Problem 10924


A regular polygon of 2001 sides is inscribed in a circle of unit radius. Prove that its side and all its diagonals have irrational lengths.

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We will prove a more general statement:
In a regular polygon of \( N \geq 2 \) sides and inscribed in a circle of unit radius, the side and all the diagonals have irrational lengths iff \( N \) is an odd integer.

First recall that for all integers \( n \geq 1 \) and for all \( \theta \in \mathbb{R} \)

\[
\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n = \sum_{k=0}^{n} \binom{n}{k} i^k (\sin \theta)^k (\cos \theta)^{n-k}.
\]

After taking the imaginary part, we have

\[
\sin n\theta = \sum_{k=1, \text{odd}}^{n} \binom{n}{k} (-1)^{\frac{k-1}{2}} (\sin \theta)^k (\cos \theta)^{n-k}.
\]

If \( n \) is odd then the exponents \( n-k \) are even and

\[
(\cos \theta)^{n-k} = (\cos^2 \theta)^{\frac{n-k}{2}} = (1 - \sin^2 \theta)^\frac{n-k}{2}.
\]

Hence, for all odd integers \( n \geq 1 \), there is a polynomial \( Q_n(x) \in \mathbb{Z}[x] \) of degree \( n-1 \), of the following form

\[
Q_n(x) = (-1)^{\frac{n-1}{2}} \left( 1 + \binom{n}{2} \right) x^{n-1} + \ldots + n,
\]

such that

\[
\sin n\theta = \sin \theta \cdot Q_n(\sin \theta) \quad \text{for all odd integers } n \geq 1. \quad (1)
\]

After this key remark, we start the proof of our statement.

If \( N \) is even then the diagonal between two opposite vertices has length \( 2 \in \mathbb{Q} \).
On the other hand, if \( N \) is odd then the side and the diagonals have lengths: \( 2 \sin(\pi m/N) \) for \( m = 1, \ldots, N-1 \). Assume that one of these numbers is rational, then we will reach a contradiction.

Since \( 1 \leq m \leq N-1 \), there are an odd prime \( p \) and some integer \( e \geq 1 \) such that \( p^e \) divides \( N \), \( p^{e-1} \) divides \( m \), but \( p^e \) does not divide \( m \). Therefore \( N/p^e \) is an odd integer and by (1)

\[
\sin \left( \frac{\pi m'}{p^e} \right) = \sin \left( \frac{N}{p^e} \cdot \frac{\pi m}{N} \right) = \sin \left( \frac{\pi m}{N} \right) \cdot Q_{p^e} \left( \sin \left( \frac{\pi m}{N} \right) \right) \in \mathbb{Q}
\]
where \( m' = m/p^{e-1} \). Since \( \text{MCD}(m', p) = 1 \), if \( k \in \mathbb{Z} \) then there are two integers \( s_0 \) and \( t_0 \) such that \( s = s_0 + j \cdot p \) and \( t = t_0 - j \cdot m' \) solve the following linear diophantine equation for all \( j \in \mathbb{Z} \):

\[
s m' + t p = k.
\]

We can always pick \( j \) such that \( s \) is an odd positive integer. Then, by (1)

\[
\sin \left( \frac{\pi k}{p} \right) = \sin \left( \frac{\pi (s m' + t p)}{p} \right) = (-1)^j \sin \left( \frac{\pi m'}{p} \right) \cdot Q_s \left( \sin \left( \frac{\pi m'}{p} \right) \right) \in \mathbb{Q}.
\]

Varying \( k = \pm 1, \ldots, \pm \frac{p-1}{2} \), we obtain \( p-1 \) distinct rational numbers which are the zeroes of the polynomial

\[
Q_p(x) = (-1)^{\frac{p-1}{2}} \left( 1 + \left( \frac{p}{2} \right) \right) x^{p-1} + \ldots + p.
\]

The polynomial \( Q_p \) has integer coefficients and therefore every rational zero has the following property: the numerator divides \( p \) whereas the denominator divides \( a = 1 + p(p-1)/2 \). It follows that

\[
S = \left\{ \sin \left( \frac{\pi k}{p} \right) : k = \pm 1, \ldots, \pm \frac{p-1}{2} \right\} \subset T = \left\{ \pm \frac{1}{d}, \pm \frac{p}{d} : d \mid a \right\} \cap (-1,1).
\]

Hence

\[
\cos \left( \frac{\pi}{2p} \right) = \sin \left( \frac{\pi}{p} \cdot \frac{p-1}{2} \right) = \max S \leq \max T \leq \frac{p}{p+1} = 1 - \frac{1}{p+1},
\]

and therefore

\[
\frac{1}{p+1} \leq 1 - \cos \left( \frac{\pi}{2p} \right) = 2 \sin^2 \left( \frac{\pi}{4p} \right) < 2 \left( \frac{\pi}{4p} \right)^2 < \frac{2}{p^2}.
\]

This inequality never holds for \( p \geq 3 \) which is a contradiction. □

Note that, if \( N = 2001 = 3 \cdot 23 \cdot 29 \) then we can get a contradiction in a different way:

- if \( p = 23 \) then \( a = 254 = 2 \cdot 127 \) and \( T = \{ \pm \frac{1}{127}, \pm \frac{1}{127}, \pm \frac{1}{127}, \pm \frac{23}{127}, \pm \frac{23}{127} \} \),
- if \( p = 29 \) then \( a = 467 = 11 \cdot 37 \) and \( T = \{ \pm \frac{1}{11}, \pm \frac{1}{37}, \pm \frac{1}{11}, \pm \frac{1}{37}, \pm \frac{29}{11}, \pm \frac{29}{37} \} \).

In both cases the number of elements of \( T \) is much less than \( p - 1 \). The remaining case \( p = 3 \) is easily solved because \( \sin(\pi/3) = \sqrt{3}/2 \notin \mathbb{Q} \).