Mathematical Analysis II, 2018/19 First semester

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We basically follow the textbook "Calculus" Vol. I,II by Tom M. Apostol, Wiley.

Oct 22. Implicit functions and partial derivatives

Recall that a **function** or a **scalar field** $f(\cdot)$ defined on a subset S of $\mathbb{R}^n$ assigns to each point $x \in S$ a real number $f(x)$, and it is represented by a curve or a surface.

Example 1. Explicitly given functions.

- $f(x) = x^2$
- $f(x, y) = \cos xe^y$
- $f(x, y, z) = e^{198x^7}z + (x - 2345)^{32} + (x^2 + 28)(y^3 - 2\pi)...$

Sometime a function is defined implicitly: consider the equation

$$x^2 + y^2 = 1.$$

This defines a circle. By solving this equation, we obtain

$$y = \pm\sqrt{1 - x^2}.$$

Namely, the curve $x^2 + y^2 = 0$ **defines implicitly** the function $y = f(x) = \pm\sqrt{1 - x^2}$. Similarly, the equation $x^2 + y^2 + z^2 = 1$ represents a sphere. It defines the function (scalar field) $z = f(x, y) = \pm\sqrt{1 - x^2 - y^2}$.

In general, if $F(x, y, z)$ is a function, the equation $F(x, y, z) = 0$ **may** define a function (but not always). Furthermore, even if it defines a function, it is **not always possible to solve it explicitly**. Can you solve the following equation in z?

$$F(x, y, z) = y^2 + xz + z^2 - e^z - 4 = 0$$

We **assume** that there is a function $f(x, y)$ such that $F(x, y, f(x, y)) = 0$. Even if **we do not know** the explicit form of $f(x, y)$, we can obtain **some information** about $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Consider $g(x, y) = F(x, y, F(x, y)) = 0$ as a function of two variables x, y. Obviously we have $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$. On the other hand, one can see it as

$$g(x, y) = F(u_1(x, y), u_2(x, y), u_3(x, y)) \quad \text{with } u_1(x, y) = x, \quad u_2(x, y) = y, \quad u_3(x, y) = f(x, y).$$

By the chain rule,

$$0 = \frac{\partial g}{\partial x}(x, y) = \frac{\partial F}{\partial x}(x, y, f(x, y)) \cdot 1 + \frac{\partial F}{\partial y}(x, y, f(x, y)) \cdot 0 + \frac{\partial F}{\partial z}(x, y, f(x, y)) \cdot \frac{\partial f}{\partial x}(x, y),$$

therefore,

$$\frac{\partial f}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}(x,y,f(x,y))}{\frac{\partial F}{\partial z}(x,y,f(x,y))}.$$

Similarly,

$$\frac{\partial f}{\partial y}(x,y) = -\frac{\frac{\partial F}{\partial y}(x,y,f(x,y))}{\frac{\partial F}{\partial z}(x,y,f(x,y))}.$$

Note that $F(x,y,z)$ is **explicitly given**.

Example 2. $F(x,y,z) = y^2 + xz + z^2 - e^z - C$, where $C \in \mathbb{R}$. Assume the existence of $f(x,y)$ such that $F(x,y,f(x,y)) = 0$. Find the value of C such that $f(0,e) = 2$ and compute $\frac{\partial f}{\partial x}(0,e)$, $\frac{\partial f}{\partial y}(0,e)$.

Solution. $F(0,e,f(0,e)) = F(0,e,2) = e^2 + 0 + 2^2 - e^2 - C = 0 \Longrightarrow C = 4$. Note that

- $\frac{\partial F}{\partial x}(x,y,z) = z$ and hence $\frac{\partial F}{\partial x}(x,y,f(x,y)) = f(x,y)$
- $\frac{\partial F}{\partial y}(x,y,z) = 2y$ and hence $\frac{\partial F}{\partial y}(x,y,f(x,y)) = 2y$
- $\frac{\partial F}{\partial z}(x,y,z) = x + 2z - e^z$ and hence $\frac{\partial F}{\partial z}(x,y,f(x,y)) = x + f(x,y) - e^{f(x,y)}$

Therefore,

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= -\frac{\frac{\partial F}{\partial x}(x,y,f(x,y))}{\frac{\partial F}{\partial z}(x,y,f(x,y))} = -\frac{f(x,y)}{x + 2f(x,y) - e^{f(x,y)}} \\
\frac{\partial f}{\partial x}(0,e) &= -\frac{f(0,e)}{0 + 2f(0,e) - e^{f(0,e)}} = \frac{2}{e^2 - 4}. \\
\frac{\partial f}{\partial y}(x,y) &= -\frac{\frac{\partial F}{\partial y}(x,y,f(x,y))}{\frac{\partial F}{\partial z}(x,y,f(x,y))} = -\frac{2y}{x + 2f(x,y) - e^{f(x,y)}} \\
\frac{\partial f}{\partial y}(0,e) &= -\frac{2e}{0 + 2f(0,e) - e^{f(0,e)}} = \frac{2e}{e^2 - 4}.
\end{aligned}$$

More generally, if $F(x_1, \cdots, x_n) = 0$ defines a function $x_n = f(x_1, \cdots, x_{n-1})$, then

$$\frac{\partial f}{\partial x_k}(x_1, \cdots, x_{n-1}) = -\frac{\frac{\partial F}{\partial x_k}(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1}))}{\frac{\partial F}{\partial x_n}(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1}))}.$$

Next, let us consider two surfaces $F(x,y,z) = 0$ and $G(x,y,z) = 0$ and assume that their intersection is a curve $(X(z), Y(z), z)$. Namely, $F(X(z), Y(z), z) = 0, G(X(z), Y(z), z) = 0$.

Example 3. The unit sphere $F(x,y,z) = x^2 + y^2 + z^2 = 0$ and the xz-plane $G(x,y,z) = y = 0$ has the intersection $x^2 + z^2 = 0 \Longrightarrow x = X(z) = \pm\sqrt{1 - z^2}, Y(z) = 0$.

Even if $X(z)$ and $Y(z)$ are only **implicitly given**, we can compute their derivatives. As before, put $f(z) = F(x,y,z), g(z) = G(x,y,z) = 0$. By the chain rule,

$$0 = f'(z) = X'(z)\frac{\partial F}{\partial x}(X(z), Y(z), z) + Y'(z)\frac{\partial F}{\partial x}(X(z), Y(z), z) + \frac{\partial F}{\partial z}(X(z), Y(z), z).$$

Similarly,

$$0 = g'(z) = X'(z)\frac{\partial G}{\partial x}(X(z), Y(z), z) + Y'(z)\frac{\partial G}{\partial x}(X(z), Y(z), z) + \frac{\partial G}{\partial z}(X(z), Y(z), z).$$

From these, we obtain

$$\begin{pmatrix} X'(z) \\ Y'(z) \end{pmatrix} = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}^{-1} (X(z), Y(z), z) \cdot \begin{pmatrix} -\frac{\partial G}{\partial z} \\ -\frac{\partial G}{\partial z} \end{pmatrix} (X(z), Y(z), z).$$

Example 4. Computations of partial derivatives.

- $x = u + v, y = uv^2$ defines $u(x, y), v(x, y)$. Compute $\frac{\partial v}{\partial x}$.

  Solution. By eliminating u, we obtain $xv^2 - v^3 - y = 0$. In other words, $F(x, y, v) = 0$ where $F(x, y, v) = xv^2 - v^3 - y$. By the formula above, with $\frac{\partial F}{\partial x} = v^2$, $\frac{\partial F}{\partial v} = 2xv - 3v^2$, $\frac{\partial v}{\partial x} = -\frac{v(x,v)^2}{2xv(x,y)-3v(x,y)^2}$.

- Assume that $g(x, y) = 0$ defines implicitly $Y(x)$. Let $f(x, y)$ be another function. Then $h(x) = f(x, Y(x))$ is a function of x. By the chain rule,

$$\begin{aligned} h'(x) &= \frac{\partial f}{\partial x}(x, Y(x)) + \frac{\partial F}{\partial y}(x, Y(x))Y'(x) \\ &= \frac{\partial f}{\partial x}(x, Y(x)) - -\frac{\frac{\partial g}{\partial x}(x, Y(x))}{\frac{\partial g}{\partial y}(x, Y(x))}\frac{\partial F}{\partial y}(x, Y(x)) \end{aligned}$$

- Let u be defined by $F(u+x, yu) = u$. Let $u = g(x, y)$, then $g(x, y) = F(g(x, y)+x, yg(x, y))$, and

$$\begin{aligned} &\frac{\partial g}{\partial x}(x, y) \\ &= \frac{\partial F}{\partial X}(g(x, y) + x, yg(x, y))\left(\frac{\partial g}{\partial x}(x, y) + 1\right) + \frac{\partial F}{\partial Y}(g(x, y) + x, yg(x, y)) \cdot y\frac{\partial g}{\partial x}(x, y) \\ &\implies \frac{\partial g}{\partial x}(x, y) = \frac{-\frac{\partial F}{\partial X}(g(x, y) + x, yg(x, y))}{\frac{\partial F}{\partial X}(g(x, y) + x, yg(x, y)) + y\frac{\partial F}{\partial X}(g(x, y) + x, yg(x, y)) - 1} \end{aligned}$$

- $2x = v^2 - u^2, y = uv$ defines implicitly $u(x, y), v(x, y)$ (it is also possible to solve them: $2x + (\frac{y}{v})^2 = v^2, (\frac{y}{u})^2 - u^2 = 2x$). Compute $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.

  Solution. By differentiating with respect to x,

$$2 = 2v\frac{\partial v}{\partial x} - 2u\frac{\partial u}{\partial x}, \qquad 0 = \frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x}$$

  From which one obtains

$$\frac{\partial u}{\partial x} = -\frac{u}{u^2 + v^2}, \qquad \frac{\partial v}{\partial x} = \frac{v}{u^2 + v^2}.$$

  Similarly,

$$\frac{\partial u}{\partial y} = \frac{v}{u^2 + v^2}, \qquad \frac{\partial v}{\partial y} = \frac{u}{u^2 + v^2}.$$

Oct 24. Minima, maxima and saddle points

Various extremal points

Let $S \subset \mathbb{R}^n$ be an open set, $f : S \to \mathbb{R}$ be a scalar field and $\boldsymbol{a} \in S$. Recall that $B(\boldsymbol{a}, r) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{a}\| < r\}$ is the r-ball centered at $\boldsymbol{a}$.

Definition 5. (Minima and maxima)

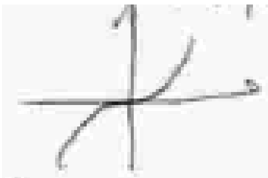
- If $f(\boldsymbol{a}) \leq f(\boldsymbol{x})$ (respectively $f(\boldsymbol{a}) \geq f(\boldsymbol{x})$) for all $\boldsymbol{x} \in S$, then $f(\boldsymbol{a})$ is said to be the **absolute** minimum (resp. maximum) of f.

- If $f(\boldsymbol{a}) \leq f(\boldsymbol{x})$ (respectively $f(\boldsymbol{a}) \geq f(\boldsymbol{x})$) for $\boldsymbol{x} \in B(\boldsymbol{a}, r)$ for some r, then $f(\boldsymbol{a})$ is said to be a **relative** minimum (resp. maximum)

Theorem 6. *If f is differentiable and has a relative minumum (resp. maximum) at $\boldsymbol{a}$, then* $\nabla f(\boldsymbol{a}) = 0$.

Proof. We prove the statement only for a relative minumum, because the other case is analogous. For any unit vector $\boldsymbol{y}$, consider $g(u) = f(\boldsymbol{a} + u\boldsymbol{y})$. As $\boldsymbol{a}$ is a relative minumum, g has a relative minumum at $u = 0$, therefore, $g'(0) = 0$, and $f'(\boldsymbol{a};\boldsymbol{y}) = 0$ for any $\boldsymbol{y}$. This implies that $\nabla f(\boldsymbol{a}) = \boldsymbol{0}$. □

Remark 7. $\nabla f(\boldsymbol{a}) = \boldsymbol{0}$ does not imply that f takes a relative minumum or maximum at $\boldsymbol{a}$. Even in $\mathbb{R}$, $f(x) = x^3$ has $f'(0) = 0$ but 0 is not a relative minumum either a relative maximum.

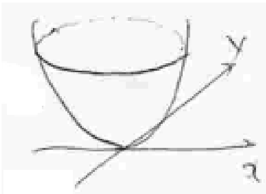


Definition 8. (Stationary points)

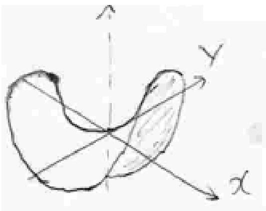
- If $\nabla f(\boldsymbol{a}) = \boldsymbol{0}$, then $\boldsymbol{a}$ is called a **stationary point**.
- If $\nabla f(\boldsymbol{a}) = \boldsymbol{0}$ and $\boldsymbol{a}$ is neither a relative minumum nor a relative maximum, then $\boldsymbol{a}$ is called a **suddle point**.

Example 9. (Stationary points)

- $f(x, y) = x^2 + y^2$. $\nabla f(\boldsymbol{x}) = (2x, 2y), \nabla f(0, 0) = (0, 0)$. $f(0, 0)$ is the absolute minumum.



- $f(x, y) = xy$. $\nabla f(\boldsymbol{x}) = (y, x), \nabla f(0, 0) = 0$. $f(0, 0)$ is a saddle:
  - $x > 0, y > 0$, then $f(x, y) > 0$.
  - $x > 0, y < 0$, then $f(x, y) < 0$.



Second-order Taylor formula

Let f be a differentiable function. We learned that $f(\boldsymbol{a} + \boldsymbol{y}) = f(\boldsymbol{a}) + \nabla f(\boldsymbol{a}) \cdot \boldsymbol{y} + \|\boldsymbol{y}\| E_1(\boldsymbol{a}, \boldsymbol{y})$ and $E_1(\boldsymbol{a}, \boldsymbol{y}) \to 0$ as $\|\boldsymbol{y}\| \to 0$.

Let f have continuous second partial derivatives and let us denote them by $D_{ij} f = \frac{\partial^2 f}{\partial x_i, \partial x_j}$. Define the **Hessian matrix** by

$$H(\boldsymbol{x}) = \begin{pmatrix} D_{11} f(\boldsymbol{x}) & D_{12} f(\boldsymbol{x}) & \cdots & D_{1n} f(\boldsymbol{x}) \\ \vdots & & & \vdots \\ D_{n1} f(\boldsymbol{x}) & D_{n2} f(\boldsymbol{x}) & \cdots & D_{nn} f(\boldsymbol{x}) \end{pmatrix}$$

This is a *real symmetric matrix*. For $\boldsymbol{y} = (y_1, \cdots, y_n)$, $\boldsymbol{y} H(\boldsymbol{x}) \boldsymbol{y}^{\mathrm{t}} \in \mathbb{R}$.

Theorem 10. *Let f be a scalar field with continuous second partial derivatives on $B(\boldsymbol{a};r)$. Then, for $\boldsymbol{y}$ such that $\boldsymbol{a}+\boldsymbol{y}\in B(\boldsymbol{a};r)$ there is $0<c<1$ such that*

$$f(\boldsymbol{a}+\boldsymbol{y}) = f(\boldsymbol{a}) + \nabla f(\boldsymbol{a})\cdot\boldsymbol{y} + \frac{1}{2}\boldsymbol{y}H(\boldsymbol{a}+c\boldsymbol{y})\boldsymbol{y}^{\mathrm{t}},$$
$$f(\boldsymbol{a}+\boldsymbol{y}) = f(\boldsymbol{a}) + \nabla f(\boldsymbol{a})\cdot\boldsymbol{y} + \frac{1}{2}\boldsymbol{y}H(\boldsymbol{a})\boldsymbol{y}^{\mathrm{t}} + \|\boldsymbol{y}\|^2\cdot E_2(\boldsymbol{a},\boldsymbol{y})$$

and $E_2(\boldsymbol{a},\boldsymbol{y})\to 0$ as $\|\boldsymbol{y}\|\to 0$.

Proof. Let us define $g(u)=f(\boldsymbol{a}+u\boldsymbol{y})$. We apply the Taylor formula to g to get $g(1)=g(0)+g'(0)+\frac{1}{2}g''(c)$ with $0<c<1$. Since we have $g(u)=f(a_1+uy_1,\cdots,a_n+uy_n)$, by chain rule,

$$g'(u) = \sum_{j=1}^{n} D_j f(a_1+uy_1,\cdots,a_n+uy_n)y_j = \nabla f(\boldsymbol{a}+u\boldsymbol{y})\cdot\boldsymbol{y},$$

where $D_j f = \frac{\partial f}{\partial x_j}$. Similarly,

$$g''(u) = \sum_{i,j=1}^{n} D_{ij} f(a_1+uy_1,\cdots,a_n+uy_n)y_i y_j = \boldsymbol{y}H(\boldsymbol{a}+u\boldsymbol{y})\boldsymbol{y}^{\mathrm{t}},$$

from which the first equation follows. As for the second equation, we define E_2 by $E_2(\boldsymbol{a},\boldsymbol{y}) = \frac{1}{2}(\boldsymbol{y}H(\boldsymbol{a}+c\boldsymbol{y}) - H(\boldsymbol{a})^{\mathrm{t}})\boldsymbol{y}^{\mathrm{t}}/\|\boldsymbol{y}\|^2$. Then

$$|E_2(\boldsymbol{a},\boldsymbol{y})| \le \frac{1}{2}\sum_{i,j=1}^{n}\frac{|y_iy_j|}{\|\boldsymbol{y}\|^2}|D_{ij}f(\boldsymbol{a}+c\boldsymbol{y}) - D_{ij}f(\boldsymbol{a})| \to 0$$

as $\|y\|\to 0$, by the continuity of $D_{ij}f(\boldsymbol{a})$. $\square$

Classifiying stationary points

We give a criterion to deterine whether $\boldsymbol{a}$ is a minumum/maximum/saddle when $\nabla f(\boldsymbol{a})=\boldsymbol{0}$.

