BSc Engineering Sciences – A. Y. 2019/20

**Written exam of the course Mathematical Analysis 2**

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Solve the following problems, motivating in detail the answers.

**1.** Find the Taylor expansion around the point $x_0 = 0$ of the function

$$f(x) = \frac{1}{x^3 - 2x^2 + x - 2},$$

determine its radius of convergence $r$.

*Solution.* The function $x^3 - x^2 + 2x - 2$ can be factorized as $(x^2+1)(x-2)$. By putting $f(x) = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$ we obtain

$$\begin{aligned} f(x) &= \frac{\frac{1}{5}}{x-2} + \frac{-\frac{x}{5} - \frac{2}{5}}{x^2+1} \\ &= \frac{1}{10} \cdot \frac{1}{\frac{x}{2} - 1} + \frac{1}{5} \cdot \frac{-x-2}{x^2+1}. \end{aligned}$$

For $-1 < y < 1$, we have the geometric series $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$. Applying this to $y = \frac{x}{2}, -x^2$ respectively, we obtain

$$\begin{aligned} f(x) &= -\frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n + \frac{1}{5}\left(-x\sum_{n=0}^{\infty}(-x^2)^n - 2\sum_{n=0}^{\infty}(-x^2)^n\right) \\ &= \sum_{n=0}^{\infty} -\frac{x^n}{10 \cdot 2^n} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n+1} - 2(-1)^n x^{2n}}{5} \end{aligned}$$

This is a power series expansion of $f(x) = \sum_{m=0}^{\infty} a_m x^m$ with

$$\begin{cases} a_{2m} = -\frac{1}{10 \cdot 2^{2m}} - \frac{2(-1)^m}{5}, \\ a_{2m+1} = -\frac{1}{10 \cdot 2^{2m+1}} - \frac{(-1)^{m+1}}{5}. \end{cases}$$

As for the radius of convergence, let us divide the series into parts: $f(x) = \sum_{n=0}^{\infty} -\frac{x^n}{10 \cdot 2^n} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n+1}}{5} - 2\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{5}$. By the ratio test, it is easy to see that the radii of convergence of these series are $2, 1, 1$, respectively. Therefore, the radius of convergence of the whole series is at least 1. On the other hand, if $1 < |x| < 2$, the first series is convergent, while for the latter, terms diverge, hence the series is not convergent. This shows that the radius of convergence is 1.

**2.** Find all the stationary points of the following scalar field, defined on $\mathbb{R}^3$,

$$f(x, y, z) = (x^2 + 2xy + 2y^2 - 2x - 4y)e^{z^3-3z}$$

and classify them into relative minima, maxima and saddle points.

*Solution.*

For the $f$ given above, it holds that

$$\begin{aligned}&\nabla f(x, y, z)\\ &= \left((2x + 2y - 2)e^{z^3-3z}, (2x + 4y - 4)e^{z^3-3z}, (x^2 + 2xy + 2y^2 - 2x - 4y)(3z^2 - 3)e^{z^3-3}\right).\end{aligned}$$

At stationary points, $\nabla f(x, y, z) = \mathbf{0}$ holds. Namely,

$$(2x+2y-2)e^{z^3-3z} = 0, (2x+4y-4)e^{z^3-3z} = 0, (x^2+2xy+2y^2-2x-4y)(3z^2-3)e^{z^3-3} = 0.$$

As $e^{-z^3-3z}$ takes never 0, this is equivalent to

$$2x + 2y - 2 = 0, 2x + 4y - 4 = 0, (x^2 + 2xy + 2y^2 - 2x - 4y)(3z^2 - 3) = 0.$$

From the first two equations, we obtain $x = 0, y = 1$. Substituting this in the last equation, $-6(z^2 - 1) = 0$, namely, $z = \pm 1$. Therefore, the stationary points are $(0, 1, \pm 1)$.

To classify these points, let us compute the Hessian matrix:

$$\begin{aligned}D_{xx} = 2e^{z^3-3z}, D_{yx} = D_{xy} = 2e^{z^3-3z}, D_{yy} = 4e^{z^3-3z}\\ D_{xz} = D_{zx} = (2x + 2y - 2)(3z^2 - 3)e^{z^3-3}\\ D_{yz} = D_{zy} = (2x + 4y - 4)(3z^2 - 3)e^{z^3-3}\\ D_{zz} = (x^2 + 2xy + 2y^2 - 2x - 4y)(6z + (3z^2 - 3)^2)e^{z^3-3}.\end{aligned}$$

(it is a good idea not to expand the formula at this point, because, at the stationary point many terms vanish). At the point $(x, y, z) = (0, 1, 1)$, this becomes

$$e^{-2}\begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & -12 \end{pmatrix}.$$

$(0, 0, 1)^{\mathrm{t}}$ is an eivenvector with eivenvalue $-12$. As for the part $\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$, the determinant is positive and the trace is positive, hence there are two positive eigenvalues. Altogether, there are positive and negative eigenvalues, hence the point $(0, 1, 1)$ is a saddle.

At the point $(x, y) = (0, 1, -1)$, this becomes

$$e^{-2}\begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 12 \end{pmatrix}.$$

In this case, arguing as before, we find that all the eigenvalues are positive, hence $(0, 1, -1)$ is a relative minumum.

