Problem 11225

Proposed by J. L. Díaz-Barrero (Spain).

Find
\[ \lim_{n \to \infty} \frac{1}{n} \int_0^n \left( \frac{x \log(1 + x/n)}{1 + x} \right) \, dx. \]

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Letting \( t = x/n \) the integral becomes
\[
\frac{1}{n} \int_0^n \left( \frac{x \log(1 + x/n)}{1 + x} \right) \, dx = \int_0^1 \frac{nt}{1 + nt} \cdot \log(1 + t) \, dt
\]
\[
= \int_0^1 \log(1 + t) \, dt - \int_0^1 \frac{t}{1 + nt} \cdot \log(1 + t) \, dt
\]
\[
= 2 \log(2) - 1 - \int_0^1 \frac{t}{1 + nt} \cdot \log(1 + t) \, dt.
\]

Taking the limit as \( n \) goes to infinity then the remaining integral goes to zero because
\[
0 \leq \int_0^1 \frac{t}{1 + nt} \cdot \log(1 + t) \, dt \leq \frac{1}{n} \int_0^1 \log(1 + t) \, dt < \frac{1}{n}.
\]

Actually \( \int_0^1 \log(1 + t)/t \, dt = \pi^2/12 \), but it suffices to note that since \( \log(1 + t) < t \) for \( t > 0 \) then this integral is less than 1. Therefore
\[
\lim_{n \to \infty} \frac{1}{n} \int_0^n \left( \frac{x \log(1 + x/n)}{1 + x} \right) \, dx = 2 \log(2) - 1.
\]

\( \square \)