Theorem 11. *Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}$ be a real symmetric matrix, and $Q(\boldsymbol{y}) = \boldsymbol{y}A\boldsymbol{y}^{\mathrm{t}}$. Then,*

- *$Q(\boldsymbol{y})>0$ for all $\boldsymbol{y}\neq\boldsymbol{0}$ if and only if all eigenvalues of A are positive.*
- *$Q(\boldsymbol{y})<0$ for all $\boldsymbol{y}\neq\boldsymbol{0}$ if and only if all eigenvalues of A are negative.*

Proof. A real symmetric matrix A can be diagonalized by an orthogonal matrix C, namely, $L = C^{\mathrm{t}}AC = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$. If all $\lambda_j>0$, then $Q(\boldsymbol{y}) = \boldsymbol{y}CC^{\mathrm{t}}ACC^{\mathrm{t}}\boldsymbol{y}^{\mathrm{t}} = \boldsymbol{v}L\boldsymbol{v}^{\mathrm{t}} = \sum_j \lambda_j v_j > 0$, where $\boldsymbol{v}=\boldsymbol{y}C$. If $Q(\boldsymbol{y})>0$ for all $\boldsymbol{y}\neq\boldsymbol{0}$, then especially for $\boldsymbol{y}_k = \boldsymbol{u}_kC$ where $\boldsymbol{u}_k = (0,\cdots,0,\underset{k\text{-th}}{1},0,\cdots,0)$, and $Q(\boldsymbol{y}_k)=\lambda_k>0$. $\square$

Theorem 12. *Let f be a scalar field with continuous second derivatives on $B(\boldsymbol{a};r)$. Assume that $\nabla f(\boldsymbol{a})=\boldsymbol{0}$. Then,* *(a) If all the eigenvalues λ_j of $H(\boldsymbol{a})$ are positive, then f has a relative minumum at $\boldsymbol{a}$.*

(b) If all the eigenvalues λ_j of $H(\boldsymbol{a})$ are negative, then f has a relative maximum at $\boldsymbol{a}$.

(c) If some $\lambda_k > 0$ and $\lambda_\ell <$, then $\boldsymbol{a}$ is a saddle.

Proof. (a) Let $Q(\boldsymbol{y}) = \boldsymbol{y}H(\boldsymbol{a})\boldsymbol{y}^{\mathrm{t}}$. Let h be te smallest eigenvalue of $H(\boldsymbol{a})$, $h > 0$ and diagonalize $h(\boldsymbol{a})$ by C. We set $\boldsymbol{y}C = \boldsymbol{v}$, then $\|\boldsymbol{y}\| = \|\boldsymbol{v}\|$. Furthermore,

$$\boldsymbol{y}H(\boldsymbol{a})\boldsymbol{y}^{\mathrm{t}} = \boldsymbol{v}CH(\boldsymbol{a})C^{\mathrm{t}}\boldsymbol{v}^{\mathrm{t}} = \sum_j \lambda_j v_j^2 > h\sum_j v_j^2 = h\|\boldsymbol{v}\|^2 = h\|\boldsymbol{y}\|^2.$$

By Theorem 10,

$$f(\boldsymbol{a}+\boldsymbol{y}) = f(\boldsymbol{a}) + \boldsymbol{y}H(\boldsymbol{a})\boldsymbol{y}^{\mathrm{t}} + \|\boldsymbol{y}\|^2 E_2(\boldsymbol{a},\boldsymbol{y}).$$

As $H_2(\boldsymbol{a},\boldsymbol{y}) \to 0$ as $\|\boldsymbol{y}\| \to 0$, there is r_1 such that if $\|\boldsymbol{y}\| < r_1$, then $|E_2(\boldsymbol{a},\boldsymbol{y})| < \frac{h}{2}$. Now

$$f(\boldsymbol{a}+\boldsymbol{y}) = f(\boldsymbol{a}) + \frac{1}{2}\boldsymbol{y}H(\boldsymbol{a})\boldsymbol{y}^{\mathrm{t}} + \|\boldsymbol{y}\|^2 E_2(\boldsymbol{a},\boldsymbol{y}) > f(\boldsymbol{a}) + \frac{h}{2}\|\boldsymbol{y}\|^2 - \frac{h}{2}\|\boldsymbol{y}\| > f(\boldsymbol{a}),$$

hence f has a relative minumum at $\boldsymbol{a}$.

(b) This case is similar as above.

(c) Let $\boldsymbol{y}_k$ be an eigenvector with eigenvalue λ_k, $\boldsymbol{y}_\ell$ be an eigenvector with eigenvalue λ_ℓ. As in (a), $f(\boldsymbol{a}+c\boldsymbol{y}_k) > f(\boldsymbol{a})$ and as in (b) $f(\boldsymbol{a}+c\boldsymbol{y}_\ell) < f(\boldsymbol{a})$ for small c, hence $\boldsymbol{a}$ is a saddle. □

Example 13. $f(x,y) = xy$. $\lambda\varphi(x,y) = (y,x), H(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $(0,0)$ is a stationary point and $H(0,0)$ has eigenvalues $1,-1$, hence $(0,0)$ is a saddle.

Oct 31. Extremal values

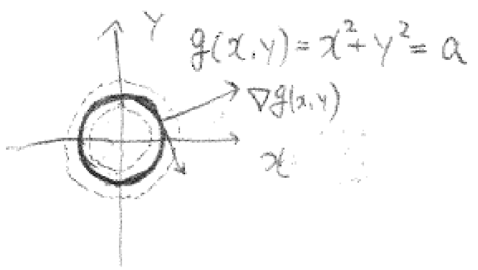
Lagrange's multiplier method

Let $f(\boldsymbol{x})$ be a scalar field. We can restrict f to the set $g(\boldsymbol{x}) = 0$. We would like to find relative minima and maxima of f on that set. We can use the **Lagrange multiplier method**.

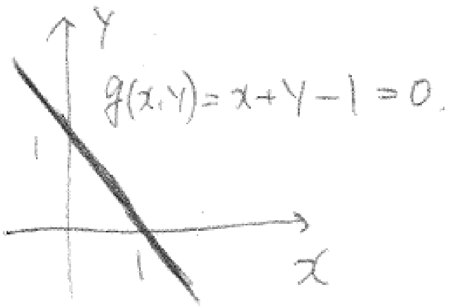
Let $n = 2$. Consider f on the set $g(x,y) = 0$. If f has an extremal value (relative minimum or maximum) at the point (x_1,y_1), f should be stationary along the tangent of $g = 0$ at (x_1,y_1). The gradient $\nabla g(x,y)$ is also orthogonal to the tangent of $g = 0$. Therefore, $\nabla f(x_1,y_1)$ is parallel to $\nabla g(x_1,y_1)$ and

$$\nabla f(x_1,y_1) = \lambda \nabla g(x_1,y_1) \quad \text{for some } \lambda \in \mathbb{R}.$$

Consider the case where $g(x,y) = x^2+y^2-1$. The gradient $\nabla g(x,y) = (2x,2y)$ points to the orthogonal direction to the circle $g(x,y) = 0$.



Example 14. Find the extrema of $f(x,y) = xy$ on $g(x,y) = x+y-1 = 0$.



Solution. $\nabla f(x,y) = (y,x), \nabla g(x,y) = (1,1)$. By the Lagrange multiplier method, at the extrema (x,y) there is λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$. Furthermore, we have $g(x,y) = 0$. Altogether,

$$x = \lambda, \quad y = \lambda, \quad x + y - 1 = 0.$$

By solving these equations, we obtain $x = y = \lambda, 2x - 1 = 0$, namely, $(x,y) = (\frac{1}{2}, \frac{1}{2}), f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$.
Alternative solution. We eliminate y by $y = 1 - x$. We must find the extrema of $f(x, 1-x) = x(1-x) = x - x^2$. $f'(x) = 1 - 2x, f'(x) = 0 \Leftrightarrow x = \frac{1}{2}$ and $f''(\frac{1}{2}) = -\frac{1}{4}$, hence this is a relative maximum and the value is $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$.

Example 15. Find the extrema of $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ on $g(x,y,z) = 0$, where g is some function. The function f is the distance from the point $(0,0,0)$, hence the extrema of f on $g(x,y,z) = 0$ contain the points which are nearest and farthest from $(0,0,0)$.

Next consider $n = 3$. Let $f(x,y,z)$ again be a function and consider the curve C defined by $g_1(x,y,z) = 0, g_2(x,y,z) = 0$, which is the intersection of two curves $g_1(x,y,z) = 0$ and $g_2(x,y,z) = 0$. As before, if $\boldsymbol{a}$ is an extrema point, where $\boldsymbol{a} = (x,y,z)$, then $\nabla f(\boldsymbol{a})$ is orthogonal to the tangent of C at $\boldsymbol{a}$. The same holds for $\nabla g_2(\boldsymbol{a})$ and $\nabla g_2(\boldsymbol{a})$. If $\nabla g_1(\boldsymbol{a})$ and $\nabla g_2(\boldsymbol{a})$ are linearly independent, then they span the orthogonal complement of the tangent of C . Therefore,

$$\nabla f(\boldsymbol{a}) = \lambda_1 \nabla g_1(\boldsymbol{a}) + \lambda_2 g_2(\boldsymbol{a}).$$

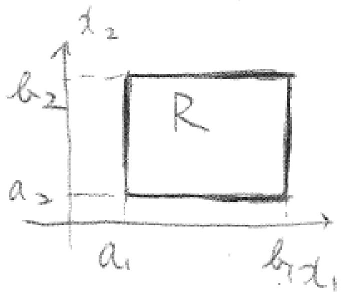
for some $\lambda_1, \lambda_2 \in \mathbb{R}$. We have 5 variables $x, y, z, \lambda_1, \lambda_2$, and 5 equations: the above equation has 3 components, and $g_1(\boldsymbol{a}) = 0$ and $g_2(\boldsymbol{a}) = 0$, hence these equations have generically (but not always) discrete solutions.

In general, in $\mathbb{R}^n$ with m constraings $g_1(\boldsymbol{x}) = 0, \cdots g_m(\boldsymbol{x}) = 0$, $m < n$, we have to solve $\nabla f(\boldsymbol{x}) = \lambda_1 \nabla g_1(\boldsymbol{x}) + \cdots + \lambda_m \lambda g_m(\boldsymbol{x})$ and $g_1(\boldsymbol{x}) = 0, \cdots, g_m(\boldsymbol{x}) = 0$, where there are $n + m$ equations for $n + m$ variables.

It is also possible to determine whether the stationary points are local minima/maxima or saddles, using the techniques of partial derivatives of implicit functions (Nov. 22). In some situations, for example if $g_m(\boldsymbol{x}) = 0$ define a compact set, then it is enough to check all the stationary points and to compare them, and determine which is the maximum and minumum.

Extreme value theorem

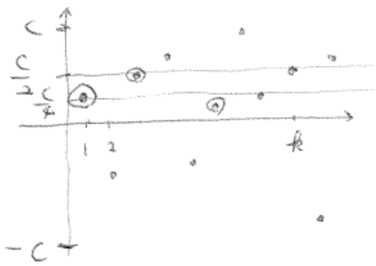
Let $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ a **closed rectangle**. We are going to show that any continuous function f on R has the maximum and minimum.



The idea is the following: we set $M = \sup_R f(\boldsymbol{x})$, then there is a sequence $\boldsymbol{x}_k$ such that $f(\boldsymbol{x}_k) \to M$. As R is closed, $\boldsymbol{x}_k$ converges to a point $\boldsymbol{x}$ in R, and by continuity $f(\boldsymbol{x}_k) \to f(\boldsymbol{x}) = M$.

Lemma 16. *Let $\{x_k\}$ be a bounded sequence, $a \le x_k \le b$. Then there is a convergent subsequence $\{x_{N_k}\}$.*

Proof. We may assume that $-c \le x_k \le c$ for some $c > 0$. As the sequence $\{x_k\}$ is infinite, one of $[-c, 0]$ and $[0, c]$ contains infinite subsequence. Take the first element x_{N_1}. If $[0, c]$ contains an infinite subsequence, then consider next the subdivision $[0, \frac{c}{2}], [\frac{c}{2}, c]$. One of them contains an infinite subsequence. Take x_{N_2}. We can repeat this procedure and obtain a subsequence $\{x_{N_k}\}$.



Let $y_k = \sup_{\ell \ge k} x_{N_\ell}$, $z_k = \inf_{\ell \ge k} x_{N_\ell}$. These are monotone and bounded sequences, hence are convergent: $y := \lim_k y_k, z := \lim_k z_k$. By construction, $|y_k - z_k| < \frac{c}{2^k}$, hence $y = z$. Since $z_k \le x_{N_k} \le y_k$, $\{x_{N_k}\}$ is also convergent to $y = z$. ☐

Lemma 17. *Let $\{\boldsymbol{x}_k\}$ be a sequence of points in R. Then there is a convergent subsequence $\{\boldsymbol{x}_{N_k}\}$.*

Proof. We apply the previous lemma to components of $\boldsymbol{x}_k$. Namely, by considering the first component, we can extract a subsequence whose first component is convergent. Then take a subsequence of it whose second component is convergent. We repeat this procedure n times and obtain a subsequence whose all components are convergent. ☐

Theorem 18. *Let f be a continuous function on R. Then f is bounded, namely, there is $c > 0$ such that $|f(\boldsymbol{x})| < c$.*

Proof. Assume the contrary. Then, for each N, there is $\boldsymbol{x}_N$ such that $|f(\boldsymbol{x}_N)| > N$. By Lemma, there is a convergent subsequence of $\{\boldsymbol{x}_k\}$, hence $\boldsymbol{x}_{N_k} \to \boldsymbol{a}$, which is in R because R is closed. By continuity of f at $\boldsymbol{a}$, there is r such that $|f(\boldsymbol{x}) - f(\boldsymbol{a})| < \epsilon$ for $B(\boldsymbol{a}; r)$. This contradicts with $|f(\boldsymbol{x}_{N_k})| > N$. ☐

Theorem 19. *Let f be a continuous function on R. Then there is $\boldsymbol{a}, \boldsymbol{b} \in R$ such that $f(\boldsymbol{a}) = \sup_{\boldsymbol{x} \in R}$ and $f(\boldsymbol{b}) = \inf_{\boldsymbol{x} \in R}$.*

Proof. By the previous theorem, $\sup_{\boldsymbol{x} \in R} f(\boldsymbol{x}) < c$. There is a sequence $\{\boldsymbol{x}_k\}$ such that $f(\boldsymbol{x}_k) \to \sup_{\boldsymbol{x} \in R} f(\boldsymbol{x})$. By lemma, there is a convergent subsequence $\{\boldsymbol{x}_{N_k}\}$, $\lim_k \boldsymbol{x}_{N_k} = \boldsymbol{a}$. By continuity of f, $f(\boldsymbol{x}_{N_k}) \to f(\boldsymbol{a}) = \sup_{\boldsymbol{x} \in R} f(\boldsymbol{x})$.

The proof for inf is similar. ☐

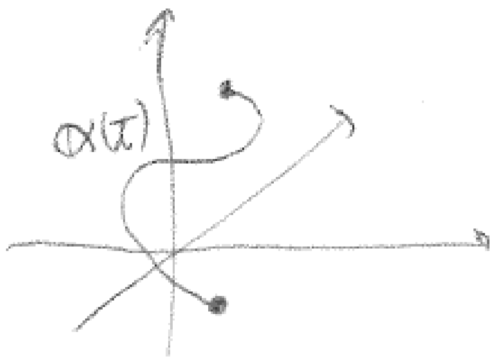
Theorem 20. *Let f be a continuous function on R. Then for any $\epsilon > 0$, there is δ such that for any $\boldsymbol{x} \in R$, $\boldsymbol{y}, \boldsymbol{z} \in B(\boldsymbol{x}; \delta)$, it holds that $|f(\boldsymbol{y}) - f(\boldsymbol{z})| < \epsilon$.*

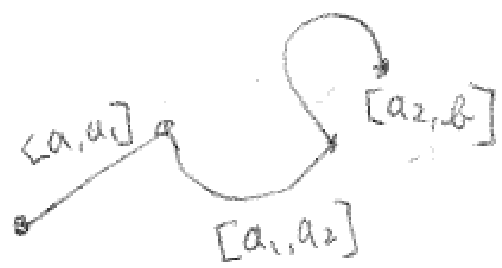
Proof. Let us assume the contrary: there is ϵ such that for each k there are $\boldsymbol{x}_k, \boldsymbol{y}_k, \boldsymbol{z}_k$ such that $\boldsymbol{y}_k, \boldsymbol{z}_k \in B(\boldsymbol{x}_k, \frac{1}{2^k})$ but $|f(\boldsymbol{y}_k) - f(\boldsymbol{z}_k)| > \epsilon$. By Lemma, there is a subsequence $\boldsymbol{x}_{N_k}$, converging to $\boldsymbol{x}$. By continuity of f, there is r such that $|f(\boldsymbol{y}) - f(\boldsymbol{z})| < \epsilon$ for $B(\boldsymbol{x}, r)$. For sufficiently large k, $B(\boldsymbol{x}, r) \supset B(\boldsymbol{x}_{N_k}, \frac{1}{2^k})$, which is a contradiction. $\square$

Nov 5. Line integrals

Definition and basic properties

Let $\boldsymbol{\alpha}(t)$ be a vector-valued continuous function on $[a, b] \to \mathbb{R}^n$. This means that $\boldsymbol{\alpha}(t) = (\alpha_1(t), \cdots, \alpha_n(t))$. We say that $\boldsymbol{\alpha}(t)$ is **continuously differentiable** if each component $\alpha_k(t)$ is differentiable on $[a, b]$ and $\alpha'_k(t)$ is continuous. $\boldsymbol{\alpha}(t)$ is **piecewise continuously differentiable** if $[a, b) = [a, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_\ell, b]$ and $\boldsymbol{\alpha}(t)$ is continuously differentiable on each of these intervals. Such an $\boldsymbol{\alpha}(t)$ **parametrizes a curve** C. We call such an $\boldsymbol{\alpha}$ a **path**.