**3.** Consider the following three formulae for vector fields

$$\boldsymbol{f}(x,y) = \begin{pmatrix} -y(x^2+y^2)^{-1} \\ x(x^2+y^2)^{-1} \end{pmatrix}, \quad \boldsymbol{g}(x,y) = \begin{pmatrix} 1+y+e^{x+y^2} \\ x+2ye^{x+y^2} \end{pmatrix}, \quad \boldsymbol{h}(x,y) = \begin{pmatrix} \frac{1}{y}\cos(\frac{x}{y}) \\ \ln y \cos(\frac{x}{y}) \end{pmatrix}.$$

One of these is a gradient on $\mathbb{R}^2$. Identify which this is and find a scalar field $\varphi$ such that this particular vector field is equal to $\nabla\varphi$.

*Solution.* Calculating the difference of derivatives $\partial_y \boldsymbol{f}_1 - \partial_x \boldsymbol{f}_2$ suggests that $\boldsymbol{f}$ might be a gradient but for the fact that the vector field blows up at $(0,0)$. Calculating the difference of derivatives $\partial_y \boldsymbol{h}_1 - \partial_x \boldsymbol{h}_2$ shows that $\boldsymbol{h}$ isn't a gradient. On the other hand $\boldsymbol{g}$ is a gradient. We can find the scalar field by the method of indeterminate integrals:

$$\int 1+y+e^{x+y^2}\,dx = x+xy+e^{x+y^2}+B(y),$$
$$\int x+2ye^{x+y^2}\,dy = xy+e^{x+y^2}+C(x).$$

Consequently we choose $\varphi(x,y) = x+xy+e^{x+y^2}$ and can double check by differentiating.

**4.** Let $D \subset \mathbb{R}^3$ be the solid pyramid which has 5 planar surfaces and vertices $(-1,0,0)$, $(0,-1,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Compute the integral

$$\iiint_D x^2 + y^2 \; dxdydz.$$

*A possible method is: 1. Let $E \subset \mathbb{R}^3$ be the solid bounded by the four surfaces $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$, $\{x + y + z = 1\}$; 2. Calculate $\iiint_E x^2 \; dxdydz$ and $\iiint_E y^2 \; dxdydz$ separately in order to compute $\iiint_E x^2 + y^2 \; dxdydz$; 3. Deduce the full answer by observing the symmetry in this problem.*

*Solution.* $D$ is a pyramid with a square base and $E$ is a quarter of that pyramid. (This integral is the moment of inertia of the solid pyramid.) Let $T = \{(x,y) : x \in [0,1], 0 \le y \le 1 - x\}$ and so

$$E = \{(x,y,z) : (x,y) \in T, 0 \le z \le 1 - (x+y)\}.$$

This means that

$$\iiint_E x^2 \; dxdydz = \iint_T \left( \int_0^{1-(x+y)} x^2 \; dz \right) \; dxdy = \iint_T x^2 - x^3 - x^2 y \; dxdy.$$

In turn

$$\iint_T x^2 - x^3 - x^2 y \; dxdy = \int_0^1 \left( \int_0^{1-x} x^2 - x^3 - x^2 y \; dy \right) dx$$

and

$$\int_0^{1-x} x^2 - x^3 - x^2 y \; dy = \left[ x^2 y - x^3 y - \tfrac{1}{2} x^2 y^2 \; dy \right]_0^{1-x} = \tfrac{1}{2} x^2 - x^3 + \tfrac{1}{2} x^4.$$

Finally

$$\int_0^1 \tfrac{1}{2} x^2 - x^3 + \tfrac{1}{2} x^4 dx = \left[ \tfrac{1}{6} x^3 - \tfrac{1}{4} x^4 + \tfrac{1}{10} x^5 \right]_0^1 = \frac{1}{60}$$

We have therefore calculated that $\iiint_E x^2 \; dxdydz = \frac{1}{60}$. Similarly $\iiint_E y^2 \; dxdydz = \frac{1}{60}$. The rotational symmetry of this problem ($x^2 + y^2$ is simply the square of the distance from the $z$-axis) means that $\iiint_D x^2 + y^2 \; dxdydz$ is equal to 4 times $\iiint_E x^2 + y^2 \; dxdydz$. This means that $\iiint_D x^2 + y^2 \; dxdydz = \frac{2}{15}$.

**5.** Let $S$ be the planar triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and consider the vector field $\boldsymbol{f}(x,y,z) = (x^2, y^2, 1)$. Compute $\iint_S \boldsymbol{f} \cdot \mathbf{n}\, dS$, where $\mathbf{n}$ is the unit normal with positive $z$ component.

*Solution.* As a representation of the surface we choose $T = \{(u,v) : u \in [0,1], 0 \le v \le 1-u\}$ and

$$\mathbf{r} : (x,y) \mapsto (u, v, 1-(u+v)).$$

We calculate $\partial_u \mathbf{r} \times \partial_v \mathbf{r} = \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$. Consequently

$$\iint_S \boldsymbol{f} \cdot \mathbf{n}\, dS = \iint_T \left(\begin{smallmatrix} u^2 \\ v^2 \\ 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)\, dudv = \int_0^1 \left[\int_0^{1-u} u^2 + v^2 + 1\, dv\right]\, du.$$

We calculate

$$\int_0^{1-u} u^2 + v^2 + 1\, dv = \left[(u^2+1)v + \frac{v^3}{3}\right]_0^{1-u} = \frac{4}{3} - 2u + 2u^2 - \frac{4}{3}u^2,$$

and

$$\int_0^1 \frac{4}{3} - 2u + 2u^2 - \frac{4}{3}u^2\, du = \left[\frac{4}{3}u - u^2 + \frac{2}{3}u^3 - \frac{4}{9}u^3\right]_0^1 = \frac{5}{9}.$$