Problem 10930

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Show that there do not exist real 2-by-2 matrices $A$ and $B$ such that their commutator is nonzero and commutes with both $A$ and $B$. (The commutator $C$ of $A$ and $B$ is defined by $C = AB - BA$.)

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

By the Cayley-Hamilton theorem, the matrices $A$ and $B$ solve their characteristic polynomial:

$$A^2 = aA + b, \quad B^2 = cB + d$$

where $a = \text{tr}(A)$, $b = -\det(A)$ and $c = \text{tr}(B)$, $d = -\det(B)$. Since $C$ commutes with $A$

$$\begin{align*}
AC - CA &= 0 \\
AC + CA &= A^2B + BA^2 = (aA + b)B - B(aA + b) = aC
\end{align*}$$

Therefore, $AC = CA = \frac{a}{2}C$ and, in the same way, $BC = CB = \frac{c}{2}C$. Thus $C^2 = (AB - BA)C = A(BC) - B(AC) = \frac{a}{2} \cdot \frac{c}{2}C - \frac{c}{2} \cdot \frac{a}{2}C = 0$.

Assume that $C \neq 0$. Then there is a vector $v \neq 0$ such that $u := Cv \neq 0$. Note that $u$ is a common eigenvector of $C$, $A$ and $B$:

$$Cu = C^2v = 0, \quad Au = ACv = \frac{a}{2}u, \quad Bu = BCv = \frac{c}{2}v = \frac{c}{2}u.$$ 

The vectors $u$ and $v$ are linearly independent: let $\alpha, \beta \in \mathbb{R}$ such that $\alpha u + \beta v = 0$ then

$$0 = C(\alpha u + \beta v) = \alpha Cu + \beta Cv = \alpha C^2v + \beta Cv = \beta u,$$

which implies that $\alpha = \beta = 0$. Since $a = \text{tr}(A)$ and $c = \text{tr}(B)$ then the representations of the matrices $A$ and $B$ with respect to the base $<u, v>$ are

$$A = \begin{pmatrix}
\frac{a}{2} & \gamma \\
0 & \frac{a}{2}
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
\frac{c}{2} & \delta \\
0 & \frac{c}{2}
\end{pmatrix}$$

for some $\gamma, \delta \in \mathbb{R}$. Then

$$C = AB - BA = \begin{pmatrix}
\frac{ac}{4} & \frac{a}{2} \delta + \frac{c}{2} \gamma \\
0 & \frac{ac}{4}
\end{pmatrix} - \begin{pmatrix}
\frac{ac}{4} & \frac{c}{2} \gamma + \frac{a}{2} \delta \\
0 & \frac{ac}{4}
\end{pmatrix} = 0$$

that is a contradiction. □

The statement does not hold in higher dimension: for example in $\mathbb{R}^{3 \times 3}$

if $A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$ and $B = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}$ then $C = AB - BA = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$

and $AC = CA$, $BC = CB$. 

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