Let $\boldsymbol{\alpha}(t)$ be a piecewise continuously differentiable path on $[a, b]$ and $\boldsymbol{f}(\boldsymbol{x})$ be a vector field: $S \to \mathbb{R}^n$, where $\boldsymbol{\alpha}(t) \in S$ for $t \in [a, b]$. If $\boldsymbol{f}$ is continuous, we define the line integral of $\boldsymbol{f}$ along $\boldsymbol{\alpha}$ by

$$\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} := \int_a^b \boldsymbol{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) dt,$$

where $\boldsymbol{\alpha}'(t) = (\alpha'_1(t), \cdots, \alpha'_n(t))$. Note that $\boldsymbol{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)$ is a continuous function of t, hence the last integral makes sense.

There are other notations in literature such as $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ if the parametrization of C is clear, or $\int f_1 d\alpha_1 + \cdots + f_n d\alpha_n$.

Example 21. Let $\boldsymbol{f}(x, y) = (y, x^2 y), \boldsymbol{\alpha}(t) = (t, t^2), t \in [0, 1]$. Then we have $\boldsymbol{\alpha}'(t) = (1, 2t)$ and $\boldsymbol{f}(\boldsymbol{\alpha}(t)) = (t^2, t^4)$, therefore,

$$\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int_0^1 (t^2, t^4) \cdot (1, 2t) dt = \int_0^1 (t^2 + 2t^5) dt = \left[\frac{t^3}{3} + \frac{t^6}{3}\right]_0^1 = \frac{2}{3}.$$

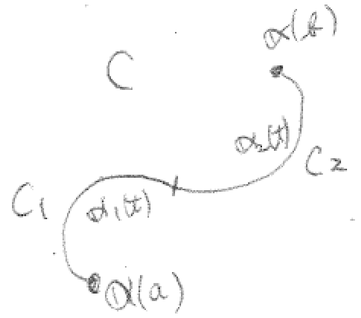
From the definition, it follows that

$$\int (c\boldsymbol{f} + d\boldsymbol{g}) \cdot d\boldsymbol{\alpha} = c \int \boldsymbol{f} \cdot d\boldsymbol{\alpha} + d \int \boldsymbol{g} \cdot d\boldsymbol{\alpha}.$$

Furthermore, if $\boldsymbol{\alpha}(t) \begin{cases} \boldsymbol{\alpha}_1(t) & t \in [a, c] \\ \boldsymbol{\alpha}_2(t) & t \in [c, b] \end{cases}$, then it holds that

$$\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int \boldsymbol{f} \cdot d\boldsymbol{\alpha}_1 + \int \boldsymbol{f} \cdot d\boldsymbol{\alpha}_2.$$

If we name the corresponding curves C, C_1, C_2, then we can write it as $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int_{C_1} \boldsymbol{f} \cdot d\boldsymbol{\alpha} + \int_{C_2} \boldsymbol{f} \cdot d\boldsymbol{\alpha}$.

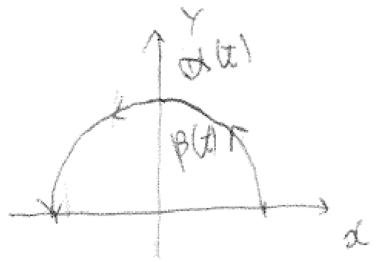


Change of parametrization

The same curve C can be parametrized by $\boldsymbol{\alpha}(t)$ on $[a, b]$ and $\boldsymbol{\beta}(t)$ on $[c, d]$. We will show that the line integral does not depend on the change of parametrization. We say that $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ are **equivalent** if there is a contiuously differentiable function $u : [c, d] \to [a, b]$ such that $\boldsymbol{\alpha}(u(t)) = \boldsymbol{\beta}(t)$. Furthermore,

- If $u(c) = a, u(d) = b$ and hence $u'(t) > 0$, then we say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are in the **same direction**.
- If $u(c) = b, u(d) = a$ and hence $u'(t) < 0$, then we say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are in the **opposite direction**.

For example, both $\boldsymbol{\alpha}(t) = (-t, \sqrt{1-t^2}), t \in [-1, 1]$ and $\boldsymbol{\beta}(t) = (\cos t, \sin t), t \in [0, \pi]$ parametrize the half-circle. With $u(t) = -\cos t$, we have $\boldsymbol{\beta}(t) = \boldsymbol{\alpha}(u(t))$.



Theorem 22. *Let $\boldsymbol{f}$ be a continuous vector field, $\boldsymbol{\alpha}, \boldsymbol{\beta}$ equivalent, piecewise continuously differentiable paths. Then*

$$\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \begin{cases} \int \boldsymbol{f} \cdot d\boldsymbol{\beta} & \textit{the same direction} \\ -\int \boldsymbol{f} \cdot d\boldsymbol{\beta} & \textit{the opposite direction} \end{cases}.$$

Proof. We may assume that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are continuously differentiable, by decomposing the integrals into subintervals where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are continuously differentiable. Now there is continuously differentiable $u(t)$ such that $\boldsymbol{\alpha}(u(t)) = \boldsymbol{\beta}(t)$. By the chain rule, we have $\boldsymbol{\beta}'(t) = \boldsymbol{\alpha}'(u(t))u'(t)$, and therefore,

$$\begin{aligned} \int \boldsymbol{f} \cdot d\boldsymbol{\beta} &= \int_c^d \boldsymbol{f}(\boldsymbol{\beta}) \cdot d\boldsymbol{\beta}(t) = \int_c^d \boldsymbol{f}(\boldsymbol{\alpha}(u(t))) \cdot \boldsymbol{\alpha}'(u(t))u'(t)dt \\ &= \begin{cases} \int_a^b \boldsymbol{f}(\boldsymbol{\alpha}(t))\boldsymbol{\alpha}'(t)dt = \int \boldsymbol{f} \cdot d\boldsymbol{\alpha} & \text{the same direction} \\ \int_b^a \boldsymbol{f}(\boldsymbol{\alpha}(t))\boldsymbol{\alpha}'(t)dt = -\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} & \text{the opposite direction} \end{cases} \end{aligned}$$

□

Applications

Gradient

Let $h(x,y)$ be the height of a mountain in the xy-plane and $\boldsymbol{f}(x,y) = \nabla h(x,y)$ be the gradient field. If one walks along a path $\boldsymbol{\alpha}(t)$ on $[0,1]$, then $h(\boldsymbol{\alpha}(t))$ is the height at time t. By the chain rule, $\frac{d}{dt}h(\boldsymbol{\alpha}(t)) = \nabla(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)$. Therefore, the line integral of the gradient gives the change of height:

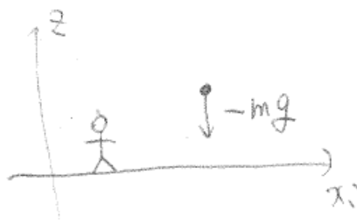
$$\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int_0^1 \boldsymbol{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)dt = \int_0^1 \frac{d}{dt} h(\boldsymbol{\alpha}(t))dt = h(\boldsymbol{\alpha}(1)) - h(\boldsymbol{\alpha}(0)).$$

Work in physics

Consider the gravitational field around the earth surface $\boldsymbol{f}(x,y,z) = (0,0,mg)$ with m the mass of a particle and $g = 9.8\mathrm{m/s}^2$. If one moves the particle from the point $\boldsymbol{a} = (a_1,a_2,a_3)$ to $\boldsymbol{b} = (b_1,b_2,b_3)$ along a path $\boldsymbol{\alpha}(t)$ on $[0,1]$, the **work** performed on this path is defined by $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$, and it can be calculated as

$$\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int_0^1 \boldsymbol{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)dt = \int_0^1 mg \cdot \alpha_3'(t)dt = mg[\alpha_3(t)]_0^1 = mg(b_3 - a_3).$$

Namely, the work depends only on the height. The value $mg(b_3 - a_3)$ is the difference of the **potiential energy** of the particle between the positions $\boldsymbol{b}$ and $\boldsymbol{a}$.



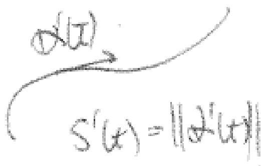
Let $\boldsymbol{f}$ be a force field and assume that a particle is moving in $\boldsymbol{f}$, with velocity $\boldsymbol{v}(t)$ at time t. Define the kinetic energy by $\frac{1}{2}\|\boldsymbol{v}(t)\|^2$. Let $\boldsymbol{r}(t)$ be the position of the particle. Hence it holds that $\boldsymbol{v}(t) = \boldsymbol{r}'(t)$. Newton's law says that $\boldsymbol{f}(\boldsymbol{r}(t)) = m\boldsymbol{r}''(t) = m\boldsymbol{v}'(t)$. The work done by the force field between time $t \in [0,1]$ is

$$\begin{aligned}\int \boldsymbol{f} \cdot d\boldsymbol{r} &= \int_0^1 \boldsymbol{f}(\boldsymbol{r}(t)) \cdot \boldsymbol{v}(t)dt = \int_0^1 m\boldsymbol{v}'(t) \cdot \boldsymbol{v}(t)dt \\ &= \int_0^1 \frac{d}{dt}\left(\frac{1}{2}\|\boldsymbol{v}(t)\|^2\right) = \left(\frac{1}{2}\|\boldsymbol{v}(1)\|^2 - \frac{1}{2}\|\boldsymbol{v}(0)\|^2\right),\end{aligned}$$

namely, the work done on the particle by the force field is equal to the change of the kinetic energy.

Length of a curve (not a line integral)

Let $\boldsymbol{\alpha}(t)$ be a parametrization of a curve. The length of the piece of the curve $u \in [a,t]$ is defined by $s(t) = \int_a^t \|\boldsymbol{\alpha}(u)\| du$, and hence $s'(t) = \|\boldsymbol{\alpha}(t)\|$. Knowing s is sometimes useful (see below).



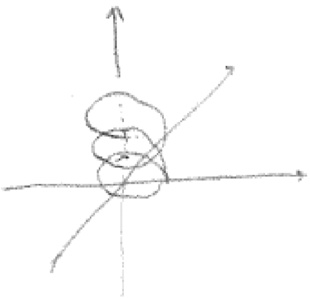
Mass of a wire

Assume that $\boldsymbol{\alpha}(t)$ represents a wire, and the mass per unit length at $\boldsymbol{\alpha}$ is given by a function $\varphi(\boldsymbol{\alpha})$. The mass M of the wire is given by

$$M = \int \varphi(\boldsymbol{\alpha}(t)) s'(t) dt,$$

where $s(t)$ is the length function above.

Example 23. With $a, b > 0$, the mass of a coil parametrized by $\boldsymbol{\alpha}(t) = (a\cos t, a\sin t, bt), t \in [0, 2\pi]$ with the density function $\varphi(x,y,z) = x^2+y^2+z^2$. Let us compute the mass. We need first $s(t)$. We have $\boldsymbol{\alpha}'(t) = (-a\sin t, a\cos t, b)$, $\|\boldsymbol{\alpha}'(t)\| = \sqrt{a^2+b^2} = s'(t)$. By the above formula, $M = \int_0^2 \pi(a^2+b^2t^2)\sqrt{a^2+b^2}dt = \sqrt{a^2+b^2}(2\pi a^2 + \frac{8\pi^3}{3}b^2)$.



Nov 9. Gradients and line integrals

The second fundamental theorem of calculus

Recall that, on $\mathbb{R}$, if φ is a real differentiable function and φ' is continuous on $[a,b]$, then $\int_a^b \varphi'(t)dt = \varphi(b) - \varphi(a)$. We have a similar result in $\mathbb{R}^n$, in terms of the gradient and line integral.

Theorem 24. *Let φ be a continuously differentiable scalar field on $S \subset \mathbb{R}^n$. Let $\boldsymbol{\alpha}$ be a piecewise continuously differentiable path in S defined on $[a,b]$. Then $\int \nabla\varphi \cdot d\boldsymbol{\alpha} = \varphi(\boldsymbol{b}) - \varphi(\boldsymbol{a})$, where $\boldsymbol{b} = \boldsymbol{\alpha}(b)$ and $\boldsymbol{a} = \boldsymbol{\alpha}(a)$.*

Proof. We first assume that $\boldsymbol{\alpha}$ is continuously differentiable. By chain rule,

$$\begin{aligned}\frac{d}{dt}\varphi(\boldsymbol{\alpha}(t)) &= \frac{d}{dt}\varphi(\alpha_1(t), \cdots, \alpha_n(t)) = \alpha_1'(t)\frac{\partial\varphi}{\partial x_1}(\boldsymbol{\alpha}(t)) + \cdots + \alpha_n'(t)\frac{\partial\varphi}{\partial x_n}(\boldsymbol{\alpha}(t)) \\ &= \boldsymbol{\alpha}'(t) \cdot \nabla\varphi(\boldsymbol{\alpha}(t)).\end{aligned}$$

Now by the result in $\mathbb{R}$, we have

$$\begin{aligned}\int \nabla\varphi \cdot d\boldsymbol{\alpha} &= \int_a^b \nabla\varphi(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)dt = \int_a^b \frac{d}{dt}\varphi(\boldsymbol{\alpha}(t))dt \\ &= [\varphi(\boldsymbol{\alpha}(t))]_a^b = \varphi(\boldsymbol{b}) - \varphi(\boldsymbol{a}).\end{aligned}$$

If $\boldsymbol{\alpha}$ is only piecewise continuously differentiable, say, smooth on each subinterval of the decomposition $[a,b] = [a,a_1] \cup [a_1,a_2] \cup \cdots \cup [a_k,b]$, then

$$\begin{aligned}\frac{d}{dt}\varphi(\boldsymbol{\alpha}(t)) &= [\varphi(\boldsymbol{a}(a_1)) - \varphi(\boldsymbol{\alpha}(a))] + [\varphi(\boldsymbol{\alpha}(a_2)) - \varphi(\boldsymbol{\alpha}(a_1)] + \cdots [\varphi(\boldsymbol{b}) - \varphi(\boldsymbol{a}(a_k))] \\ &= \varphi(\boldsymbol{b}) - \varphi(\boldsymbol{a}).\end{aligned}$$

□

Example 25. Let us assume that there is a force field $\boldsymbol{f}$ and it is given by the potential φ, namely, $\boldsymbol{f} = \nabla\varphi$. Let us say that a particle is moving along a path $\boldsymbol{\alpha}$ on $[a,b]$ according to Newton's law $\boldsymbol{f}(\boldsymbol{\alpha}(t)) = m\boldsymbol{\alpha}''(t) = m\boldsymbol{v}'(t)$. In this case, we learned that

- the work done by the force field is $\int \nabla\varphi \cdot d\boldsymbol{\alpha} = \varphi(\boldsymbol{\alpha}(b)) - \varphi(\boldsymbol{\alpha}(a))$.
- the change in the kinetic energy is $\frac{m}{2}\|\boldsymbol{v}(b)\|^2 - \frac{m}{2}\|\boldsymbol{v}(a)\|^2 = \int \nabla\varphi \cdot d\boldsymbol{\alpha}$.

If we define the potential energy by $V(\boldsymbol{x}) = -\varphi(\boldsymbol{x})$, then it holds that $\frac{m}{2}\|\boldsymbol{v}(b)\|^2 + V(\boldsymbol{\alpha}(b)) = \frac{m}{2}\|\boldsymbol{v}(a)\|^2 + V(\boldsymbol{\alpha}(a))$. This is the conservation of energy $\frac{m}{2}\|\boldsymbol{v}(t)\|^2 + V(\boldsymbol{\alpha}(t))$.

Example 26. Let us put the center of earth at $(0,0,0)$ with mass M and throw a small particle with mass m. The force field of the gravitation of the earth is given by

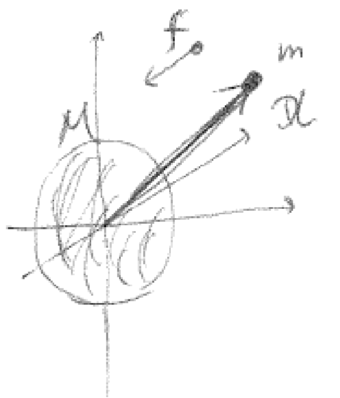
$$\boldsymbol{f}(\boldsymbol{x}) = \frac{-GmM\boldsymbol{x}}{\|\boldsymbol{x}\|^3}.$$

We can take the potential energy as

$$\varphi(\boldsymbol{x}) = \frac{GmM}{\|\boldsymbol{x}\|} = \frac{GmM}{\sqrt{x_1^2 + x_2^2 + x_3^2}},$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$. Indeed, by a straightforward computation, we have

$$\nabla\varphi(\boldsymbol{x}) = \left(-\frac{GmM \cdot 2x_1}{2(x_1^2 + x_2^2 + x_3^2)^{\frac{3}{2}}}, -\frac{GmM \cdot 2x_2}{2(x_1^2 + x_2^2 + x_3^2)^{\frac{3}{2}}}, -\frac{GmM \cdot 2x_3}{2(x_1^2 + x_2^2 + x_3^2)^{\frac{3}{2}}}\right) = \frac{-GmM\boldsymbol{x}}{\|\boldsymbol{x}\|^3}.$$



The first fundamental theorem of calculus

Recall again what we know from $\mathbb{R}$. Let f be a continuous function and $\varphi(x) = \int_a^x f(t)dt$, then $\varphi'(x) = f(x)$.

In $\mathbb{R}^n$, with a given vector field $\boldsymbol{f}$, we want to define $\varphi(\boldsymbol{x}) = \int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ for a path $\boldsymbol{\alpha}(t)$ such that $\boldsymbol{\alpha}(x) = \boldsymbol{x}$ and $\boldsymbol{\alpha}(a) = \boldsymbol{a}$. But this definition may depend on the choice of $\boldsymbol{\alpha}$. As a consequence, we can have a similar correspondence when the line integral actually does not depend on the path.

