

**Problem 10974**

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The digital root  $\rho(n)$  of a positive integer  $n$  is the eventual image of  $n$  under the mapping that carries an integer  $n$  to the sum of its base-ten digit. Thus  $\rho(10974) = \rho(21) = 3$ . Find  $\rho(F_n)$ , where  $F_n$  is the  $n$ th Fibonacci number, with  $F_1 = F_2 = 1$ .

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is well known that the digital root of a positive integer  $n$  is the remainder of the division of  $n$  by 9, so we define

$$f_n = \text{mod}(F_n, 9) = \rho(F_n).$$

The sequence  $f_n$  can be generated by the recurrence relation

$$\begin{cases} f_{n+1} = f_n + f_{n-1} & (\text{mod } 9) \\ f_1 = 1, f_2 = 1 \end{cases}$$

and it is periodic with a period length of 24. Indeed

$$\begin{array}{cccccc} f_1 = 1 & f_2 = 1 & f_3 = 2 & f_4 = 3 & f_5 = 5 & f_6 = 8 \\ f_7 = 4 & f_8 = 3 & f_9 = 7 & f_{10} = 1 & f_{11} = 8 & f_{12} = 0 \\ f_{13} = 8 & f_{14} = 8 & f_{15} = 7 & f_{16} = 6 & f_{17} = 4 & f_{18} = 1 \\ f_{19} = 5 & f_{20} = 6 & f_{21} = 2 & f_{22} = 8 & f_{23} = 1 & f_{24} = 0. \end{array}$$

and the sequence goes on with  $f_{25} = f_1 = 1$  and  $f_{26} = f_2 = 1$ .

A first explicit formula for  $f_n$  is the following

$$f_n = \frac{(5+i)^n - (5-i)^n}{2i} = \text{Im}((5+i)^n) \quad (\text{mod } 9).$$

Note that the complex numbers  $5+i$  and  $5-i$  allow to split in the Galois field  $\text{GF}(9)$  the characteristic polynomial  $x^2 - x - 1$  associated to the Fibonacci recurrence equation. The formula can be proven by induction:

$$\begin{aligned} f_1 &= \text{Im}(5+i) = 1 && (\text{mod } 9), \\ f_2 &= \text{Im}((5+i)^2) = \text{Im}(6+i) = 1 && (\text{mod } 9). \end{aligned}$$

moreover, for  $n \geq 2$ , we have that

$$\begin{aligned} f_n + f_{n-1} &= \text{Im}((5+i)^n) + \text{Im}((5+i)^{n-1}) && (\text{mod } 9) \\ &= \text{Im}((5+i)^n + (5+i)^{n-1}) && (\text{mod } 9) \\ &= \text{Im}((5+i)^{n-1} \cdot (5+i+1)) && (\text{mod } 9) \\ &= \text{Im}((5+i)^{n-1} \cdot (5+i)^2) = f_{n+1} && (\text{mod } 9). \end{aligned}$$

Using the formula one can easily show that the sequence is periodic: since by the Fermat's Little Theorem  $\text{mod}(2^6, 9) = 1$ , then

$$(5 + i)^{24} = ((5 + i)^3)^8 = (2(1 + i))^8 = 2^8 \cdot 2^4 = 2^{12} = 1 \pmod{9}$$

and for  $n \geq 1$

$$f_{n+24} = \text{Im}((5 + i)^n \cdot (5 + i)^{24}) = \text{Im}((5 + i)^n) = f_n \pmod{9}.$$

Hence the computation of  $f_n$  can be further on simplified:

$$f_n = \text{Im}\left((5 + i)^{(n,24)}\right) \pmod{9}$$

where  $(n, 24) = \text{mod}(n, 24)$  and, since

$$5 + i = \sqrt{26} \exp\left(i \arcsin\left(\frac{1}{\sqrt{26}}\right)\right),$$

we have our final formula

$$f_n = (\sqrt{26})^{(n,24)} \sin\left((n, 24) \cdot \arcsin\left(\frac{1}{\sqrt{26}}\right)\right) \pmod{9}.$$