Problem 11508
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Proposed by M. Bencze (Romania).

Prove that for all positive integers k there are infinitely many positive integers n such that $kn+1$ and $(k+1)n+1$ are both perfect squares.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let us consider the Pell equation

$$u^2 - k(k+1)v^2 = 1$$

which has infinite positive solutions $\{(u_i, v_i)\}$ such that

$$u_1 = 2k+1, v_1 = 2 \quad \text{and} \quad u_i + v_i\sqrt{k(k+1)} = \left(u_1 + v_1\sqrt{k(k+1)}\right)^i \quad \text{for } i \geq 1.$$

By letting

$$n_i = ((u_i + kv_i)^2 - 1)/k = (2k+1)v_i^2 + 2u_iv_i$$

we have a sequence of positive integers $\{n_i\}$ such that

$$kn_i + 1 = (u_i + kv_i)^2 \quad \text{and} \quad (k+1)n_i + 1 = (u_i + (k+1)v_i)^2.$$