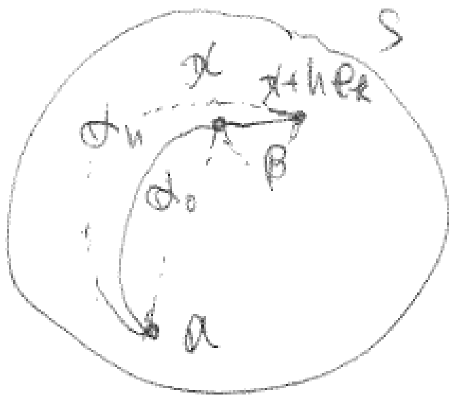
Theorem 27. *Let $\boldsymbol{f}$ be a continuous vector field on $S \subset \mathbb{R}^n$, where S is connected. Assume that, given $\boldsymbol{x}, \boldsymbol{a} \in S$, the line integral $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ along a path $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha}(a) = \boldsymbol{a}$ and $\boldsymbol{\alpha}(x) = \boldsymbol{x}$ does not depend on $\boldsymbol{\alpha}$. We fix $\boldsymbol{a} \in S$ and define $\varphi(\boldsymbol{x}) = \int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ for some path $\boldsymbol{\alpha}$ with the above condition. Then φ is continuously differentiable and $\nabla\varphi = \boldsymbol{f}$.*

Proof. Recall that $\boldsymbol{e}_k = (0, \cdots 0, \underset{k\text{-th}}{1}, 0, \cdots, 0)$. Note that, for a small h,

$$\varphi(\boldsymbol{x} + h\boldsymbol{e}_k) - \varphi(\boldsymbol{x}) = \int \boldsymbol{f} \cdot d\boldsymbol{\alpha}_h - \int \boldsymbol{f} \cdot d\boldsymbol{\alpha}_0 = \int \boldsymbol{f} \cdot d\boldsymbol{\beta},$$

where $\boldsymbol{\alpha}_h$ is a path from $\boldsymbol{a}$ through $\boldsymbol{x}$ to $\boldsymbol{x} + h\boldsymbol{e}_k$ on $[a, b+h]$ and $\boldsymbol{\beta}(t) = \boldsymbol{x} + t\boldsymbol{e}_k$ is a path from $\boldsymbol{x}$ to $\boldsymbol{x} + h\boldsymbol{e}_k$ on $[0, h]$. Therefore, by noting that $\boldsymbol{\beta}'(t) = \boldsymbol{e}_k$,

$$\begin{aligned} \frac{\partial\varphi}{\partial x_k} = \lim_{h\to 0} \frac{\varphi(\boldsymbol{x} + h\boldsymbol{e}_k)}{h} &= \lim_{h\to 0} \frac{1}{h} \int_0^h \boldsymbol{f}(\boldsymbol{\beta}(t)) \cdot \boldsymbol{\beta}'(t) = \lim_{h\to 0} \frac{1}{h} \int_0^h \boldsymbol{f}(\boldsymbol{\beta}(t)) \cdot \boldsymbol{e}_k dt \\ &= \lim_{h\to 0} \frac{1}{h} \int_0^h f_k(\boldsymbol{x} + h\boldsymbol{e}_k)dt = f_k(\boldsymbol{x}). \end{aligned}$$



Namely, $\nabla\varphi(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})$.

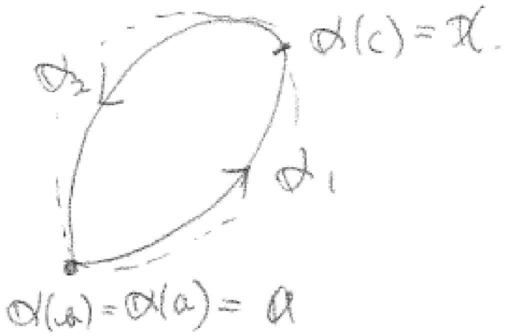
To show that φ is differentiable, we take $\boldsymbol{y}$ such that $\boldsymbol{x} + t\boldsymbol{y} \in S$ for $t \in [0, 1]$. With $\boldsymbol{\alpha}(t) = \boldsymbol{x} + t\boldsymbol{y}, t \in [0, 1]$, we have

$$\begin{aligned} \varphi(\boldsymbol{x} + \boldsymbol{y}) - \varphi(\boldsymbol{x}) &= \int_0^1 \boldsymbol{f}(\boldsymbol{x} + \boldsymbol{y}) \cdot \boldsymbol{\alpha}'(t)dt = \int_0^1 \boldsymbol{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{y}dt \\ &= \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{y} + \int_0^1 (\boldsymbol{f}(\boldsymbol{\alpha}(t)) - \boldsymbol{f}(\boldsymbol{x})) \cdot \boldsymbol{y}dt \end{aligned}$$

and $\|\boldsymbol{f}(\boldsymbol{\alpha}(t)) - \boldsymbol{f}(\boldsymbol{x})\| \to 0$ as $\boldsymbol{y} \to 0$, which is the differentiability of φ. □

We say that a path is **closed** if $\boldsymbol{\alpha}(b) = \boldsymbol{\alpha}(a)$. Any closed path $\boldsymbol{\alpha}$ can be written as

$$\boldsymbol{\alpha}(t) = \begin{cases} \boldsymbol{\alpha}_1(t) & \text{for } t \in [a, c] \\ \boldsymbol{\alpha}_2(t) & \text{for } t \in [c, b]. \end{cases}.$$



Theorem 28. *The following are equivalent for a vector field* $\boldsymbol{f}$*:*

(a) There is φ *such that* $\boldsymbol{f} = \nabla\varphi$*.*

(b) $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ *does not depend on* $\boldsymbol{\alpha}$ *so long as* $\boldsymbol{\alpha}(x) = \boldsymbol{x}$ *and* $\boldsymbol{\alpha}(a) = \boldsymbol{a}$*.*

(c) For any closed path $\boldsymbol{\alpha}$*,* $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} = 0$*.*

Proof. We have shown already (a)⇔(b). To show (b)⇔(c), there is a one-to-one correspondence between closed paths $\boldsymbol{\beta}$ starting at $\boldsymbol{a}$ and passing $\boldsymbol{x}$, and a pair of paths $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ starting at $\boldsymbol{a}$ and ending at $\boldsymbol{x}$. By recalling that, by reversing the direction of the path, the line integral gets multiplied by -1. Therefore, $\int \boldsymbol{f} \cdot d\boldsymbol{\beta} = \int \boldsymbol{f} \cdot d\boldsymbol{\alpha}_1 - \int \boldsymbol{f} \cdot d\boldsymbol{\alpha}_2$. □

Let us write $\partial_k f = \frac{\partial f}{\partial x_k}$.

Theorem 29. *Let* $\boldsymbol{f}$ *be a continuously differentiable vector field. If* $\boldsymbol{f} = \varphi$*, then* $\partial_k f_\ell = \partial_\ell f_k$*.*

Proof. By assumption, φ has continuous second partial derivatives, and therefore,

$$\partial_k f_\ell = \frac{\partial^2 \varphi}{\partial x_k \partial x_\ell} = \frac{\partial^2 \varphi}{\partial x_\ell \partial x_k} = \partial_\ell f_k.$$

□

Example 30. $\boldsymbol{f}(x, y) = (3x^2 y, x^3 y)$. $\partial_1 f_2(x, y) = 3x^2 y, \partial_2 f_1(x, y) = 3x^2$, which do not coincide. Therefore, $\boldsymbol{f}$ is not a gradient.

Nov 12. Potential function

Constructing a potential by line integral

Let us assume that $\boldsymbol{f}$ is a gradient. We learned that $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ is independent of a path as long as $\boldsymbol{\alpha}(a) = \boldsymbol{a}$ and $\boldsymbol{\alpha}(x) = \boldsymbol{x}$. We can use this fact in order to find a function φ such that $\nabla\varphi = \boldsymbol{f}$.

A rectangular region

Let $\boldsymbol{f}$ be a continuous vector field on $R = [a, \tilde{a}] \times [b, \tilde{b}]$. For $\boldsymbol{x} = (x, y) \in R$, we can take the path $\boldsymbol{\alpha}$ as the union of the following two paths:

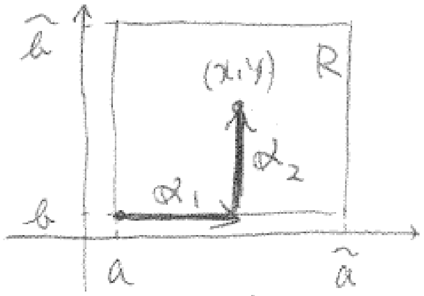
$$\boldsymbol{\alpha}_1 = (t, b), t \in [a, x], \boldsymbol{\alpha}_2(t) = (x, t), t \in [b, y].$$

Therefore, by noting that $\boldsymbol{\alpha}_1'(t) = (1, 0), \boldsymbol{\alpha}_2'(t) = (0, 1)$, we have

$$\varphi(\boldsymbol{x}) = \int \boldsymbol{f} \cdot d\boldsymbol{f} = \int_a^x \boldsymbol{f}(\boldsymbol{\alpha}_1(t)) \cdot \boldsymbol{\alpha}_1(t) dt + \int_b^y \boldsymbol{f}(\boldsymbol{\alpha}_2(t)) \cdot \boldsymbol{\alpha}_2'(t) dt = \int_a^x f_1(t, b) dt + \int_b^y f_2(x, t) dt.$$

Similarly, we can also take $\boldsymbol{\alpha}_1 = (a,t), t \in [b,y], \boldsymbol{\alpha}_2(t) = (t,y), t \in [a,x]$ and

$$\varphi(\boldsymbol{x}) = \int \boldsymbol{f} \cdot d\boldsymbol{f} = \int_a^x f_1(t,y)dt + \int_b^y f_2(a,t)dt.$$



A similar formula holds for rectangular regions in $\mathbb{R}^n$.

Example 31. Find a potential for $\boldsymbol{f}(x,y) = (e^x y^2 + 1, 2e^x y)$ on $\mathbb{R}^2$. We can take

$$\varphi(x,y) = \int_0^x f_1(t,0)dt + \int_0^y f_2(x,t)dt = \int_0^x dt + \int_0^y 2e^x t dt = x + [e^x t^2]_0^y = x + e^x y^2.$$

By indefinite integrals

If $\nabla\varphi = \boldsymbol{f}$, namely $\frac{\partial\varphi}{\partial x} = f_1, \frac{\partial\varphi}{\partial y} = f_2$, we have the following.

$$\int_a^x f_1(t,y)dt + A(y) = \varphi(x,y) = \int_b^y f_2(x,t)dt + B(x).$$

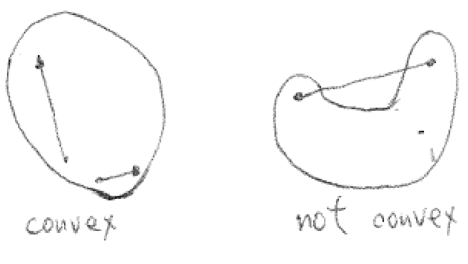
By comparing $A(x)$ and $B(x)$ by hand, we can sometimes determine them.

Example 32. Find a potential for $\boldsymbol{f}(x,y) = (e^x y^2 + 1, 2e^x y)$ on $\mathbb{R}^2$.

$$\int f_1(t,y)dt + A(y) = e^x y^2 + x + A(y) = \int f_2(x,y)dt + B(x) = e^x y^2 + B(x),$$

and if we take $A(y) = 0, B(x) = x$, the equality holds. Hence the potential is $\varphi(x,y) = e^x y^2 + x$.

Definition 33. A set $S \subset \mathbb{R}^n$ is said to be **convex** if for any pair $\boldsymbol{x}, \boldsymbol{y} \in S$, the segment $t\boldsymbol{x} + (1-t)\boldsymbol{y}$ is contained in S, where $t \in [0,1]$.



Example 34. If $\boldsymbol{f}$ is a continuous vector field which is a gradient of φ on a convex set S, then we fix $\boldsymbol{a} \in S$ and for each $\boldsymbol{x} \in S$ we take the path $\boldsymbol{\alpha}(t) = t(\boldsymbol{x} - \boldsymbol{a}) + \boldsymbol{a}$.

If f is a continuous function, then the vector field $\boldsymbol{f} = (xf(\sqrt{x^2+y^2}), yf(\sqrt{x^2+y^2}))$ is a gradient. Indeed, we can take $\boldsymbol{\alpha}(t) = (tx, ty)$, and

$$\varphi(x,y) = \int_0^1 \boldsymbol{f}(tx,ty) \cdot (x,y)dt = \int_0^1 t(x^2+y^2)f(t\sqrt{(x^2+y^2)})dt = \int_0^{\sqrt{x^2+y^2}} tf(t)dt.$$

It is easy to show that $\nabla\varphi = \boldsymbol{f}$.

Sufficient condition for a vector field to be a gradient

Theorem 35. *Let R be a closed rectangular region in $\mathbb{R}^n$. Let $J = [a, b] \subset \mathbb{R}$, $J_{n+1} = R \times J \subset \mathbb{R}^{n+1}$.*

Assume that ψ is a scalar field on J_{n+1}, $\frac{\partial\psi}{\partial x_k}$ is continuous for each $k = 1, \cdots, n$. If we define $\varphi(\boldsymbol{x}) = \int_a^b \psi(\boldsymbol{x}, t)dt$, then φ has partial derivatives and and $\frac{\partial\varphi}{\partial x_k}(\boldsymbol{x}) = \int_a^b \frac{\partial\psi}{\partial x_k}(\boldsymbol{x}, t)dt$.

Proof. Let $\boldsymbol{x} \in S$ and h small such that $\boldsymbol{x} + h\boldsymbol{e}_k \in S$. By definition, we have

$$\varphi(\boldsymbol{x} + h\boldsymbol{e}_k) - \varphi(\boldsymbol{x}) = \int_a^b (\psi(\boldsymbol{x} + h\boldsymbol{e}_k, t) - \psi(\boldsymbol{x}, t))\, dt$$

By mean value theorem applied to ψ, we have

$$\varphi(\boldsymbol{x} + h\boldsymbol{e}_k) - \varphi(\boldsymbol{x}) = h \int_a^b \frac{\partial\psi}{\partial x_k}(\boldsymbol{x} + z\boldsymbol{e}_k, t)dt,$$

where $z \in [0, h]$, or equivalently,

$$\frac{1}{h}(\varphi(\boldsymbol{x} + h\boldsymbol{e}_k) - \varphi(\boldsymbol{x})) = \int_a^b \frac{\partial\psi}{\partial x_k}(\boldsymbol{x} + z\boldsymbol{e}_k, t)dt.$$

By taking the limit, the left-hand side tends to $\frac{\partial\varphi}{\partial x_k}(\boldsymbol{x})$, while the right-hand side is, by the uniform continuity (Theorem 20) applied to the continuous function $\frac{\partial\psi}{\partial x_k}$, we have that

$$\left| \frac{\partial\psi}{\partial x_k}(\boldsymbol{x} + z\boldsymbol{e}_k, t) - \frac{\partial\psi}{\partial x_k}(\boldsymbol{x}, t) \right| < \epsilon$$

if h is small. Then the integral of it is smaller than $\epsilon(b - a)$, which tends to 0 as $h \to 0$. □

Theorem 36. *Let $\boldsymbol{f} = (f_1, \cdots, f_n)$ be a continuously differentiable vector on a convex region $S \subset \mathbb{R}^n$.*

$\boldsymbol{f}$ is a gradient if and only if $\partial_k f_\ell = \partial_\ell f_k$ for all $k, \ell = 1, \cdots, n$.

Proof. We know that if $\boldsymbol{f}$ is a gradient, them the condition $\partial_k f_\ell = \partial_\ell f_k$ holds.

Conversely, let us assume $\partial_k f_\ell = \partial_\ell f_k$ for $k, \ell = 1, \cdots, n$, and construct a potential φ. By translation, we may assume that $\boldsymbol{0} \in S$. We take $\boldsymbol{x} \in S$ and define

$$\varphi(\boldsymbol{x}) = \int \boldsymbol{f} \cdot d\boldsymbol{\alpha}, \qquad \boldsymbol{\alpha}(t) = t\boldsymbol{x}, t \in [0, 1].$$

As $\boldsymbol{\alpha}'(t) = \boldsymbol{x}$, we have $\varphi(\boldsymbol{x}) = \int_0^1 \boldsymbol{f}(t\boldsymbol{x}) \cdot \boldsymbol{x}$. By Theorem 35 applied to the function $\psi(\boldsymbol{x}, t) = \boldsymbol{f}(t\boldsymbol{x}) \cdot \boldsymbol{x}$, we obtain

$$\frac{\partial\varphi}{\partial x_k}(\boldsymbol{x}) = \int_0^1 \left(t\frac{\partial \boldsymbol{f}}{\partial x_k}(t\boldsymbol{x}) \cdot \boldsymbol{x} + f_k(t\boldsymbol{x}) \right) dt.$$

By assumption $\partial_k f_\ell = \partial_\ell f_k$, this is equal to

$$\int_0^1 (t\nabla f_k(t\boldsymbol{x}) \cdot \boldsymbol{x} + f_k(t\boldsymbol{x}))\, dt.$$

By putting $g(t) = f_k(t\boldsymbol{x})$, and by chain rule $g'(t) = \nabla f_k(t\boldsymbol{x}) \cdot \boldsymbol{x}$, we have

$$\frac{\partial\varphi}{\partial x_k}(\boldsymbol{x}) = \int_0^1 tg'(t) + g(t))dt = [tg(t)]_0^1 = g(1) = f_k(\boldsymbol{x}).$$

□

Applications to differential equations

Let us consider a differential equation of the form $P(x,y) + Q(x,y)\frac{dy}{dx}$.

Theorem 37. *If there is $\varphi(x_1,x_2)$ such that $\nabla\varphi(x_1,x_2) = (P(x_1,x_2), Q(x_1,x_2))$, then the solution $Y(x)$ of the equation $P(x,y) + Q(x,y)\frac{dy}{dx}$ satisfies $\varphi(x, Y(x)) = C$ for some $C \in \mathbb{R}$. Conversely, if $\varphi(x,y) = C$ defines implicitly a function $Y(x)$, then $Y(x)$ is a solution of the equation $Q(x,y)\frac{dy}{dx} = P(x,y)$.*

Proof. If $Y(x)$ satiefies $\varphi(x, Y(x)) = C$, then by chain rule and $\nabla\varphi = (P,Q)$,

$$P(x, Y(x)) + Y'(x)Q(x, Y(x)) = 0.$$

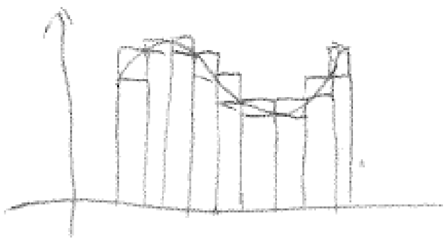
Conversely, if $Y(x)$ is a solution, $\varphi(x, Y(x))$ must be a constant of x. □

Example 38. Solve $y^2 + 2xyy' = 0$.

