Mathematical Analysis II, 2018/19 First semester

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We basically follow the textbook "Calculus" Vol. I,II by Tom M. Apostol, Wiley.

Sep 27. Pointwise and uniform convergence

1. (a) Study the convergence of $f_n(x) = e^{-nx^2}, x \in \mathbb{R}$.
   (b) Show that $f_n(x) = \sum_{k=1}^n \frac{\sin x}{3^k}, x \in \mathbb{R}$ is uniformly convergent.
   (c) Compute $\lim_{n\to\infty} \int_0^s x^n dx$ for $s \in [0, 1)$.

2. Let $f_n(x) = nxe^{-nx^2}$ for $n = 1, 2, \cdots$ and $x \in \mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \lim f_n(x)dx.$$

3. Let $f_n(x) = \frac{\sin x}{n}$, $f(x) = \lim_{n\to\infty} f(x)$. Show that $\lim_{n\to\infty} f_n'(x) \neq f'(x)$.

4. Study the convergence, as $n \to \infty$, of $f_n(x) := \sin(x + 2\pi\sqrt{n^2 + 1})$, $\quad x \in \mathbb{R}, n \in \mathbb{N}$.

5. Determine the radius of convergence.
   (a) $\sum \frac{z^n}{2^n}$
   (b) $\sum \frac{z^{2n}}{(n+1)2^n}$
   (c) $\sum \frac{(-1)^n 2^{2n} z^n}{2n}$
   (d) $\sum (1 - (-2)^n) z^n$
   (e) $\sum_{n=1}^\infty \frac{1}{n} \left(1 + \frac{1}{n}\right)^{n^2} x^n$
   (f) $\sum_{n=0}^\infty \frac{1}{3^n} (\sqrt{n+1} - \sqrt{n}) x^n$

6. Prove the expansion.
   (a) $\frac{1}{x+1} = \sum_{n=0}^\infty (-1)^n x^n$ for $|x| < 1$ and $\log(x + 1) = \sum_{n=0}^\infty \frac{(-1)^n}{n+1} x^{n+1}$.
   (b) $\frac{1}{x^2+1} = \sum_{n=0}^\infty (-1)^n x^{2n}$ for $|x| < 1$ and $\arctan x = \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} x^{2n+1}$.

Oct 4. Power series, scalar and vector fields

1. Find the radius of convergence adn compute the sum.
   (a) $\sum_{n=0}^\infty \frac{x^n}{3^{n+2}}$.
   (b) $\sum_{n=0}^\infty (-1)^n x^{2n}$.

2. Prove the expansion.

   (a) With $a > 0$, $a^x = \sum_{n=0}^{\infty} \frac{(\log a)^n}{n!} x^n$ for $x \in \mathbb{R}$.

   (b) $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2x+1}}{(2n+1)!}$, where $\sinh x = \frac{e^x - e^{-x}}{2}$.

3. Let $f(x) = e^{-\frac{1}{x}}$ for $x > 0$ and 0 for $x \le 0$. Show that $f^{(n)}(0) = 0$, hence Taylor's series does not converge to $f(x)$.

4. Solve $(1 - x^2)y''(x) - 2xy'(x) + 6y(x) = 0$ with $y(0) = 1, y'(0) = 0$.

5. Let $y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$. Prove that $xy''(x) + y'(x) - y(x) = 0$.

6. Are the following set open in $\mathbb{R}^2$?

   (a) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

   (b) $\{(x, y) \in \mathbb{R}^2 : x \ge 0, y > 1\}$

   (c) $\{(x, y) \in \mathbb{R}^2 : 3x^2 + 2y^2 < 1, |x| \le 2\}$

   (d) $\{(x, y) \in \mathbb{R}^2 : y < \operatorname{sign} x\}$, where $\operatorname{sign} x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$.

7. Determine where the following functions are defined and are continuous

   (a) $f(x, y) = x^4 + y^4 - 4x^2y^2$

   (b) $f(x, y) = \log(x^2 + y^2)$

8. Let $f(x, y) = \frac{x-y}{x+y}$ if $x + y \ne 0$. Show that f is discontinuous at $(0, 0)$.

Oct 11. Partial derivatives, chain rule, level sets.

1. Compute the gradient.

   (a) $f(x, y) = x^2 + y^2 \sin(xy)$.

   (b) $f(x, y) = e^x \cos y$.

   (c) $f(x, y, z) = x^2y^3z^4$.

   (d) $f(x, y, z) = x^{y^z}$ for $x, y, z > 0$.

2. Evaluate the directional derivative. $f(x, y, z) = x^2 + 2y^2 + 3z^2$ at $\boldsymbol{a} = (1, 1, 2)$ in the direction $(1, -1, 2)$.

3. Check that $D_1D_2f = D_2D_1f$ for the following function. $f(x, y) = e^{x^2+y}(x + y^2)$.

4. Compute the derivative of the composed function $f(\boldsymbol{a}(t))$.

   (a) $f(x, y) = x^2 + y^2, \boldsymbol{a}(t) = (t, t^2)$.

