**Problem 11277**

Proposed by Prithwijit De, Ireland.

**Find**
\[
\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{\log(2 - \sin \theta \cos \phi) \sin \theta}{2 - 2 \sin \theta \cos \phi + \sin^2 \theta \cos^2 \phi} \, d\theta \, d\phi.
\]

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

We will show that
\[
I = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{\log(2 - \sin \theta \cos \phi)}{1 + (1 - \sin \theta \cos \phi)^2} \cdot \sin \theta \, d\theta \, d\phi = \frac{\pi^2 \log(2)}{16}.
\]

This is a surface integral over the first octant of the unit sphere with respect to the spherical coordinates. The same integral with respect to the cartesian coordinates is
\[
I = \int_{y=0}^{\sqrt{1-x^2}} \int_{x=0}^{1} \frac{\log(2-x)}{1 + (1-x)^2} \cdot \frac{1}{\sqrt{1-x^2-y^2}} \, dy \, dx.
\]

After performing the improper integral with respect to \(y\) we obtain
\[
\int_{x=0}^{1} \frac{\log(2-x)}{1 + (1-x)^2} \left[ \arctan \left( \frac{y}{\sqrt{1-x^2-y^2}} \right) \right]_{y=0}^{\sqrt{1-x^2}} \, dx = \frac{\pi}{2} \int_{0}^{1} \frac{\log(1+x)}{1 + x^2} \, dx = \frac{\pi}{2} \int_{0}^{1} \log(1+x) \, dx.
\]

The Catalan’s constant \(K \approx 0.9159655942\) has several integral representation. Some of them are the following ones:
\[
K = -\int_{0}^{1} \frac{\log(x)}{1 + x^2} \, dx = -\int_{0}^{1} \frac{\log((1-x^2)/2)}{1 + x^2} \, dx = -\int_{0}^{1} \frac{\log((1-x)/\sqrt{2})}{1 + x^2} \, dx.
\]

Therefore
\[
K = \frac{\pi \log(2)}{4} - \int_{0}^{1} \frac{\log(1-x)}{1 + x^2} \, dx - \int_{0}^{1} \frac{\log(1+x)}{1 + x^2} \, dx = \frac{\pi \log(2)}{8} - \int_{0}^{1} \frac{\log(1-x)}{1 + x^2} \, dx
\]

and we find that
\[
\int_{0}^{1} \frac{\log(1+x)}{1 + x^2} \, dx = \frac{\pi \log(2)}{4} - \frac{\pi \log(2)}{8} = \frac{\pi \log(2)}{8}.
\]

Finally
\[
I = \frac{\pi}{2} \int_{0}^{1} \frac{\log(1+x)}{1 + x^2} \, dx = \frac{\pi}{2} \cdot \frac{\pi \log(2)}{8} = \frac{\pi^2 \log(2)}{16}.
\]