By putting $P(x,y) = y^2, Q(x,y) = 2xy$, we find $\varphi(x,y) = xy^2$ such that $\nabla\varphi = (P,Q)$. Therefore, a solution $Y(x)$ satisfies $xY(x)^2 = C$. In other words. $Y(x) = \sqrt{C/x}$ is a solution, for $C/x > 0$.

Nov 14. Multiple integrals

In the one dimensional case, we defined $\int_a^{\tilde{a}} f(x)dx$ first for step functions and then for "integrable" functions. Continuous functions are integrable.



We define first double integrals by step functions. Higher multiple integrals are analogous. The definition will allow us to compute it by iterated integrale,

$$\int_Q f(x,y)dxdy = \int_a^{\tilde{a}} \left[\int_b^{\tilde{b}} f(x,y)dy\right] dx.$$

Partitions of a rectangle

Let $Q = [a,\tilde{a}] \times [b,\tilde{b}]$ be a rectangle. A partition of Q is a pair $P_1 \times P_2$ of finite subsets P_1 of $[a,\tilde{a}]$ and P_2 of $[b,\tilde{b}]$ such that $P_1 = \{x_0, \cdots, x_n\}$ where $a = x_0 < x_1 < \cdots x_n = \tilde{a}$ and $b = y_0 < y_1 < \cdots < y_m = \tilde{b}$. Q is decomposed in to nm open subrectangles. The edges of these subrectangles are not contained in these subrectangles. If P and P' are two partitions, then $P \cup P'$ is a refined partition.

Step functions and their integrals

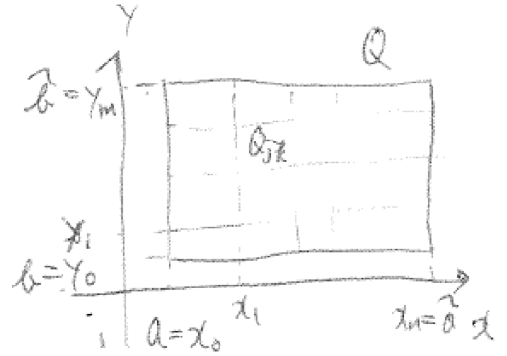
A function f on Q is said to be a step function if there is a partition $P = P_1 \times P_2$ of Q such that f is constant on each subrectangle of P. f is defined also on the edges of the subrectangles, but they do not matter for integration. If f and g are step functions on partitions P and P' respectively, then $cf + dg$ is also a step function on the partition $P \cup P'$.

Let f be a step function onf P. Recall that Q is divided into subrectangles of the form $[x_j, x_{j+1}] \times [y_k, y_{k+1}]$. Its area is $(x_{j+1} - x_j)(y_{k+1} - y_k)$. The value of f on each rectangle is constant, say c_{jk}.

Definition 39. The double integral of f on Q is

$$\iint_Q f dxdy = \sum_{j=0}^{n-1}\sum_{k=0}^{m-1} c_{jk}(x_{j+1}-x_j)(y_{k+1}-y_k).$$

This does not depend on the partition P so long as f is constant on each subrectangle. Furthermore, this does not depend on the value of f on the edges of subrectangles.



Let us call $Q_{jk} = (x_j, x_{j+1}) \times (y_k, y_{k+1})$. By definition, we have

$$\begin{aligned}\iint_{Q_{jk}} f dxdy &= (x_{j+1}-x_j)(y_{k+1}-y_k) \\ &= \int_{x_j}^{x_{j+1}}\left[\int_{y_k}^{y_{k+1}} f(x,y)dy\right]dx = \int_{y_k}^{y_{k+1}}\left[\int_{x_j}^{x_{j+1}} f(x,y)dx\right]dy.\end{aligned}$$

As we remarked, we can set the value of f on the edges of subrectangles as we wish. Let us set it to some constant C. Then, for a fixed y, the partial integral $\int_{x_j}^{x_{j+1}} f(x,y)dx$ is well-defined. Furthermore, the value of the partial integral for y in each open interval does not depend on that choice, as we know from the one-dimensional integrals. Therefore, we have for the partial integral

$$\int_a^{\tilde{a}} f(x,y)dx = \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} f(x,y)dx$$

for y in some open interval (y_k, y_{k+1}), because then $f(x,y)$ as a function of x is a step function. Now this is a step function of y, hence

$$\int_b^{\tilde{b}}\left[\int_a^{\tilde{a}} f(x,y)dx\right]dy = \sum_{k=0}^{m-1}\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} f(x,y)dx = \iint_Q f dxdy.$$

Similarly, we have $\iint_Q f dxdy = \int_a^{\tilde{a}}\left[\int_b^{\tilde{b}} f(x,y)dy\right]dx$. Namely, for step functions, we can perform the integral by iteration of one-dimensional integrals.

It is straightforward to show the following.

Theorem 40. *Let f, s, t be step functions.*

- *For $c, d \in \mathbb{R}$, $\iint_Q (cf+dg)dxdy = c\iint_Q f dxdy + d\iint_Q g dxdy$*
- *If Q is subdivided into Q_1, Q_2, then $\iint_Q f dxdy = \iint_{Q_1} f dxdy + \iint_{Q_2} f dxdy$*
- *If $s(x,y) \le t(x,y)$, then $\iint_Q s dxdy \le \iint_Q t dxdy$.*

Upper and lower integrals

Let f be a function on Q and bounded, namely, $-M \le f(x,y) \le M$ for some M. For step functions such that $s \le f \le t$, we have $\iint_Q s dx dy \le \iint_Q t dx dy$.

Definition 41. (integrability)

- $\sup \iint_Q s dx dy$ where s runs all step functions such that $s \le f$ is called the lower integral of f.
- $\inf \iint_Q t dx dy$ where t runs all step functions such that $f \le t$ is called the upper integral of f.
- If the lower and upper integrals coincide, then f is said to be **integrable** and we define $\iint_Q f dx dy$ to be that value.

Theorem 42. *Let f be a bounded integrable function on $Q = [a, \tilde{a}] \times [b, \tilde{b}]$. Assume that for each $y \in [b, \tilde{b}]$, the integral $\int_a^{\tilde{a}} f(x,y)dx = A(y)$ exists. Then $\int_b^{\tilde{b}} A(y)dy$ exists and*

$$\iint_Q f dx dy = \int_b^{\tilde{b}} \left[\int_a^{\tilde{a}} f(x,y)dx \right] dy.$$

Proof. Let us choose step functions $s \le f \le t$. By assumption,

$$\int_a^{\tilde{a}} s(x,y)dx \le \int_a^{\tilde{a}} f(x,y)dx = A(y) \le \int_a^{\tilde{a}} t(x,y)dx.$$

Note that $\int_a^{\tilde{a}} s(x,y)dx$ and $\int_a^{\tilde{a}} t(x,y)dx$ are step functions. By integrability of f, we can take s, t such that $\int_b^{\tilde{b}} \int_a^{\tilde{a}} s(x,y)dxdy$ and $\int_b^{\tilde{b}} \int_a^{\tilde{a}} T(x,y)dx$ are arbitrarily close. Note that they are step functions in y. This means that $A(y)$ is integrable and

$$\int_b^{\tilde{b}} \int_a^{\tilde{a}} s(x,y)dx \le \int_b^{\tilde{b}} \int_a^{\tilde{a}} f(x,y)dx \le \int_b^{\tilde{b}} \int_a^{\tilde{a}} t(x,y)dx.$$

As the sup of the left-hand side and the inf of the right-hand side coincides and is $\iint_Q f dx dy$, we obtain the claim. □

Example 43. We will show that continuous functions are integrable.

Let $Q = [0,1] \times [0,1], f(x,y) = e^{x+y}$. $\int_0^1 e^{x+y} dx = e^{y+1} - e^y$ and hence $\iint_Q e^{x+y} dx dy = \int_0^1 (e^{y+1} - e^y) dy = e^2 - 1$.

Volume of a solid

Let $f(x,y) > 0$ be a (continuous) function, $Q = [a, \tilde{a}] \times [b, \tilde{b}]$ a rectangle and $V = \{(x,y,z) : (x,y) \in Q, 0 \le z \le f(x,y)\}$. This V is a solid in the xyz-space. Its volume was defined by $v(V) = \int_b^{\tilde{b}} A(y)dy$, where $A(y) = \int_a^{\tilde{a}} f(x,y)dx$ is the area of the xz-slice at y. Now, under the condition of integrability (which is satisfied if f is continuous as we will see), we have $v(V) = \iint_Q f(x,y)dxdy$, namely, the volume is given by the double integral.

Nov 19. Integrability of some functions

Continuous function

Let us recall that a function f in a closed rectangle $Q = [a, \tilde{a}] \times [b, \tilde{b}]$ is **uniformly continuous** by Theorem 20, namely, for any ϵ there is δ such that if $\boldsymbol{x}, \boldsymbol{x}' \in Q$, $\|\boldsymbol{x} - \boldsymbol{x}'\| < \delta$ then $|f(\boldsymbol{x}) - f(\boldsymbol{x}')| < \epsilon$.

Theorem 44. *If f is a continuous function on Q, then f is integrable on Q and*

$$\iint_Q f dxdy = \int_b^{\tilde{b}} \left[\int_a^{\tilde{a}} f(x,y)dx\right] dy = \int_a^{\tilde{a}} \left[\int_b^{\tilde{b}} f(x,y)dy\right] dx.$$

Proof. Note that we know that f is bounded by Theorem 18, therefore, the upper and lower integrals exist. Let us fix ϵ, then there is δ such that if $\boldsymbol{x}, \boldsymbol{x'} \in Q$, $\|\boldsymbol{x}-\boldsymbol{x'}\| < \delta$ then $|f(\boldsymbol{x})-f(\boldsymbol{x'})| < \epsilon$. We take a partition P of Q such that for $\boldsymbol{x}, \boldsymbol{x'} \in Q_{kj}$ it holds that $\|\boldsymbol{x}-\boldsymbol{x'}\| < \delta$. Then $\sup_{\boldsymbol{x}\in Q_{kj}} f(\boldsymbol{x}) - \inf_{\boldsymbol{x}\in Q_{kj}} f(\boldsymbol{x}) \leq \epsilon$.

Let us take the step functions s, t such that $s(\boldsymbol{x}) = \inf_{\boldsymbol{x}\in Q_{kj}} f(\boldsymbol{x})$ and $t(\boldsymbol{x}) = \sup_{\boldsymbol{x}\in Q_{kj}} f(\boldsymbol{x})$ for $\boldsymbol{x} \in Q_{kj}$. Then it is clear that $s(\boldsymbol{x}) \leq f(\boldsymbol{x}) \leq t(\boldsymbol{x})$ (we do not have to specify the values for $\boldsymbol{x} \notin Q_{jk}$). By additivity of integrals with respect to regions, we have

$$\left|\iint_Q tdxdy - \iint_Q sdxdy\right| = \iint |t(x,y) - s(x,y)|dxdy \leq \epsilon(\tilde{a}-a)(\tilde{b}-b).$$

The right-hand side can be arbitrarily small, therefore, $\sup_s \iint sdxdy = \inf_t \iint tdxdy$, where the step functions run s, t such that $s \leq f \leq t$ and hence f is integrable by Theorem 42.

Furthermore, by contiuity of f, for each $y \in [b, \tilde{b}]$, $f(x,y)$ is a continuous function in x, hence again by Theorem 42 we have

$$\iint_Q f dxdy = \int_b^{\tilde{b}} \left[\int_a^{\tilde{a}} f(x,y)dx\right] dy.$$

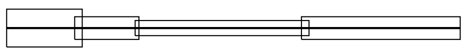
By exchanging the roles of x and y, we obtain the remaining claim. □

Functions with discontinuity

A function f might be integrable even if it has some discontinuity in a "small" set.

Definition 45. Let A be a bounded subset of $\mathbb{R}^2$. A is said to have **content zero** if for every $\epsilon > 0$ there is a finite set of rectangles whose union includes A and the sum of whose areas do not exceed ϵ.

The easiest examples of A are bounded segments. We will see that continuous curves have content zero.



Namely, the set A has content zero if it can be covered by small rectangles whose total area is arbitrarily small.

Theorem 46. *Let f be a bounded function on Q and the set D of discontinities of f has content zero. Then the double integral $\iint_Q f dxdy$ exists.*

Proof. Let M be such that $|f(\boldsymbol{x})| < M$ on Q. Take a cover of D by rectangles with total area δ, and let P be a partition of Q which is finer than the cover, namely, each subrectangle Q_{jk} is a subset of one of the rectanlges. Let us call R the union of the rectangles. Furthermore, by taking a finer partition, we may assume that $\sup_{\boldsymbol{x}\in Q_{kj}} f(\boldsymbol{x}) - \inf_{\boldsymbol{x}\in Q_{kj}} f(\boldsymbol{x}) \leq \epsilon$ on each Q_{kj} which does not contain discontinuity of f, again by uniform continuity on $Q \setminus R$. This time we take step functions s, t such that

$$s(\boldsymbol{x}) = \begin{cases} -M & \text{if } \boldsymbol{x} \in R \\ \inf_{\boldsymbol{x}\in Q_{kj}} f(\boldsymbol{x}) & \text{if } \boldsymbol{x} \notin R, \in Q_{jk} \end{cases}, \qquad t(\boldsymbol{x}) = \begin{cases} M & \text{if } \boldsymbol{x} \in R \\ \sup_{\boldsymbol{x}\in Q_{kj}} f(\boldsymbol{x}) & \text{if } \boldsymbol{x} \notin R, \in Q_{jk} \end{cases}$$

Then we have the estimate

$$\left|\iint_Q tdxdy - \iint_Q sdxdy\right| = \iint_{Q\setminus R} |t(x,y) - s(x,y)|dxdy + 2M\delta \leq \epsilon(\tilde{a}-a)(\tilde{b}-b) + 2M\delta,$$

and the right-hand side can be arbitrarily small. This implies that f is integrable. □

Double integrals over regions bounded by continuous functions

Let S be a region included in a rectangle Q. Let f be a bouded function on S. We can extend f to Q by setting

$$f_Q(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in S \\ 0 & \text{if } (x,y) \in Q \setminus S \end{cases}.$$

We say that f (defined on S) is integrable if f_Q is integrable, and we set

$$\iint_S f dxdy = \iint_Q f_Q dxdy.$$

Definition 47. A region S is said to be of **type I** if there are two continuous functions φ_1, φ_2 such that

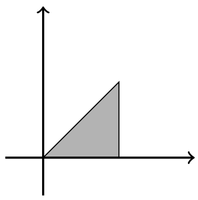
$$S = \{(x,y) : x \in [a, \tilde{a}], \varphi_1(x) \le y \le \varphi_2(x)\}.$$

Similarly, S is of **type II** if there are two continuous functions ψ_1, ψ_2 such that

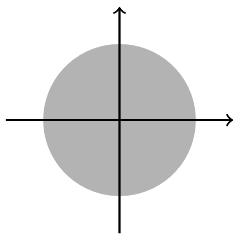
$$S = \{(x,y) : y \in [b, \tilde{b}], \psi_1(y) \le x \le \psi_2(y)\}.$$

Example 48.

- $S = \{(x,y) : 0 \le x \le 1, 0 \le y \le x\}$ is a region of type I with $\varphi_1(x) = 0, \varphi_2(x) = x$. This is also a type II region, because $S_1 = \{(x,y) : 0 \le y \le 1, y \le x \le 1\}$, with $\psi_1(y) = y, \psi_2(y) = 1$.



- $S = \{(x,y) : -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$ is a region of type I with $\varphi_1(x) = -\sqrt{1-x^2}, \varphi_2(x) = \sqrt{1-x^2}$. This is also a type II region, because $S = \{(x,y) : -1 \le y \le 1, -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}\}$, with $\psi_1(y) = -\sqrt{1-y^2}, \psi_2(y) = \sqrt{1-y^2}$.



Lemma 49. *Let φ be a continuous function on $[a, \tilde{a}]$. Then the graph $A = \{(x,y) : x \in [a, \tilde{a}], y = \varphi(x)\}$ has content zero.*

Proof. We apply uniform continuity (in $\mathbb{R}$) to φ, namely, for any $\epsilon > 0$, there is δ such that $|\varphi(x) - \varphi(y)| < \epsilon$ if $|x - y| < \delta$. We take a partition $P = \{x_0, \cdots, x_n\}$ into intervals of lenght less than δ, and take the rectangles $[x_k, x_{k+1}] \times [f(x_k) - \epsilon, f(x_k) + \epsilon]$. Then these rectangles cover A and have the total area $2\epsilon(\tilde{a} - a)$, which can be arbitrarily small. $\square$

Theorem 50. *Let $S = \{(x,y) : x \in [a, \tilde{a}], \varphi_1(x) \le y \le \varphi_2(x)\}$ be a region of type I, and f be a bounded continous function of the interior of S. Then f is integrable and*

$$\iint_S f dxdy = \int_a^{\tilde{a}} \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y) dy \right] dx.$$

Proof. The set of discontinuity of f is the boundary of S in Q, which consists of the graphs of φ_1 and φ_2. They have content zero by Lemma 49. Therefore, by Theorem 46, f is integrable. Furthermore, for each x, $f(x,y)$ is integrable in y because it has only two discontinous points. As $f_Q(x,y) = 0$ outside S, we have $\int_b^{\tilde{b}} f_Q(x,y)dy = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dx$, and hence

$$\iint_S f dxdy = \int_a^{\tilde{a}} \left[\int_b^{\tilde{b}} f_Q(x,y)dy\right] dx. = \int_a^{\tilde{a}} \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right] dx.$$

□

A similar result holds for type II regions.

