Problem 11133

Proposed by Paul Bracken, University of Texas Pan-American, Edinburg, TX.

Let $f$ be a nonnegative, continuous, concave function on $[0, 1]$ with $f(0) = 1$. Prove that

$$2 \int_0^1 x^2 f(x) \, dx + \frac{1}{12} \leq \left( \int_0^1 f(x) \, dx \right)^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Let $l(x) = 1 - x$ then, since $f$ is a nonnegative concave function with $f(0) = 1$, $f(x) \geq f(0) + (f(1) - f(0))x = 1 + f(1)x - x \geq l(x)$ for $x \in [0, 1]$.

Moreover, it is easy to see that

$$2 \int_0^1 x^2 l(x) \, dx + \frac{1}{12} = \left( \int_0^1 l(x) \, dx \right)^2.$$

Therefore letting $g(x) = f(x) - l(x)$ it suffices to prove that

$$2 \int_0^1 x^2 g(x) \, dx \leq \left( \int_0^1 f(x) \, dx \right)^2 - \left( \int_0^1 l(x) \, dx \right)^2 \leq \left( \int_0^1 g(x) \, dx \right) \left( \int_0^1 (g(x) + 2l(x)) \, dx \right) \leq \left( \int_0^1 g(x) \, dx \right)^2 + \int_0^1 g(x) \, dx.$$

or

$$\int_0^1 (2x^2 - 1)g(x) \, dx \leq \left( \int_0^1 g(x) \, dx \right)^2.$$

We will show that the first term is nonnegative that is for $a = 1/\sqrt{2} \in [0, 1]$

$$\int_0^1 (2x^2 - 1)g(x) \, dx \leq \int_0^a (1 - 2x^2)g(x) \, dx.$$

Since $g$ is a nonnegative continuous concave function with $g(0) = 0$ then $g(x) \geq m' x$ for $x \in [0, a]$ with $m' = g(a)/a$

and

$$g(x) \leq mx + q \quad \text{for } x \in [0, 1]$$

where $y = mx + q$ is a support line to the graph of $g$ at $(a, g(a))$. Now

$$\int_a^1 (2x^2 - 1)g(x) \, dx \leq \int_a^1 (2x^2 - 1)(mx + q) \, dx = m/8 + (\sqrt{2} - 1)q/3$$

on the other hand

$$\int_0^a (1 - 2x^2)g(x) \, dx \geq \int_0^a (1 - 2x^2)m' x \, dx = m'/8.$$

Since $m'a = g(a) = ma + q$ we have that $m' - m = \sqrt{2}q$ and our inequality holds as soon as

$$(\sqrt{2} - 1)q/3 \leq (m' - m)/8 = \sqrt{2}q/8$$

which is true because $q \geq g(0) = 0$ and $8 \geq 5\sqrt{2} \approx 7.07$. 