BSc Engineering Sciences – A. Y. 2019/20

Written exam of the course Mathematical Analysis 2

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Solutions

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1. Find a power series solution $y(x)$ around $x_0 = 0$ of the differential equation

$$xy''(x) + 2y'(x) + xy(x) = 1,$$

such that $y(0) = 1$ and determine its radius of convergence.

Solution.

Let us put $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then we have $y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. If $y(x)$ satisfies the above equation, then

$$\begin{aligned}
1 &= x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=1}^{\infty} na_n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} 2na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= \sum_{n=1}^{\infty} (n+1)na_{n+1} x^n + \sum_{n=0}^{\infty} 2(n+1)a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n
\end{aligned}$$

where in the 3rd equality we shifted the index by $n \to n+1$ in the first and the second summations while we shifted the index by $n+1 \to n$ in the last summation.

If this equality holds as power series, then all the coefficients must coincide. In particular, if we look at the coefficient of x^0 (constant), we obtain $1 = 2a_1$, hence $a_1 = \frac{1}{2}$. On the other hand, from $y(0) = 1$, it follows that $a_0 = 1$.

Now, again by comparison of coefficients for $n \geq 1$, we have $(n+1)na_{n+1} + 2(n+1) + a_{n-1} = 0$, or equivalently,

$$a_{n+1} = -\frac{a_{n-1}}{(n+2)(n+1)}.$$

By this recursion relation, we have

$$a_{2n} = \frac{(-1)^n a_0}{(2n+1)!} = \frac{(-1)^n}{(2n+1)!}, a_{2n+1} = \frac{(-1)^n a_1}{(2n+2)!} = \frac{(-1)^n}{2(2n+2)!}.$$

Altogether, we obtain

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+2)!} x^{2n+1}.$$

To see the convergence of the infinite sum, we apply the ratio test to each part of the sum:

$$\frac{(2n+1)!|x^{2(n+1)}|}{(2(n+1)+1)!|x|^{2n}} = \frac{|x|^2}{(2n+2)(2n+3)} \to 0, \quad \frac{2(2n+2)!|x^{2(n+1)+1}|}{2(2(n+1)+2)!|x|^{2n+1}} = \frac{|x|^2}{2n+2} \to 0,$$

Therefore, the radius of convergence is ∞.

2. Find all the stationary points of the following scalar field, defined on $\mathbb{R}^2$,

$$f(x,y) = e^{-3x^2+2xy-3y^2}(x-y)$$

and classify them into relative minima, maxima and saddle points.

Solution.

For the f given above, it holds that

$$\nabla f(x,y) = \left(e^{-3x^2+2xy-3y^2}(1+(x-y)(-6x+2y)), e^{-3x^2+2xy-3y^2}(-1+(x-y)(2x-6y))\right).$$

At stationary points, $\nabla f(x,y) = \mathbf{0}$ holds. Namely,

$$e^{-3x^2+2xy-3y^2}(1+(x-y)(-6x+2y) = 0, e^{-3x^2+2xy-3y^2}(-1+(x-y)(2x-6y)) = 0$$

As $e^{-3x^2+2xy-3y^2}$ takes never 0, this is equivalent to

$$1+(x-y)(-6x+2y) = 0, -1+(x-y)(2x-6y) = 0$$

By summing these equations, we have $(x-y)(-4x-4y) = 0$, hence $x = y$ or $x = -y$

Case 1. $x = y$**.** This with the first equation gives $1 = 0$, which is impossible.

Case 2. $x = -y$**.** This with the first equation gives $1 - 16x^2 = 0$, or $x = \pm\frac{1}{4}$. Correspondingly, $(x,y) = (\frac{1}{4}, -\frac{1}{4})$ and $(-\frac{1}{4}, \frac{1}{4})$.

To classify these points, let us compute the Hessian matrix:

$$\begin{aligned}
Dxx &= e^{-3x^2+2xy-3y^2}((1+(x-y)(-6x+2y))(-6x+2y) + (-6x+2y) - 6(x-y)) \\
Dyx &= e^{-3x^2+2xy-3y^2}((1+(x-y)(-6x+2y))(2x-6y) - (-6x+2y) + 2(x-y)) \\
Dxy &= e^{-3x^2+2xy-3y^2}(-1+(x-y)(2x-6y))(-6x+2y) + (-6x+2y) + 2(x-y)) \\
Dyy &= e^{-3x^2+2xy-3y^2}((-1+(x-y)(2x-6y))(2x-6y) - (2x-6y) - 6(x-y))
\end{aligned}$$

(it is a good idea not to expand the formula at this point, because, at the stationary point we have $1+(x-y)(-6x+2y) = -1+(x-y)(2x-6y) = 0$). At the point $(x,y) = (\frac{1}{4}, -\frac{1}{4})$, this becomes

$$e^{-\frac{1}{2}}\begin{pmatrix} -5 & 3 \\ 3 & -5 \end{pmatrix}.$$

Its determinant is $e^{-\frac{1}{2}}16 > 0$, and its trace is $-e^{-\frac{1}{2}}10 < 0$, therefore, its eigenvalues are negative, and the point $(\frac{1}{4}, -\frac{1}{4})$ is a relative maximum.