Example 51. $S = \{(x,y) : x \in [-1,1], -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$. We take the function $f(x,y) = 1$. Then the integral $\iint_S f dxdy$ gives the area.

$$\begin{aligned}\iint_S dxdy &= \int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\right] dx \\ &= \int_{-1}^{1} 2\sqrt{1-x^2}dx = 2\pi.\end{aligned}$$

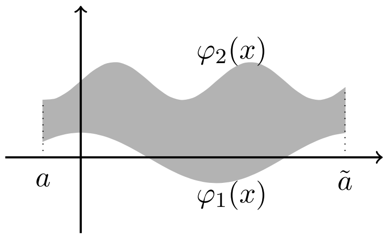
Nov 21. Volume, area and further applications

Area and volume

Let S be a type I region, namely $S = \{(x,y) : a \le x \le \tilde{a}, \varphi_1(x) \le x \le \varphi_2(x)\}$ with continuous functions φ_1, φ_2. By integrating the function 1, we have

$$\iint_S dxdy = \int_a^{\tilde{a}} \varphi_2(x) - \varphi_1(x)dx$$

and the right-hand side was the area of S.



Let $f(x,y) > g(x,y)$ be continuous function on S. Then, for a given x, the integral $\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dx$ represents the area of the region $\{(y,z) : \varphi_1(x) \le y \le \varphi_2(x), g(x,y) \le z \le f(x,y)\}$ and

$$\iint_S f dxdy = \int_a^{\tilde{a}} \left[\int_{\varphi_1(x)}^{\varphi_2(x)} (f(x,y) - g(x,y))dy\right] dx$$

gives the volume of the solid $V := \{(x,y,z) : a \le x \le \tilde{a}, \varphi_1(x) \le y \le \varphi_2(x), g(x,y) \le z \le f(x,y)\}$.

Example 52. • Let $a,b,c > 0$. Compute the volume of $V = \{(x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$.

Solution. We take $S = \{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\}$. Then V can be written as

$$V = \left\{(x,y,z) : (x,y) \in S : -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \le z \le c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\right\}$$

Now S is a type I region, indeed, $S = \left\{(x,y) : -a \le x \le a, -b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}}\right\}$.
By the remark above,

$$\begin{aligned} V &= \iint_S c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}dxdy = c\int_{-a}^{a}\left[\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}dy\right]dy \\ &= c\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}\left[\int_{-1}^{1}\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-s^2}dx\right]dy \quad \text{(by substitution } y = sb\sqrt{1-\frac{x^2}{a^2}}) \\ &= \pi bc\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)dx = \pi bc\left[x-\frac{x^3}{3a^2}\right]_{-a}^{a} = \frac{4\pi abc}{3}. \end{aligned}$$

- Compute the area of a type I region $S = \{(x,y) : 0 \le x \le 1, x^2 \le y \le x\}$. Show that it is also a type II region and compute the area accordingly.

  Solution. The area can be computed by

  $$\iint_S dxdy = \int_0^1\left[\int_{x^2}^{x} dy\right]dx = \int_0^1 (x-x^2)dx = \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

  The region can be equivalently written as

  $$S = \{(x,y) : 0 \le x \le 1, y \le x \le \sqrt{y}\} = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1, y \le x \le \sqrt{y}\}.$$

  With this respec to this expression, we have

  $$\iint_S dxdy = \int_0^1\left[\int_y^{\sqrt{y}} dx\right]dy = \int_0^1 (\sqrt{y}-y)dy = \left[\frac{2y^{\frac{3}{2}}}{3}-\frac{x^2}{2}\right]_0^1 = \frac{1}{6}.$$

Mass and center of mass

If there are particles of mass $m_1, \cdots, m_n$ and are located at points $\boldsymbol{P}_1, \cdots, \boldsymbol{P}_n$, then the **total mass** is $\sum_{k=1}^n m_k$ and the **centre of mass** is $\boldsymbol{C} = \frac{\sum_k m_k \boldsymbol{P}_k}{\sum_{k=1}^n m_j}$.

When we treat a continuum solid, similar concepts can be defined by multiple integrals. Let us imagine that there is a plate on a plain which has the shape of a region S. Furthermore, let us assume that the density of the plate at (x,y) is $f(x,y)$. Then the total mass is given by the double integral $\iint_S f(x,y)dxdy$. Similarly, the center of mass $(\bar{x},\bar{y})$ is

$$\bar{x} = \frac{\iint_S xf(x,y)dxdy}{\iint_S f(x,y)dxdy}, \qquad \bar{y} = \frac{\iint_S yf(x,y)dxdy}{\iint_S f(x,y)dxdy}.$$

If the density is constant $f(x,y) = c$, then $(\bar{x},\bar{y})$ is called the **centroid**.

Example 53. Let $S = \{(x,y) : 0 \le x \le \pi, 0 \le y \le \sin x\}$. This is a type I region. Putting the density to 1, its total mass is

$$\iint_S dxdy = \int_0^\pi\left[\int_0^{\sin x} dy\right]dx = \int_0^\pi \sin x dx = [-\cos x]_0^\pi = 2.$$

The centroid can be computed from

$$\begin{aligned} \iint_S xdxdy &= \int_0^\pi\left[\int_0^{\sin x} xdy\right]dx = \int_0^\pi x\sin xdx = [-x\cos x + \sin x]_0^\pi = \pi, \\ \iint_S ydxdy &= \int_0^\pi\left[\int_0^{\sin x} ydy\right]dx = \int_0^\pi \frac{\sin^2 x}{2}dx = \frac{1}{8}[2x-\sin 2x]_0^\pi = \frac{\pi}{4}, \end{aligned}$$

hence $(\bar{x},\bar{y}) = (\frac{\pi}{2},\frac{\pi}{8})$.

Let $0 \le g \le f$ be two functions on the interval $[a, \tilde{a}]$. Let V be the solid obtained by rotating the region Q between the graphs of g and f around the x-axis. Let $v(V)$ be the volume of V and $a(Q)$ be the area of Q. Then it holds that $v(V) = 2\pi\bar{y}a(Q)$ (Pappus' theorem). Indeed, the volume is given by $\int_a^{\tilde{a}} \pi(f(x)^2 - g(x)^2)dx$ and

$$\bar{y}a(Q) = \iint_Q ydxdy = \int_a^{\tilde{a}} \left[\int_{g(x)}^{f(x)} ydy\right] dx = \int_a^{\tilde{a}} \left[\frac{y^2}{2}\right]_{g(x)}^{f(x)} dx = \frac{1}{2}\int_a^{\tilde{a}} (f(x)^2 - g(x)^2)dx.$$

Example 54. Volume of a torus. A torus V can be obtained by rotating a disk $Q = \{(x, y, 0) : x^2 + (y-b)^2 \le R^2\}$ around the x-axis. The volume is given by Pappus' theorem as $2\pi^2 bR^2$. One can obtain this by the integral

$$\int_{-R}^{R} \pi((b + \sqrt{R^2 - x^2})^2 - (b - \sqrt{R^2 - x^2})^2)dx.$$

If we have two disjoint regions A, B, then the centroid of the union $A \cup B$ is

$$\boldsymbol{C} = \frac{a(A)\boldsymbol{C}_A + a(B)\boldsymbol{C}_B}{a(A) + a(B)},$$

where $\boldsymbol{C}_\bullet$ is the centroid of the region. This follows from the additivity of integral with respect to the domain of integral.

Nov 26. Green's theorem

For a region bounded by a single curve

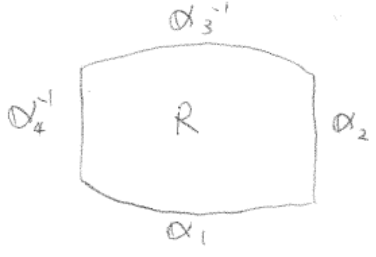
Theorem 55. *Let $\boldsymbol{f}(x, y) = (P(x, y), Q(x, y))$ be a continuously differentialble vector field on S. Let R be a region both of type I and II in S, or R can be written as a finite union of type I+II regions whose intersection is just one side of the boundary, such that the boundary of R is a single differentiable curve C. Then*

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha},$$

where $\boldsymbol{\alpha}$ is a parametrization of C going counter clockwise.

Proof. Let R be a type I region $R = \{(x, y) : a \le x \le \tilde{a}, \varphi_1(x) \le y \le \varphi_2(x)\}$. Let us assume first that $Q = 0$. Then

$$\iint_R -\frac{\partial P}{\partial y}dxdy = \int_a^{\tilde{a}} \left[\int_{\varphi_1(x)}^{\varphi_2(x)} -\frac{\partial P}{\partial y}dy\right] dx = \int_a^{\tilde{a}} (P(x, \varphi_1(x)) - P(x, \varphi_2(x)))dx$$



On the other hand, for R we can take, up to a shift of parametrization and reversing $\boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4$,

$$\boldsymbol{\alpha}_1(t) = (t, \varphi_1(x)), \boldsymbol{\alpha}_2(t) = (a, t), \boldsymbol{\alpha}_3(t) = (t, \varphi_2(t)), \boldsymbol{\alpha}_4(t) = (\tilde{a}, t),$$

and with this $\boldsymbol{\alpha}$, since $Q = 0$ we have

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int \boldsymbol{f} \cdot d\boldsymbol{\alpha}_1 + \int \boldsymbol{f} \cdot d\boldsymbol{\alpha}_3 = \int_a^{\tilde{a}} P(t, \varphi_1(t))dt + \int_{\tilde{a}}^a P(t, \varphi_2(t))dt$$
$$= \int_a^{\tilde{a}} (P(t, \varphi_1(t))dt - P(t, \varphi_2(t)))dt,$$

which completes the proof of this case.

If R is also of type II region, then we can prove the claim if $P = 0$. If R is both of type I and II at the same time, then the claim follows by linearity in $\boldsymbol{f}$, namely $f = (0, Q) + (P, 0)$.

If R can be written as a union of regions R_k both of type I and II with boundary C_k as in the claim, then we saw that the thesis holds on each of these regions: $\iint_{R_k} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{C_k} \boldsymbol{f} \cdot d\boldsymbol{\alpha}$. For the left-hand side we have

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \sum_k \iint_{R_k} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

by the additivity of integral, and for the right-hand side

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \sum_k \int_{C_k} \boldsymbol{f} \cdot d\boldsymbol{\alpha}$$

because except the boundary of C, the line integrals are taken in the opposite directions hence they cancel. This concludes the proof. □

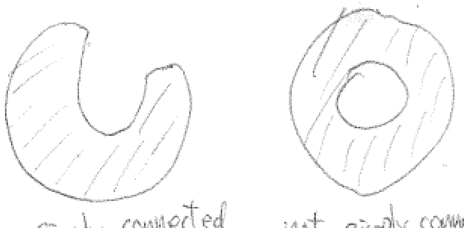
Example 56.
- Compute the line integral $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha}$, where $\boldsymbol{f}(x, y) = (y + 3x, 2y - x)$ and C is the boundary of the square $S = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$, where $\boldsymbol{\alpha}$ goes counterclockwise.

  By Green's theorem, $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$ and $\frac{\partial P}{\partial y} = 1, \frac{\partial Q}{\partial x} = -1$, therefore, $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \iint_S -2dxdy = -2$.
- Recall that the area of a region S is given by $a(S) = \iint_S 1dxdy$. If we take $P(x, y) = 0, Q(x, y) = x$, then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, therefore, by Green's theorem, $a(S) = \int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha}$, where C is the boundary of S (counterclockwise) and $\boldsymbol{f} = (P, Q)$. One can also take $P(x, y) = -y, Q(x, y) = 0$.

Application to gradients

Definition 57. A path α is a **step-polygon** if each component is parallel to the x-axis or the y-axis.

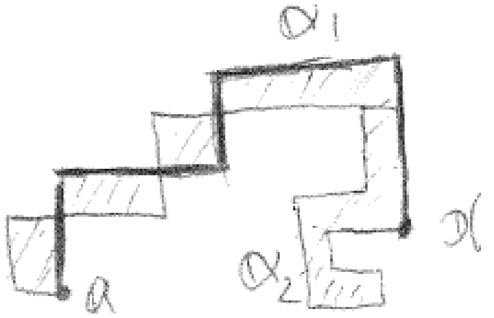
A region S is simply connected if two step-polygons $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ can be always continuously transformed to each other within step-polygons in S.



simply connected　　not simply connected

Theorem 58. *Let $\boldsymbol{f}(x, y) = (P(x, y), Q(x, y))$, continuously differentialbe in S, and S simply connected. Then, $\boldsymbol{f}$ is a gradient if and only if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, or in other notation, $\partial_1 f_2 = \partial_2 f_1$.*

Proof. We know already that if $\boldsymbol{f}$ is a gradient, then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.



Conversely, let us take two step-polygons $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ on $[a,b]$ in S such that $\boldsymbol{\alpha}_1(a) = \boldsymbol{\alpha}_2(a), \boldsymbol{\alpha}_1(b) = \boldsymbol{\alpha}_2(b)$. They may intersect at most a finite number of times, and when they do not coincide, they form polygons. Each R_k of these polygons is a union of type I+II regions, hence we can apply Theorem 55 and obtain

$$\iint_{R_k} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = (-1)^{n_k} \left(\int_{C_{k,1}} \boldsymbol{f} \cdot d\boldsymbol{\alpha} - \int_{C_{k,2}} \boldsymbol{f} \cdot d\boldsymbol{\alpha}\right),$$

where $C_{k,j}$ is the part of the boundary belonging to $\boldsymbol{\alpha}_j$, and $n_k = 0$ if C_k is taken counterclockwise and $n_k = 1$ if clockwise. By summing up and by the assumption, we obtain $\int_{C_1} \boldsymbol{f} \cdot d\boldsymbol{\alpha} - \int_{C_2} \boldsymbol{f} \cdot d\boldsymbol{\alpha} = 0$.

By fixing $\boldsymbol{\alpha}(a) = a$, we can take $\varphi(\boldsymbol{x}) = \int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ where $\boldsymbol{\alpha}$ is a step-polygons with $\boldsymbol{\alpha}(b) = \boldsymbol{x}$ on $[a,b]$ in S. This does not depend on the choice of $\boldsymbol{\alpha}$ and $\nabla\boldsymbol{\alpha} = \boldsymbol{f}$ as in Theorem 27. □

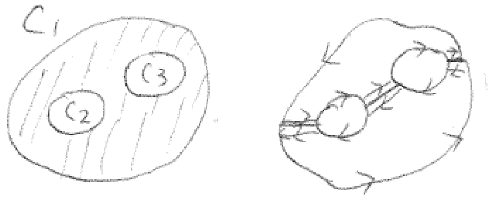
Non-simply connected regions

When the region S is not simply connected but have the boundary consisting of multiple curves, one can generalize Green's theorem.

Namely, let $\boldsymbol{f}(x,y) = (P(x,y), Q(x,y))$ be a continuously differentialble vector field on S. Let $R \subset S$ be a region whose boundary consists of curves C_k such that each of them is continuously differentiable, they do not intersect, the interior of C_1 includes all the rest C_k, C_k lies in the exterior of C_j if $j, k > 1$. Then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{C_1} \boldsymbol{f} \cdot d\boldsymbol{\alpha_1} - \sum_{k \neq 1} \int_{C_k} \boldsymbol{f} \cdot d\boldsymbol{\alpha_k},$$

where α_k is a parametrization of C_k going counter clockwise.



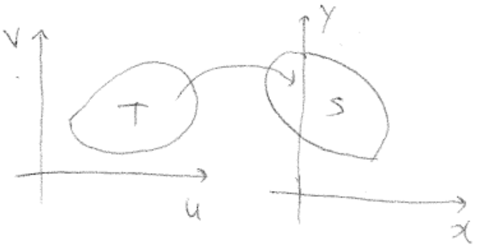
Nov 28. Change of variables

Jacobian determinant

In one dimension, with g which maps $[b, \tilde{b}]$ to $[a, \tilde{a}]$, we have

$$\int_a^{\tilde{a}} f(x)dx = \int_b^{\tilde{b}} f(g(t))g'(t)dx.$$

In two dimensions, we consider maps $\begin{cases} x = X(u,v) \\ y = Y(u,v) \end{cases}$ which maps one region T to another region S.



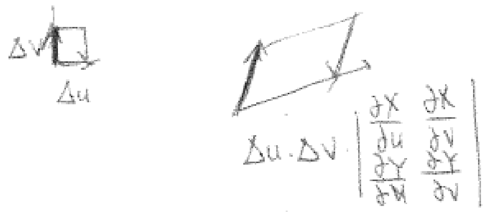
Example 59. • Linear transformations: $\begin{cases} x = X(u,v) = Au + Bv \\ y = Y(u,v) = Cu + Dv \end{cases}$

- The polar coordinates: $\begin{cases} x = X(r,\theta) = r\cos\theta \\ y = Y(r,\theta) = r\sin\theta \end{cases}$

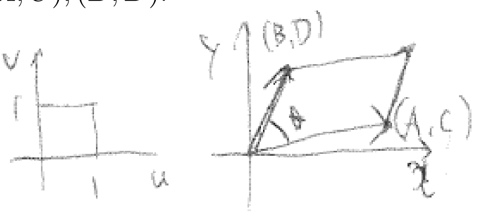
In a good situation, we have

$$\iint_S f(x,y)dxdy = \iint_T f(X(u,v),Y(u,v))|J(u,v)|dudv,$$

where $J(u,v) = \begin{vmatrix} \frac{\partial X}{\partial u}(u,v) & \frac{\partial Y}{\partial u}(u,v) \\ \frac{\partial X}{\partial v}(u,v) & \frac{\partial Y}{\partial v}(u,v) \end{vmatrix}$ is called the Jacobian determinant.



In the case of linear transformation $\begin{cases} x = X(u,v) = Au + Bv \\ y = Y(u,v) = Cu + Dv \end{cases}$, it is clear that $J(u,v) = AD - BC = (A,C)\cdot(D,-B) = \|(A,C)\|\cdot\|(B,D)\|\sin\theta$ is the area of the parallelogram formed by $(A,C),(B,D)$.

