**Problem 11553**

Proposed by Mihaly Bencze (Romania).

For a positive integer $k$, let $\alpha(k)$ be the largest odd divisor of $k$. Prove that for each positive integer $n$,

$$\frac{n(n+1)}{3} \leq \sum_{k=1}^{n} \frac{n-k+1}{k} \alpha(k) \leq \frac{n(n+3)}{3}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Let $S_0(n) = \sum_{k=1}^{n} \alpha(k)$ and $S_1(n) = \sum_{k=1}^{n} \frac{\alpha(k)}{k}$.

Since $\alpha(2k-1) = 2k-1$ and $\alpha(2k) = \alpha(k)$ it follows that for $n \geq 1$

$$S_0(n) = \sum_{k=1}^{\lfloor n/2 \rfloor} \alpha(2k) + \sum_{k=1}^{\lfloor n/2 \rfloor} \alpha(2k-1) = S_0(\lfloor n/2 \rfloor) + \lfloor n/2 \rfloor^2,$$

and in a similar way we have that

$$S_1(n) = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{\alpha(2k)}{2k} + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{\alpha(2k-1)}{2k-1} = \frac{1}{2} S_1(\lfloor n/2 \rfloor) + \lfloor n/2 \rfloor.$$

Therefore

$$T(n) := \sum_{k=1}^{n} \frac{n-k+1}{k} \alpha(k) = (n+1)S_1(n) - S_0(n)$$

$$= T(\lfloor n/2 \rfloor) + \lfloor n/2 \rfloor(\lfloor n/2 \rfloor + 1) - \frac{2 \lfloor n/2 \rfloor}{2} S_1(\lfloor n/2 \rfloor)$$

where $[2 \mid n]$ is equal to 1 if $n$ is even and it is 0 otherwise.

By using the recurrence for $S_1(n)$, it is easy to verify by induction that

$$S_1(n) \leq \frac{2\lfloor n/2 \rfloor + n + 1}{3}.$$

Hence, since $0 \leq S_1(\lfloor n/2 \rfloor) = 2(S_1(n) - \lfloor n/2 \rfloor)$ and $n = \lfloor n/2 \rfloor + \lfloor n/2 \rfloor$,

$$T(\lfloor n/2 \rfloor) + \lfloor n/2 \rfloor(\lfloor n/2 \rfloor + 1) - \frac{2 \lfloor n/2 \rfloor}{3} (\lfloor n/2 \rfloor + 1) \leq T(n) \leq T(\lfloor n/2 \rfloor) + \lfloor n/2 \rfloor(\lfloor n/2 \rfloor + 1).$$

The upper bound follows by induction by verifying that

$$\lfloor n/2 \rfloor(\lfloor n/2 \rfloor + 3) + 3\lfloor n/2 \rfloor(\lfloor n/2 \rfloor + 1) \leq n(n+3).$$

Similarly the lower bound follows by induction by verifying that

$$n(n+1) \leq \lfloor n/2 \rfloor(\lfloor n/2 \rfloor + 1) + 3\lfloor n/2 \rfloor(\lfloor n/2 \rfloor + 1) - [2 \mid n] \lfloor n/2 \rfloor + 1).$$

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