At the point $(x,y) = (-\frac{1}{4}, \frac{1}{4})$, this becomes

$$e^{-\frac{1}{2}}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}.$$

Its determinant is $e^{-\frac{1}{2}}16 > 0$, and its trace is $e^{-\frac{1}{2}}10 > 0$, therefore, its eigenvalues are positive, and the point $(-\frac{1}{4}, \frac{1}{4})$ is a relative minimum.

3. Consider the curve $C = \left\{(x,y) : \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1, y \geq 0\right\}$.

1. Find a parametrization $\boldsymbol{\alpha}(t)$ of C starting from $(-2,0)$ and ending at $(2,0)$.
2. Compute $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$ for the vector field $\mathbf{f}(x,y) = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$
3. Compute $\int \mathbf{f} \cdot d\boldsymbol{\beta}$ for the path $\boldsymbol{\beta}(t) = (t,0)$, $t \in (-2,2)$ where $\mathbf{f}$ is the vector field above.

Solution. There are many possible choices of parametrization. One possibility is (note the correct orientation)

$$\boldsymbol{\alpha}(t) = (-2\cos t, 3\sin t)\,, t \in [0,\pi].$$

(Another obvious choice is $\boldsymbol{\alpha}(t) = \left(t, 3\sqrt{1 - x^2/4}\right), t \in [-2,2]$.) In preparation for evaluating the line integral we calculate

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 2\sin t \\ 3\cos t \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} 4\cos^2 t \\ 9\sin^2 t \end{pmatrix}$$

and so $\boldsymbol{\alpha}'(t) \cdot \mathbf{f}(\boldsymbol{\alpha}(t)) = 8\sin t\cos^2 t + 27\cos t\sin^2 t$. Consequently

$$\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int_0^\pi 8\sin t\cos^2 t + 27\cos t\sin^2 t \; dt = \left[-\frac{8}{3}\cos^3 t + \frac{27}{3}\sin^3 t\right]_0^\pi = \frac{16}{3}.$$

Now we calculate the other line integral. In this case

$$\boldsymbol{\beta}'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\beta}(t)) = \begin{pmatrix} t^2 \\ 9(1 - t^2/2) \end{pmatrix}$$

and so $\boldsymbol{\beta}'(t) \cdot \mathbf{f}(\boldsymbol{\beta}(t)) = t^2$. Consequently

$$\int \mathbf{f} \cdot d\boldsymbol{\beta} = \int_{-2}^2 t^2 \; dt = \left[\frac{1}{3}t^3\right]_{-2}^2 = \frac{16}{3}.$$

That the answers are identical follows from Green's theorem since, if we write $\mathbf{f}(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$ where $f_1(x,y) = x^2$ and $f_2(x,y) = y^2$, we observe that $\partial_y f_1 = \partial_x f_2$. This means that we could just have done the second simpler integral. However by doing both it allows to check for possible errors.

4. The set $V = \{(x, y, z) : x^2 + y^2 \le 4, 0 \le z \le 2 - \sqrt{x^2 + y^2}\}$ is a cone of height 2 with base in the xy-plane. The set $W = \{(x, y, z) : (x - 1)^2 + y^2 \le 1\}$ is a cylinder. Let $D \subset \mathbb{R}^3$ be the subset of the cone V which is contained within the cylinder W. Calculate the volume of D (hint: the volume is approximately ≈ 2.7).

Solution. If we define $S = \{(x, y) : (x - 1)^2 + y^2 \le 1\}$ then

$$D = \left\{(x, y, z) : (x, y) \in S, 0 \le z \le 2 - \sqrt{x^2 + y^2}\right\}$$

and so the volume of D is equal to $\iint_S 2 - \sqrt{x^2 + y^2}\, dxdy$. To proceed we use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ which means that the Jacobian is $J(r, \theta) = r$ and the corresponding region is (it helps to sketch a picture here)

$$\widetilde{S} = \left\{(r, \theta) : \theta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right], 0 \le r \le 2\cos\theta\right\}.$$

The condition on r is because $(x - 1)^2 + y^2 \le 1$ implies $(r\cos\theta - 1)^2 + r^2\sin^2\theta \le 1$ which in turn implies that $-2\cos\theta + r \le 0$. This all means that the volume of D is equal to

$$\iint_{\widetilde{S}} r(2 - r)\; drd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_0^{2\cos\theta} 2r - r^2\; dr\right] d\theta$$