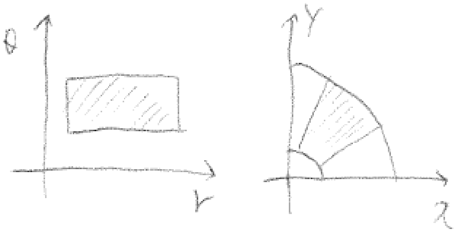


Example 60. For the polar coordinates $\begin{cases} x = X(r,\theta) = r\cos\theta \\ y = Y(r,\theta) = r\sin\theta \end{cases}$, we have

$$J(r,\theta) = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r.$$

We can compute the area of $S = \{(x,y) : 0 \le x, 0 \le y, x^2 + y^2 \le a^2\}$ which corresponds in the polar coordinates to $T = \{(r,\theta) : 0 \le r \le a, 0 \le \theta \le \frac{\pi}{2}\}$.

$$\iint_S dxdy = \iint_T J(r,\theta)drd\theta = \int_0^a \left[\int_0^{\frac{\pi}{2}} rd\theta\right] dr = \frac{\pi}{2}\left[\frac{r^2}{2}\right]_0^a = \frac{\pi a^2}{4}.$$



Proof of the formula for the special case

Let us prove the formula when $f(x,y)=1$ and S is a rectangle R in the xy-coordinates, which corresponds to R^* in the uv-coordinates. We assume that $J(u,v)>0$ on R^* and X,Y have continuous second derivatives.

We have to show

$$\iint_R dxdy = \iint_{R^*} |J(u,v)| dudv.$$

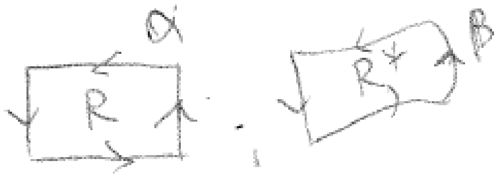
By Green's theorem, we have $\iint_R dxdy = \int \boldsymbol{f}\cdot d\boldsymbol{\alpha}$, where $\boldsymbol{f}(x,y)=(0,x)$ and $\boldsymbol{\alpha}$ goes along the boundary of R counterclockwise. On the other hand, it holds that

$$J(u,v) = \frac{\partial X}{\partial u}\frac{\partial Y}{\partial v} - \frac{\partial X}{\partial v}\frac{\partial Y}{\partial u} = \frac{\partial}{\partial u}\left(X\frac{\partial Y}{\partial v}\right) - \frac{\partial}{\partial v}\left(X\frac{\partial Y}{\partial u}\right)$$

Let us put $\boldsymbol{g}(u,v) = \left(X(u,v)\frac{\partial Y}{\partial u}, X(u,v)\frac{\partial Y}{\partial v}\right)$. By Green's theorem, we have

$$\iint_{R^*} J(u,v)dudv = \int \boldsymbol{g}\cdot d\boldsymbol{\beta},$$

where $\boldsymbol{\beta}$ goes along the boundary of R^* counterclockwise.



The boundary of R^* is mapped to the boundary of R by X,Y. Therefore, the curve $\boldsymbol{\alpha}(t) = (X(\boldsymbol{\beta}(t)), Y(\boldsymbol{\beta}(t)))$ parametrizes the boundary of R. By direct computations,

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} \frac{\partial X}{\partial u}\boldsymbol{\beta}_1'(t) + \frac{\partial X}{\partial v}\boldsymbol{\beta}_2'(t) \\ \frac{\partial Y}{\partial u}\boldsymbol{\beta}_1'(t) + \frac{\partial Y}{\partial v}\boldsymbol{\beta}_2'(t) \end{pmatrix}$$

Therefore, with $\boldsymbol{f}(x,y)=(0,x)$,

$$\begin{aligned}\iint_R dxdy = \int \boldsymbol{f}\cdot d\boldsymbol{\alpha} &= \int_a^{\tilde a} X(\boldsymbol{\beta}(t))\left(\frac{\partial Y}{\partial u}\boldsymbol{\beta}_1'(t) + \frac{\partial Y}{\partial v}\boldsymbol{\beta}_2'(t)\right)dt \\ &= \int_a^{\tilde a} \boldsymbol{g}(t)\cdot\boldsymbol{\beta}'(t)dt = \int \boldsymbol{g}\cdot d\boldsymbol{\beta} = \iint_{R^*} J(u,v)dudv.\end{aligned}$$

General case

Let us prove with a rectangle R and a function $f(x,y)\neq 1$. We take a decomposition of R into smaller rectangles R_{jk}. For a constant function s on R_{jk}, we have

$$\iint_{R_jk} sdxdy = \iint_{R^*_{jk}} s(X(u,v),Y(u,v))|J(u,v)|dudv.$$

As integral is addivive, we have for step functions

$$\iint_R s dx dy = \iint_{R^*} s(X(u,v), Y(u,v))|J(u,v)| du dv.$$

Let us take step functions $s(x,y) \le f(x,y) \le t(x,y)$. Correspondingly, we have the inequality $s(X(u,v), Y(u,v)) \le f(X(u,v), Y(u,v)) \le t(X(u,v), Y(u,v))$, therefore,

$$\begin{aligned}\iint_{R^*} s(X(u,v), Y(u,v))|J(u,v)| du dv &\le \iint_{R^*} f(X(u,v), Y(u,v))|J(u,v)| du dv \\ &\le \iint_{R^*} t(X(u,v), Y(u,v))|J(u,v)| du dv.\end{aligned}$$

Now, it follows that

$$\begin{aligned}\iint_R s(x,y) dx dy &= \iint_{R^*} s(X(u,v), Y(u,v))|J(u,v)| du dv \\ &\le \iint_{R^*} f(X(u,v), Y(u,v))|J(u,v)| du dv \\ &\le \iint_{R^*} t(X(u,v), Y(u,v))|J(u,v)| du dv = \iint_R t(x,y) dx dy.\end{aligned}$$

and if f is integrable, the sup and inf of the left and right hand sides coincide, as they run all possible step functions as above, and it is equal to $\iint_R f(x,y) dx dy$. This implies that $\iint_R f(x,y) dx dy = \iint_{R^*} f(X(u,v), Y(u,v))|J(u,v)| du dv$.

To summarize, we proved the following.

Theorem 61. *Let $X(u,v), Y(u,v)$ have continuous second derivaties, and $J(u,v) > 0$ on T. Assume that X, Y maps T to another region S surjectively, and the boundary of T is mapped to the boundary of S. If f is integrable in S, then*

$$\iint_S f(x,y) dx dy = \iint_T f(X(u,v), Y(u,v))|J(u,v)| du dv.$$

Example 62. (Gauss' integral) Compute $\int_\infty^\infty e^{-x^2} dx$ (this involves an improper integral).

Let us put $G = \int_\infty^\infty e^{-x^2} dx$. Then

$$\begin{aligned}G^2 &= \int_\infty^\infty e^{-x^2} dx \int_\infty^\infty e^{-y^2} dy = \int_{R^2} e^{-x^2-y^2} dx dy \\ &= \int_0^\infty \left[\int_0^{2\pi} e^{-r^2} r d\theta\right] dr = \pi[-e^{-r^2}]_0^\infty = \pi,\end{aligned}$$

therefore, $G = \sqrt{\pi}$.

Dec. 10. Surface

Parametric representation of a surface

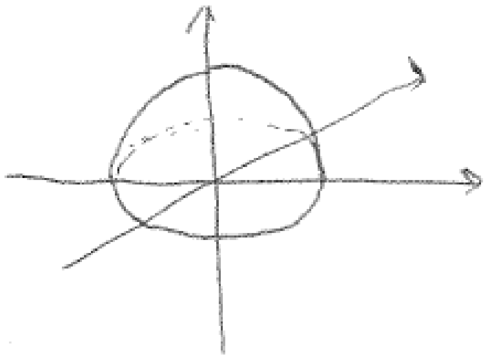
Recall that the unit circle $x^2 + y^2 = 1$ can be parametrized in two ways: $y = \pm\sqrt{1 - x^2}$, or $x = \cos t, y = \sin t$.

Consider the equation $x^2 + y^2 + z^2 = 1$. This equation defines the unit sphere, and can be solved with respect to z as $z = \pm\sqrt{1 - x^2 - y^2}$. In other words, the upper/lower hemisphere can be parametrized with $(x,y), x^2 + y^2 \le 1$ by

$$x = X(x,y), \qquad y = Y(x,y), \qquad z = Z(x,y) = \pm\sqrt{1 - x^2 - y^2}.$$

Alternatively, we can take

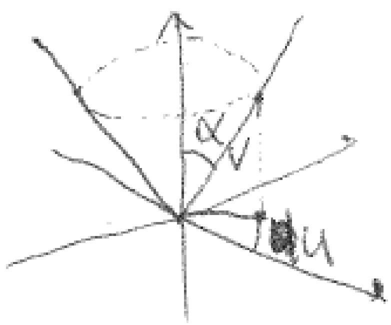
$$x = X(u,v) = \cos u \sin v, \quad y = Y(u,v) = \sin u \sin v, \quad z = Z(u,v) = \cos v.$$



In general, a two-dimensional surface in $\mathbb{R}^3$ can be parametrized by two parameters (u,v) by $x = X(u,v), y = Y(u,v), z = Z(u,v)$. Each point on the surface is represented by $\boldsymbol{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$.

Example 63. A cone with angle α can be parametrized by

$$x = v\cos\alpha\cos u, \qquad y = v\cos\alpha\sin u, \qquad z = v\sin u.$$



Vector product

Let $\boldsymbol{x} = (x_1,x_2,x_3), \boldsymbol{y} = (y_1,y_2,y_3)$. Recall that $\boldsymbol{x}\times\boldsymbol{y} = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$ is a vector orthogonal to $\boldsymbol{x},\boldsymbol{y}$ and of length $\|\boldsymbol{x}\|\|\boldsymbol{y}\|\sin\theta$, where θ is the angle between $\boldsymbol{x},\boldsymbol{y}$. Let $\boldsymbol{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$ be a parametrized surface. Then $\frac{\partial\boldsymbol{r}}{\partial u}, \frac{\partial\boldsymbol{r}}{\partial v}$ represents tangent vectors to the surface. Their vector product

$$\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v} = \left(\frac{\partial Y}{\partial u}\frac{\partial Z}{\partial v} - \frac{\partial Z}{\partial u}\frac{\partial Y}{\partial v}, \quad \frac{\partial Z}{\partial u}\frac{\partial X}{\partial v} - \frac{\partial X}{\partial u}\frac{\partial Z}{\partial v}, \quad \frac{\partial X}{\partial u}\frac{\partial Y}{\partial v} - \frac{\partial Y}{\partial u}\frac{\partial X}{\partial v}\right)$$

is orthogonal to the surface. Such a vector is called a normal vector.

If $\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v} \neq \boldsymbol{0}$ at a point $\boldsymbol{r}(u,v)$ (namely, if $\frac{\partial\boldsymbol{r}}{\partial u}$ and $\frac{\partial\boldsymbol{r}}{\partial v}$ are linearly independent), then we say that $\boldsymbol{r}$ is a regular point.

Example 64. (Normal vector on the sphere) Consider $x^2+y^2+z^2 = a^2$ and let us take the parametrization $\boldsymbol{r}(u,v) = (a\cos u\sin v, a\sin u\sin v, a\cos v)$. The tangent vectors and the normal vector are

$$\begin{aligned}\frac{\partial\boldsymbol{r}}{\partial u} &= (-a\sin u\sin v, a\cos u\sin v, 0)\\ \frac{\partial\boldsymbol{r}}{\partial v} &= (a\cos u\cos v, a\sin u\cos v, -\sin v)\\ \frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v} &= (-a^2\cos u\sin^2 v, -a^2\sin u\sin^2 v, -a^2\sin v\cos v) = -a\sin v\boldsymbol{r}.\end{aligned}$$

Area of a surface

Recall that $\|\boldsymbol{x} \times \boldsymbol{y}\|$ is the area of the parallelogram spanned by $\boldsymbol{x}$ and $\boldsymbol{y}$. A small rectangle $\Delta u \Delta v$ corresponds to a parallelogram spanned by $\frac{\partial \boldsymbol{r}}{\partial u}\Delta u$ and $\frac{\partial \boldsymbol{r}}{\partial v}\Delta v$.

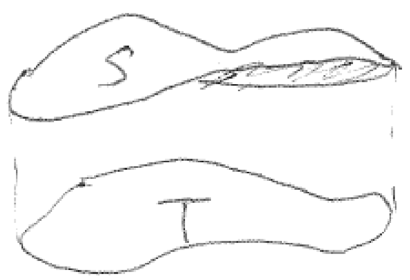
Definition 65. If a surface S is parametrized as $\boldsymbol{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$ by $(u,v) \in T$, then the area $a(S)$ is defined by

$$a(S) = \iint_T \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| dudv.$$

Example 66. If a surface is given explicitly by $x = u, y = v, z = f(u,v)$, then

$$\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial u} &= \left(1, 0, \frac{\partial f}{\partial u}\right) \\
\frac{\partial \boldsymbol{r}}{\partial v} &= \left(0, 1, \frac{\partial f}{\partial v}\right) \\
\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} &= \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right).
\end{aligned}$$

Therefore, $\|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\| = \sqrt{(\frac{\partial f}{\partial u})^2 + (\frac{\partial f}{\partial v})^2 + 1}$ and $a(S) = \iint_T \sqrt{(\frac{\partial f}{\partial u})^2 + (\frac{\partial f}{\partial v})^2 + 1}dudv$. If f is constant, this gives the area of T itself.



Example 67. (the area of hemisphere)

- By spherical coordinate $\boldsymbol{r}(u,v) = (a\cos u \sin v, a \sin u \sin v, \cos u)$ and the upper hemisphere is $T = \{(u,v) : 0 \le u \le 2\pi, 0 \le v \le \frac{\pi}{2}\}$.

  We have computed that $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = -a \sin v \boldsymbol{r}$, therefore, $\|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\| = a^2 \sin v$ and

$$a(S) = \iint_T a^2 \sin v dudv = a^2 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin v dudv = 2\pi a^2.$$

- By explicit parametrization. We can take $\tilde{T} = \{(u,v) : u^2 + v^2 \le a^2\}$ and $f(u,v) = \sqrt{a^2 - u^2 - v^2}$. By straightforward computations,

$$\frac{\partial f}{\partial u} = \frac{u}{\sqrt{a^2 - u^2 - v^2}}, \quad \frac{\partial f}{\partial v} = \frac{v}{\sqrt{a^2 - u^2 - v^2}}$$

  and hence

$$\begin{aligned}
a(S) &= \iint_{\tilde{T}} \sqrt{\frac{u^2}{a^2 - u^2 - v^2} + \frac{v^2}{a^2 - u^2 - v^2} + 1dudv} = \iint_{\tilde{T}} \frac{a}{\sqrt{a^2 - u^2 - v^2}}dudv \\
&= \iint_{\tilde{T}} \frac{a}{\sqrt{a^2 - u^2 - v^2}}dudv = \iint_{\hat{T}} \frac{a}{\sqrt{a^2 - r^2}}rdrd\theta = \int_0^a \int_0^{2\pi} \frac{a}{\sqrt{a^2 - r^2}}rdrd\theta \\
&= 2\pi \left[-a\sqrt{a^2 - r^2}\right]_0^a = 2\pi a^2 \text{ (improper integral).}
\end{aligned}$$

Another theorem of Pappus

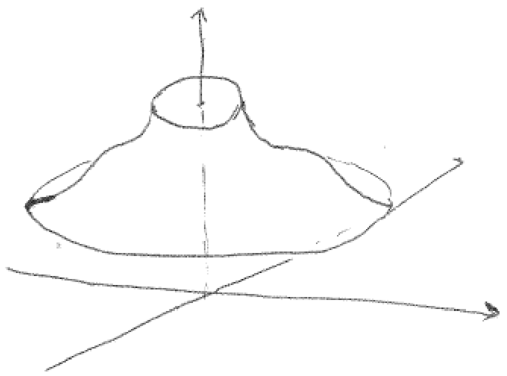
Let C be a curve in the xz-plane, represented by a function $f(x), x \in [a, \tilde{a}]$. If we rotate it around the z-axis, we obtain a surface S. The Pappus' theorem says that $a(S) = 2\pi L\bar{x}$, where L is the length of the curve and $\bar{x}$ is the centroid.

Recall that, with the parametrization of the curve $\alpha(x) = (x, 0, f(x)), \|\alpha'(x)\| = \sqrt{1+f'(x)^2}$.

$$L = \int_a^{\tilde{a}} \sqrt{1+f'(x)^2}dx, \bar{x} = \frac{1}{L}\int_a^{\tilde{a}} \sqrt{1+f'(x)^2}dx.$$

We can parametrize the surface by $\boldsymbol{r}(u,\theta) = (u\cos\theta, u\sin\theta, f(u))$ with $T = \{(u,\theta) : a \le u \le \tilde{a}, 0 \le \theta \le 2\pi\}$. By straightforward computations, we have

$$\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial u} &= (\cos\theta, \sin\theta, f'(u)) \\
\frac{\partial \boldsymbol{r}}{\partial \theta} &= (-u\sin\theta, u\cos\theta, 0) \\
\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial \theta} &= (-u\cos\theta f'(u), -u\sin\theta f'(u), u) \\
\left\|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial \theta}\right\| &= u\sqrt{1+f'(u)^2} \\
a(S) &= \iint_T u\sqrt{1+f'(u)^2}dud\theta = \int_a^{\tilde{a}}\int_0^{2\pi} u\sqrt{1+f'(u)^2}d\theta du = 2\pi L\bar{x}.
\end{aligned}$$



Dec. 12. Surface integrals

Definition

Let S be a surface, parametrized by $\boldsymbol{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$ on T. Let f be a scalar field on S. We define its surface integral by

$$\iint_S f dS := \iint_T f(\boldsymbol{r}(u,v)) \left\|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right\| dudv$$

Example 68.

- With $f = 1$, one obtains the area of S.
- If a plate has the form of S and its density at $\boldsymbol{r}(u,v)$ is $f(\boldsymbol{r}(u,v))$, then its mass is given by $\iint_T f(\boldsymbol{r}(u,v)) \left\|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right\| dudv$. The center of mass is given by $(\bar{x}, \bar{y}, \bar{z})$, where $\bar{x} = \iint_T X(u,v) f(\boldsymbol{r}(u,v)) \left\|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right\| dudv$ etc..

