Problem 11951
(American Mathematical Monthly, Vol.124, January 2017)

Proposed by O. Kouba (Syria).

Let ABC be a triangle that is not obtuse. Denote by a, b, and c the lengths of the sides opposite A, B, and C, respectively, and denote by h_a, h_b, and h_c the lengths of the altitudes dropped from A, B, and C, respectively. Prove that

$$\frac{a^2}{h_b^2+h_c^2}+\frac{b^2}{h_c^2+h_a^2}+\frac{c^2}{h_a^2+h_b^2}<\frac{5}{2}.$$

Show also that $5/2$ is the smallest possible constant in this inequality.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Since

$$h_a = b\sin C = c\sin B, \quad h_b = a\sin C = c\sin A, \quad h_c = a\sin B = b\sin A,$$

it follows that the inequality is equivalent to

$$\frac{1}{\sin^2 C+\sin^2 B}+\frac{1}{\sin^2 A+\sin^2 C}+\frac{1}{\sin^2 B+\sin^2 A}<\frac{5}{2}.$$

Note that if ABC is right-angled at C, then $s := \sin^2(A) \in (0,1)$, $\sin^2 B = \sin^2(\pi/2 - A) = 1 - s$ and the inequality is

$$\frac{1}{1+(1-s)}+\frac{1}{s+1}+\frac{1}{(1-s)+s}=\frac{3}{(2-s)(s+1)}+1<\frac{5}{2} \Leftrightarrow s(1-s)>0$$

which is true. Moreover, if $s \to 0^+$ then the left-hand side tends to $5/2$ which implies that $5/2$ is the smallest possible constant in this inequality.
So we may assume that ABC is an acute triangle. By letting

$$\alpha = \pi - 2A > 0, \quad \beta = \pi - 2B > 0, \quad \gamma = \pi - 2C > 0,$$

we have that $\alpha + \beta + \gamma = \pi$ and the inequality becomes

$$\frac{1}{\cos^2\frac{\gamma}{2}+\cos^2\frac{\beta}{2}}+\frac{1}{\cos^2\frac{\alpha}{2}+\cos^2\frac{\gamma}{2}}+\frac{1}{\cos^2\frac{\beta}{2}+\cos^2\frac{\alpha}{2}}<\frac{5}{2}.$$

Let a', b', and c' be the sides of a triangle with the corresponding angles α, β, and γ. By using the substitutions

$$x = s - a' > 0, \quad y = s - b' > 0, \quad z = s - c' > 0$$

where $2s = a' + b' + c'$, and the formulas

$$\cos^2\frac{\alpha}{2}=\frac{s(s-a)}{bc}=\frac{x(x+y+z)}{(x+z)(x+y)},\ \cos^2\frac{\beta}{2}=\frac{y(x+y+z)}{(y+x)(y+z)},\ \cos^2\frac{\gamma}{2}=\frac{z(x+y+z)}{(z+x)(z+y)}$$

we transform once more the inequality into

$$\frac{1}{1+\frac{yz}{xy+yz+zx}}+\frac{1}{1+\frac{zx}{xy+yz+zx}}+\frac{1}{1+\frac{xy}{xy+yz+zx}}<\frac{5}{2}\cdot\frac{(x+y+z)(xy+yz+zx)}{(x+y)(y+z)(z+x)}$$

which holds because

$$\frac{1}{1+\frac{yz}{xy+yz+zx}}+\frac{1}{1+\frac{zx}{xy+yz+zx}}+\frac{1}{1+\frac{xy}{xy+yz+zx}}\le 3-\frac{1}{2}=\frac{5}{2},$$

(note that $1/(1+t) \le 1 - \frac{t}{2}$ for $t \in [0,1]$), and

$$(x+y+z)(xy+yz+zx) = (x+y)(y+z)(z+x) + xyz > (x+y)(y+z)(z+x).$$

□