Problem 12083

Proposed by A. Belabess (Morocco).

Let $x$, $y$, and $z$ be positive real numbers. Prove
\[
\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{3\sqrt{3}}{2\sqrt{x^2 + y^2 + z^2}}.
\]

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By the Cauchy-Schwarz inequality,
\[
((x+y) + (y+z) + (z+x)) \cdot \left( \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \right) \geq (1 + 1 + 1)^2
\]
that is
\[
\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{9}{2(x+y+z)}.
\]
So, it remains to show that
\[
\frac{9}{2(x+y+z)} \geq \frac{3\sqrt{3}}{2\sqrt{x^2 + y^2 + z^2}}
\]
that is
\[
\sqrt{1^2 + 1^2 + 1^2} \cdot \sqrt{x^2 + y^2 + z^2} \geq (x+y+z)
\]
which holds again by the Cauchy-Schwarz inequality. □