   (b) $f(x, y) = e^{xy} \cos(xy^2), \boldsymbol{\alpha}(t) = (\cos t, \sin t)$.

5. Compute the derivative of the following functions of x:

   (a) $f(x) = x^{x^x}$ for $x > 0$.

   (b) $f(x) = \int_{-x^2}^{x^2} e^{t^2} dt$.

6. Find a parametrization $\boldsymbol{a}(t)$ for the curve $x^2 - y^2 = 1$ and compute the tangent at $\boldsymbol{a}(t)$.

7. Find an equation for the line which is tangent to $x^2 + y^2 + 2z^2 = 4$, $z = e^{x-y}$ at $(1, 1, 1)$.
8. Let $x = e^u \cos v, y = e^u = \sin v$. Show that for any $(x, y) \neq (0, 0)$ there is (u, v) satisfying these equations. Write it as $u = U(x, y), v = V(x, y)$. Show that $\nabla U(x, y)$ and $\nabla V(x, y)$ are orthogonal.

Oct 18. Derivatives of vector fields, partial differential equations.

1. $\boldsymbol{f}(x, y) = (a_1x + b_1y, c_1x + d_1y), \boldsymbol{g}(a_2x + b_2y, c_2x + d_2y)$. Compute $\boldsymbol{f}', \boldsymbol{g}'$ and $(\boldsymbol{f} \circ \boldsymbol{g})'$.
2. Let $\varphi(r, \theta) = f(r\cos\theta, r\sin\theta)$. Express $\frac{\partial^2\varphi}{\partial r^2}$ in terms of the partial derivatives of f.
3. Let $f(x, y) = g(\sqrt{x^2 + y^2})$. Prove that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{r}g'(r) + g''(r)$.
4. Determine the solution of $4\frac{\partial f}{\partial x} + 3\frac{\partial f}{\partial y} = 0$ with $f(x, 0) = \sin x$.
5. Find the solution $u(x, t)$ of the wave equation
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2} \\ u(x, 0) = \dfrac{1}{1 + x^2} \\ \dfrac{\partial u}{\partial t}(x, 0) = \dfrac{-2x}{(1 + x^2)^2} \end{cases}$$
6. Find $\alpha > 0$ for which the function $g(x, y, z) = \dfrac{1}{(x^2 + y^2 + z^2)^\alpha}$ satisfies the partial differential equation on $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$:
$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = 0.$$
7. For $n \in \mathbb{Z}$, find $\alpha > 0$ for which the function $f(x, y) = \sin(nx)e^{-\alpha t}$ satisfies the heat equation on $x \in [0, \pi], t \geq 0$:
$$k\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}.$$
8. If $k > 0$, and $g(x, t) = \frac{x}{2\sqrt{kt}}$, then set $f(x, t) = \int_0^{g(x,t)} e^{-u^2}du$.
   - Show that $\frac{\partial f}{\partial x} = e^{-g^2}\frac{\partial g}{\partial x}, \frac{\partial f}{\partial t} = e^{-g^2}\frac{\partial g}{\partial t}$.
   - Show that $f(x, t)$ satisfies the heat equation.

Oct 25. Implicit functions and partial derivatives

1. The equation $x + z + (y + z)^2 = 6$ defines implicitly a function $f(x, y) = z$. Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ in terms of x, y, z. Check that $(1, 1, 1)$ satisfies the equation, and compute $\frac{\partial f}{\partial x}(1, 1), \frac{\partial f}{\partial y}(1, 1)$.
2. Consider two surfaces $2x^2 + 3y^2 - z^2 - 25 = 0$, $x^2 + y^2 - z^2 = 0$. The intersection C can be parametrized as $(X(z), Y(z), z)$.
   (a) Check that C passes the point $P = (\sqrt{7}, 3, 4)$.
   (b) Find a tangent vector of C at P.
3. Locate and classify the stationary points.

(a) $f(x,y) = x^2 + (y-1)^2$

(b) $f(x,y) = 2x^2 - xy - 3y^2 - 3x + 7y$

(c) $f(x,y) = \sin x \cosh y$

(d) $f(x,y) = e^{x+y}(x^2 + xy)$

4. Let $x_1, \cdots, x_n$ be distinct numbers, $y_1, \cdots, y_n \in \mathbb{R}$. Let $a, b \in \mathbb{R}, f(x) = ax + b$. With $E(a,b) = \sum_{j=1}^{n} |f(x_j) - y_j|^2$. Find a, b which minimize $E(a,b)$.

5. Let $g(x,y)$ the function implicitly defined by $x^2 + y^2 + g(x,y)^2 = 1, g(x,y) > 0$ and $h(x,y) = x + 2y + 2g(x,y)$. Compute $\frac{\partial h}{\partial x}(\frac{1}{3}, \frac{2}{3}), \frac{\partial h}{\partial y}(\frac{1}{3}, \frac{2}{3})$. What does the result mean?