots, n_j \text{ with } j \geq 1
\]

where \( n_0 = 0 \). Then \( \{b_n\}_{n \geq 1} \) is a decreasing nonnegative sequence such that \( \sum_{n=1}^{\infty} b_n \) is finite:

\[
\sum_{n=1}^{\infty} b_n = \sum_{j=1}^{\infty} (n_j - n_{j-1}) \sum_{k=j}^{\infty} \frac{1}{n_k k^2} = \sum_{k=1}^{\infty} \frac{1}{n_k k^2} \sum_{j=1}^{k} (n_j - n_{j-1}) = \sum_{i=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},
\]

and, by (1), \( \sum_{n=1}^{\infty} a_n b_n \) is infinite

\[
\sum_{n=1}^{\infty} a_n b_n \geq \sum_{n=1}^{\infty} A_n \cdot (b_n - b_{n+1}) \geq \sum_{k=1}^{\infty} k n_k \cdot \frac{1}{n_k k^2} = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty.
\]

Yes, the word “decreasing” is essential: let

\[
a_n = \begin{cases} k & \text{if } n = 2^k \text{ with } k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad b_n = \begin{cases} 1 & \text{if } n = 2^k \text{ with } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}
\]

then \( \{b_n\}_{n \geq 1} \) is a nonnegative sequence (which is not decreasing) such that \( \lim_{n \to \infty} b_n = 0 \), \( \sum_{n=1}^{\infty} b_n = \sum_{k=1}^{\infty} \frac{1}{k^2} \) is finite, and \( \sum_{n=1}^{\infty} a_n b_n = \sum_{k=1}^{\infty} \frac{1}{k} \) is infinite, but \( \frac{1}{n} \sum_{k=1}^{n} a_k \) is bounded

\[
0 \leq \frac{1}{n} \sum_{k=1}^{n} a_k = \frac{1}{n} \sum_{k=1}^{\log_2(n)} k \leq \frac{\log_2(n)(\log_2(n) + 1)}{2n} < 1.
\]