Problem 11428

Proposed by W. Blumberg, USA.

Let \( p \) a prime congruent to 3 mod 4, and let \( a \) and \( q \) be integers, such that \( p \) does not divide \( q \). Show that

\[
\sum_{k=1}^{p} \left\lfloor \frac{(qk^2 + a)}{p} \right\rfloor = 2a + 1 + \sum_{k=1}^{p} \left\lfloor \frac{(qk^2 - a - 1)}{p} \right\rfloor.
\]

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Let \( r_p(n) \) be the remainder of the division of \( n \) by \( p \) then, since \( n = p[n/p] + r_p(n) \), the above equality is equivalent to

\[
\sum_{k=1}^{p} r_p(qk^2 + a) = \sum_{k=1}^{p} r_p(qk^2 - a - 1).
\]

Moreover

\[
r_p(qk^2 + a) = r_p(qk^2) + r_p(a) - px(k) \quad \text{and} \quad r_p(qk^2 - a - 1) = r_p(qk^2) - r_p(a) - 1 + py(k)
\]

with

\[
x(k) = [r_p(qk^2) \geq r_p(a)] \quad \text{and} \quad y(k) = [r_p(qk^2) \leq r_p(a)]
\]

where \( [Q] \) is equal to 1 if proposition \( Q \) is true and it is 0 otherwise.

Hence, since \( x(p) = 0 \) and \( y(p) = 1 \) then it suffices to show that

\[
\sum_{k=1}^{p-1} (x(k) + y(k)) = 2r_p(a).
\]

Note that \( r_p(qk^2) = j \) for some \( j \in \{1, \ldots, p - 1\} \) iff \( k^2 = jq^{-1} \) mod \( p \) (\( q \) is invertible mod \( p \)), that is iff \( jq^{-1} \) is a square mod \( p \) which means that \( \left( \frac{jq^{-1}}{p} \right) = 1 \) where \( \left( \frac{\cdot}{p} \right) \) is the Legendre symbol.

Therefore the number of \( k \in \{1, \ldots, p - 1\} \) such that \( r_p(qk^2) = j \) is \( \left( \frac{jq^{-1}}{p} \right) + 1 \) (it gives 0 or 2).

Since \( p = 3 \mod 4 \) then \( \left( \frac{-1}{p} \right) = (-1)^{(p-1)/2} = -1 \), and it follows that

\[
\sum_{k=1}^{p-1} x(k) = \sum_{j=p-r_p(a)}^{p-1} \left( \left( \frac{jq^{-1}}{p} \right) + 1 \right) = r_p(a) + \left( \frac{-1}{p} \right) \sum_{j=1}^{p-r_p(a)} \left( \frac{p-j}{p} \right) = r_p(a) + \left( \frac{-1}{p} \right) \sum_{j=1}^{r_p(a)} \left( \frac{j}{p} \right).
\]

In a similar way

\[
\sum_{k=1}^{p-1} y(k) = \sum_{j=1}^{r_p(a)} \left( \left( \frac{jq^{-1}}{p} \right) + 1 \right) = r_p(a) + \left( \frac{-1}{p} \right) \sum_{j=1}^{r_p(a)} \left( \frac{j}{p} \right).
\]

Finally,

\[
\sum_{k=1}^{p-1} (x(k) + y(k)) = r_p(a) - \left( \frac{-1}{p} \right) \sum_{j=1}^{r_p(a)} \left( \frac{j}{p} \right) + r_p(a) + \left( \frac{-1}{p} \right) \sum_{j=1}^{r_p(a)} \left( \frac{j}{p} \right) = 2r_p(a).
\]