For the inner integral we calculate

$$\int_0^{2\cos\theta} 2r - r^2\; dr = \left[r^2 - \frac{1}{3}r^3\right]_0^{2\cos\theta} = 4\cos^2\theta - \frac{8}{3}\cos^3\theta.$$

Consequently the volume of D is equal to

$$4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\; d\theta - \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta\; d\theta.$$

Either from memory or from calculation $\int \cos^2\; d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta)$ and $\int \cos^3\theta\; d\theta = \sin\theta - \frac{1}{3}\sin^3\theta$. It is also convenient to note that both $\cos^2$ and $\cos^3$ are even. Putting everything together we have calculated that the volume of D is equal to

$$2\left[2\theta + 2\sin\theta\cos\theta - \tfrac{8}{3}\sin\theta + \tfrac{8}{9}\sin^3\theta\right]_0^{\frac{\pi}{2}} = 2(\pi - \tfrac{8}{3} + \tfrac{8}{9}) = 2\pi - \tfrac{32}{9}.$$

5. Consider surface $S = \{(x,y,z) : x^2+y^2 = z, z \leq 3\}$ and vector field $\mathbf{f}(x,y,z) = \left(\begin{smallmatrix} xy \\ xy \\ 1 \end{smallmatrix}\right)$. Calculate $\iint_S \mathbf{f}\cdot\mathbf{n}\; dS$ where $\mathbf{n}$ is the unit normal to S which has negative z-component (hint: the answer is approximately ≈ -9.4).

Solution. It is convenient to define the surface $\widetilde{S} = \{(x,y,z) : x^2+y^2 \leq 3, z = 3\}$ and the solid $V = \{(x,y,z) : z \in [0,3], x^2+y^2 \leq z\}$. We observe that together S and $\widetilde{S}$ form a closed surface which encloses V (a sketch might be useful). The Theorem of Gauss and the additivity of the integral means that

$$\iint_S \mathbf{f}\cdot\mathbf{n}\; dS + \iint_{\widetilde{S}} \mathbf{f}\cdot\mathbf{n}\; dS = \iiint_V \nabla\cdot\mathbf{f}\; dV.$$

Note that here we use $\mathbf{n}$ as the outward unit normal which coincides with the normal of the question. We calculate that $\nabla\cdot\mathbf{f} = x+y$ and consider the integral

$$\iiint_V \nabla\cdot\mathbf{f}\; dV = \iiint_V (x+y)\; dxdydz = \iiint_V x\; dxdydz + \iiint_V y\; dxdydz = 0.$$

The integrals are equal to zero because of integrating an odd function over a symmetric region. Consequently $\iint_S \mathbf{f}\cdot\mathbf{n}\; dS = -\iint_{\widetilde{S}} \mathbf{f}\cdot\mathbf{n}\; dS$. By observation, the outward normal on $\widetilde{S}$ is the constant vector $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$ and so (still on this flat surface which we added) $\mathbf{f}\cdot\mathbf{n} = \left(\begin{smallmatrix} xy \\ xy \\ 1 \end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) = 1$. This means that $\iint_{\widetilde{S}} \mathbf{f}\cdot\mathbf{n}\; dS$ is equal to the area of $\widetilde{S}$ and so is equal to 3π and so $\iint_S \mathbf{f}\cdot\mathbf{n}\; dS = -3\pi$.

Alternatively we can calculate the surface integral directly. A possible choice for the parametric form of the surface S is to let $T = \{(r,\theta) : r \in [0,\sqrt{3}], \theta \in [0,2\pi]\}$ and

$$\mathbf{r} : (r,\theta) \mapsto (r\cos\theta, r\sin\theta, r^2).$$

We calculate

$$\frac{\partial\mathbf{r}}{\partial r} \times \frac{\partial\mathbf{r}}{\partial\theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2r \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix}$$

and observe that this corresponds to the opposite normal compared to the one that we want so we will need to add a minus sign.

$$\begin{aligned}
\iint_S \mathbf{f}\cdot\mathbf{n}\; dS &= -\int_0^{\sqrt{3}}\int_0^{2\pi} \begin{pmatrix} r^2\cos\theta\sin\theta \\ r^2\cos\theta\sin\theta \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix}\; drd\theta \\
&= -\int_0^{\sqrt{3}}\int_0^{2\pi} 2r^4\left(\cos^2\theta\sin\theta + \cos\theta\sin^2\theta\right) + r\; drd\theta \\
&= -\int_0^{\sqrt{3}}\int_0^{2\pi} r\; drd\theta = -2\pi\left[\frac{1}{2}r^2\right]_0^{\sqrt{3}} = -3\pi.
\end{aligned}$$