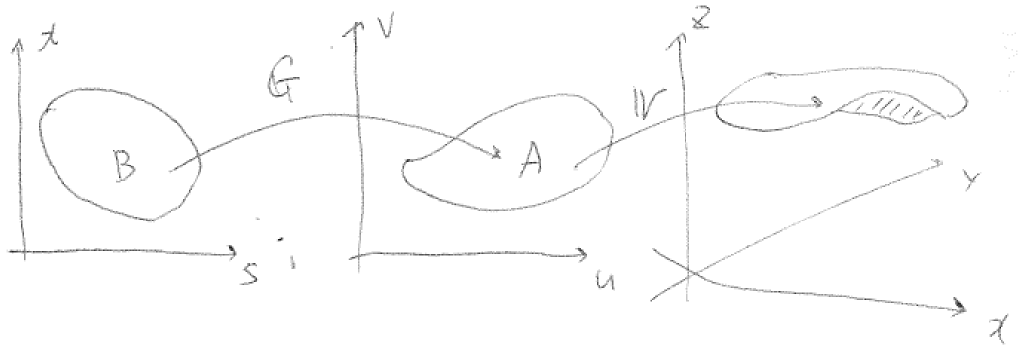
Change of parametrization

Let S be a surface, parametrized by $\boldsymbol{r}$ on A, respectively, One can consider a change of variables, namely, $\boldsymbol{G}(s,t) = (U(s,t), V(s,t))$ is a map from are region B to A, then $\boldsymbol{r}'(s,t) := \boldsymbol{r}(\boldsymbol{G}(s,t))$ is another parametrization of S.

Example 69. The upper hemisphere can be parametrized as

- $X(u,v) = u, Y(u,v) = v, Z(u,v) = \sqrt{1-u^2-v^2}$.
- $X(s,t) = \cos s \sin t, Y(s,t) = \sin s \sin t, Z(s,t) = \cos t$.

We can take $\boldsymbol{G}(s,t) = (\cos s \sin t, \sin s \sin t)$.



Lemma 70. *Let $\boldsymbol{r}, \boldsymbol{r}'$ be two parametrization of S connected by a continuouly differentiable map $\boldsymbol{G}$ as above. Then we have*

$$\begin{aligned}&\frac{\partial \boldsymbol{r}'(s,t)}{\partial s} \times \frac{\partial \boldsymbol{r}'(s,t)}{\partial t} \\ &= \frac{\partial \boldsymbol{r}(U(s,t),V(s,t))}{\partial s} \times \frac{\partial \boldsymbol{r}(U(s,t),V(s,t))}{\partial s} \left(\frac{\partial U(s,t)}{\partial s}\frac{\partial V(s,t)}{\partial t} - \frac{\partial U(s,t)}{\partial t}\frac{\partial V(s,t)}{\partial s} \right)\end{aligned}$$

Proof. By chain rule,

$$\begin{aligned}\frac{\partial \boldsymbol{r}'(s,t)}{\partial s} &= \frac{\partial \boldsymbol{r}(U(s,t),V(s,t))}{\partial u}\frac{\partial U(s,t)}{\partial s} + \frac{\partial \boldsymbol{r}(U(s,t),V(s,t))}{\partial v}\frac{\partial U(s,t)}{\partial s}, \\ \frac{\partial \boldsymbol{r}'(s,t)}{\partial t} &= \frac{\partial \boldsymbol{r}(U(s,t),V(s,t))}{\partial u}\frac{\partial U(s,t)}{\partial t} + \frac{\partial \boldsymbol{r}(U(s,t),V(s,t))}{\partial v}\frac{\partial U(s,t)}{\partial t}\end{aligned}$$

By taking the vector product, we obtain

$$\begin{aligned}&\frac{\partial \boldsymbol{r}'(s,t)}{\partial s} \times \frac{\partial \boldsymbol{r}'(s,t)}{\partial t} \\ &= \frac{\partial \boldsymbol{r}(U(s,t),V(s,t))}{\partial s} \times \frac{\partial \boldsymbol{r}(U(s,t),V(s,t))}{\partial s} \left(\frac{\partial U(s,t)}{\partial s}\frac{\partial V(s,t)}{\partial t} - \frac{\partial U(s,t)}{\partial t}\frac{\partial V(s,t)}{\partial s} \right).\end{aligned}$$

□

Theorem 71. *Let $\boldsymbol{r}, \boldsymbol{r}'$ be two parametrization of S connected by a continuouly differentiable map $\boldsymbol{G}$ as above. If f is integrable with respect to $\boldsymbol{r}$, then so is it with respect to $\boldsymbol{r}'$ and*

$$\iint_A f(\boldsymbol{r}(u,v)) \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| dudv = \iint_B f(\boldsymbol{r}'(s,t)) \left\| \frac{\partial \boldsymbol{r}'}{\partial s} \times \frac{\partial \boldsymbol{r}'}{\partial t} \right\| dsdt$$

Proof. Note that $\frac{\partial U(s,t)}{\partial s}\frac{\partial V(s,t)}{\partial t} - \frac{\partial U(s,t)}{\partial t}\frac{\partial V(s,t)}{\partial s}$ is the Jacobian determinant of the map $G(s,t) = (U(s,t), V(s,t))$. Therefore, by the change of variable (Theorem 61), we have

$$\begin{aligned}
&\iint_A f(\boldsymbol{r}(u,v)) \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| dudv \\
&= \iint_B f(\boldsymbol{r}'(s,t)) \left\| \frac{\partial \boldsymbol{r}}{\partial s} \times \frac{\partial \boldsymbol{r}}{\partial t} \right\| \left| \frac{\partial U(s,t)}{\partial s}\frac{\partial V(s,t)}{\partial t} - \frac{\partial U(s,t)}{\partial t}\frac{\partial V(s,t)}{\partial s} \right| dsdt \\
&= \iint_B f(\boldsymbol{r}'(s,t)) \left\| \frac{\partial \boldsymbol{r}'}{\partial s} \times \frac{\partial \boldsymbol{r}'}{\partial t} \right\| dsdt.
\end{aligned}$$

□

Surface integral of a vector field

Let $\boldsymbol{r}(u,v)$ parametrize a surface S and $\boldsymbol{N} := \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \neq \boldsymbol{0}$. We introduce $\boldsymbol{n}_1 := \frac{\boldsymbol{N}}{\|\boldsymbol{N}\|}, \boldsymbol{n}_2 := \boldsymbol{n}_1$. If $\boldsymbol{F}$ is a vector field defined on S, then $\boldsymbol{F} \cdot \boldsymbol{n}$ is a scalar field on S, hence we can consider its surface integral on S. It holds that (we omit that $\boldsymbol{F}, \boldsymbol{N}$ etc. depend on (u,v))

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}_1 dS = \iint_T \boldsymbol{F} \cdot \boldsymbol{n}_1 \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| dudv = \iint_T \boldsymbol{F} \cdot \boldsymbol{N} dudv$$

If we use $\boldsymbol{n}_2$, the integral gets -1.

If $\boldsymbol{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ by components, then the surface integral is sometimes denoted as

$$\iint_S \boldsymbol{F} dS = \iint P dy \wedge dz + Q dz \wedge dz + R dx \wedge dy.$$

We can represent the flow of liquid by a vector field $\boldsymbol{f}(x,y,z)$, with the density $\rho(x,y,z)$. Namely, at each point (x,y,z) the liquid has the velocity $\boldsymbol{f}(x,y,z)$. The flux density is defined by $\boldsymbol{F}(x,y,z) = \rho(x,y,z)\boldsymbol{f}(x,y,z)$. If we take a surface S, then the integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$ is the mass of the liquid passing through S at a given time.

Surface integrals appear also as electric flux.

Example 72. Let $S : x^2 + y^2 + z^2 = 1, z \geq 0$ and $\boldsymbol{F}(x,y,z) = (x,y,0)$. Compute $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$, where $\boldsymbol{n}$ is outgoing.

Solution. We take the parametrization $\boldsymbol{r}(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$. We computed: $\boldsymbol{N} = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = -\sin v \boldsymbol{r}(u,v)$, therefore, $\boldsymbol{F} \cdot \boldsymbol{N} = -\sin v(\cos^2 u \sin^2 v + \sin^2 u \sin^2 v) = -\sin^3 v$, so we take $\boldsymbol{n}_2$.

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS = -\iint_T \boldsymbol{F} \cdot \boldsymbol{N} dudv \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin^3 v \; dudv = 2\pi \cdot \left[(v - \frac{1}{3}v^3) \right]_0^1 = \frac{4\pi}{3}.$$

Curl and divergence

Let $\boldsymbol{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field. Define

$$\operatorname{curl} \boldsymbol{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right), \qquad \operatorname{div} \boldsymbol{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Example 73.

- $\boldsymbol{F}(x,y,z) = (x,y,z)$. $\operatorname{curl} \boldsymbol{F}(x,y,z) = (0,0,0), \operatorname{div} \boldsymbol{F}(x,y,z) = 1+1+1 = 3$.
- $\boldsymbol{F}(x,y,z) = (xy^2z^3, z^2 \sin y, x^2 e^y)$.$\operatorname{curl} \boldsymbol{F}(x,y,z) = (x^2 e^y - 2z \sin y, 3xy^2z^2 - 2xe^y, z^2 \cos y - y^2 z^3), \operatorname{div} \boldsymbol{F}(x,y,z) = y^2z^3 + z^2 \cos y$.

- $\boldsymbol{F}(x,y,z) = \nabla\varphi, \varphi$ has continuous second derivatives.

$$\operatorname{curl}\boldsymbol{F}(x,y,z) = \left(\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}, \frac{\partial^2\varphi}{\partial z\partial x} - \frac{\partial^2\varphi}{\partial x\partial z}, \frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x}\right) = (0,0,0).$$

  $\operatorname{div}\boldsymbol{F}(x,y,z) = \frac{\partial^2\varphi}{\partial^2 x} + \frac{\partial^2\varphi}{\partial^2 y} + \frac{\partial^2\varphi}{\partial^2 z}$ is called the Laplacian.

- $\boldsymbol{F}(x,y,z) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0)$. $\operatorname{curl}\boldsymbol{F}(x,y,z) = (0,0,\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}) = (0,0,0)$ (as we computed before), $\operatorname{div}\boldsymbol{F}(x,y,z) = \frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} + 0 = 0$.

- In general, $\operatorname{div}(\operatorname{curl}\boldsymbol{F}) = 0$, $\operatorname{curl}(\operatorname{curl}\boldsymbol{F}) = \nabla(\operatorname{div}\varphi)$.

The following notations are also used:

$$\operatorname{curl}\boldsymbol{F} = \nabla\times\boldsymbol{F}, \operatorname{div}\boldsymbol{F} = \nabla\cdot\boldsymbol{F}.$$

Theorem 74. *Let V be a convex region in $\mathbb{R}^3$. Then,* $\operatorname{curl}\boldsymbol{F} = \mathbf{0} \iff \boldsymbol{F}$ *is a gradient.*

Proof. We have seen that $\boldsymbol{F}$ is a gradient $\iff$ $\operatorname{curl}\boldsymbol{F} = \mathbf{0}$. Conversely, if $\operatorname{curl}\boldsymbol{F} = \mathbf{0}$, then $\partial_j F_k = \partial_k F_j$. □

Dec. 17. Stokes' theorem

Stokes' theorem

Recall Green's theorem: for continuously differentiable vector field $\boldsymbol{f}(x,y) = (P(x,y), Q(x,y))$ in $S\subset\mathbb{R}^2$, where S is a plane region bounded by a simple closed curve C, parametrized by $\boldsymbol{\alpha}$ going in the counterclockwise direction, we have

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_C \boldsymbol{f}\cdot d\boldsymbol{\alpha}.$$

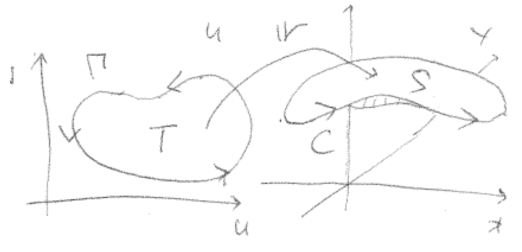
Stokes' theorem is its generalization to the three dimensional space. Recall that, for $\boldsymbol{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$, we defined

$$\operatorname{curl}\boldsymbol{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

Theorem 75 (Stokes)**.** *Let S be a surface parametrized by $\mathbf{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$ on a simply connected region T and the boundary Γ is mapped to the boundary C of S. Take a parametrization $\boldsymbol{\beta}$ of Γ which goes counterclockwise, and let $\boldsymbol{\alpha}(t) = \mathbf{r}(\boldsymbol{\beta})(t)$. Assume that $\mathbf{r}$ and $\boldsymbol{\beta}$ have continuous second derivatives, and let $\boldsymbol{F}$ be a continuously differentiably vector field on S. Then,*

$$\iint_S \operatorname{curl}\boldsymbol{F}\cdot\boldsymbol{n}_1 dS = \int_C \boldsymbol{F}\cdot d\boldsymbol{\alpha},$$

where $\boldsymbol{n}_1 = \frac{\boldsymbol{N}}{\|\boldsymbol{N}\|}, \boldsymbol{N} = \frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}$.



Proof. Let us first assume that $Q = R = 0$. In this case, it holds that

$$\begin{aligned}\iint_S \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n}_1 dS &= \iint_T \left(0, \frac{\partial P}{\partial z}, -\frac{\partial P}{\partial y}\right) \cdot \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} dudv \\ &= \iint_T \left(\frac{\partial P}{\partial z}\left(\frac{\partial Z}{\partial u}\frac{\partial X}{\partial v} - \frac{\partial X}{\partial u}\frac{\partial Z}{\partial v}\right) - \frac{\partial P}{\partial y}\left(\frac{\partial X}{\partial u}\frac{\partial Y}{\partial v} - \frac{\partial Y}{\partial u}\frac{\partial X}{\partial v}\right)\right) dudv.\end{aligned}$$

Note that

$$\begin{aligned}\frac{\partial}{\partial u}\left(P(\boldsymbol{r})\frac{\partial X}{\partial v}\right) &= \left(\frac{\partial P}{\partial x}\frac{\partial X}{\partial u} + \frac{\partial P}{\partial y}\frac{\partial Y}{\partial u} + \frac{\partial P}{\partial z}\frac{\partial Z}{\partial u}\right)\frac{\partial X}{\partial v} + P(\boldsymbol{r})\frac{\partial^2 X}{\partial u \partial v} \\ \frac{\partial}{\partial v}\left(P(\boldsymbol{r})\frac{\partial X}{\partial u}\right) &= \left(\frac{\partial P}{\partial x}\frac{\partial X}{\partial v} + \frac{\partial P}{\partial y}\frac{\partial Y}{\partial v} + \frac{\partial P}{\partial z}\frac{\partial Z}{\partial v}\right)\frac{\partial X}{\partial u} + P(\boldsymbol{r})\frac{\partial^2 X}{\partial v \partial u}\end{aligned}$$

By setting $\boldsymbol{g}(u, v) = \left(P(\boldsymbol{r})\frac{\partial X}{\partial u}, P(\boldsymbol{r})\frac{\partial X}{\partial v}\right)$, we have

$$\begin{aligned}\iint_S \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n}_1 dS &= \iint_T \frac{\partial}{\partial u}\left(P(\boldsymbol{r})\frac{\partial X}{\partial v}\right) - \frac{\partial}{\partial v}\left(P(\boldsymbol{r})\frac{\partial X}{\partial u}\right) dudv \\ &= \int_\Gamma \boldsymbol{g} \cdot d\boldsymbol{\beta} = \int_a^{\tilde{a}} \boldsymbol{g}(\boldsymbol{\beta}(t)) \cdot \boldsymbol{\beta}'(t) dt\end{aligned}$$

by Green's theorem. Finally, with $\boldsymbol{\alpha}(t) = \boldsymbol{r}(\boldsymbol{\beta}(t))$ hence

$$\boldsymbol{\alpha}'(t) = \left(\frac{\partial X}{\partial u}(\boldsymbol{\beta}(t))\beta_1' + \frac{\partial X}{\partial u}(\boldsymbol{\beta}(t))\beta_2', \frac{\partial Y}{\partial u}(\boldsymbol{\beta}(t))\beta_1' + \frac{\partial Y}{\partial u}(\boldsymbol{\beta}(t))\beta_2', \frac{\partial Z}{\partial u}(\boldsymbol{\beta}(t))\beta_1' + \frac{\partial Z}{\partial u}(\boldsymbol{\beta}(t))\beta_2'\right),$$

and

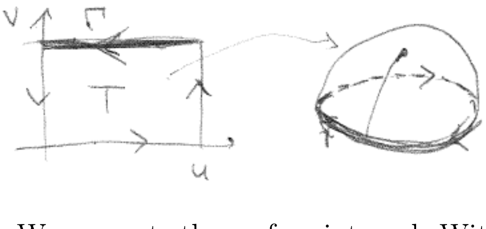
$$\begin{aligned}\boldsymbol{g}(\boldsymbol{\beta}) \cdot \boldsymbol{\beta}'(t) &= P(\boldsymbol{r}(\boldsymbol{\beta}(t)))\frac{\partial X}{\partial u}(\boldsymbol{\beta}(t))\beta_1' + P(\boldsymbol{r}(\boldsymbol{\beta}(t)))\frac{\partial X}{\partial v}(\boldsymbol{\beta}(t))\beta_2' \\ &= P(\boldsymbol{r}(\boldsymbol{\beta}(t)))\left(\frac{\partial X}{\partial u}(\boldsymbol{\beta}(t))\beta_1' + \frac{\partial X}{\partial v}(\boldsymbol{\beta}(t))\beta_2'\right) = \boldsymbol{F}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t),\end{aligned}$$

therefore,

$$\iint_S \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n}_1 dS = \int_a^{\tilde{a}} \boldsymbol{F}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) dt = \int_C \boldsymbol{F} \cdot d\boldsymbol{\alpha}.$$

The proof for the cases where $R = P = 0$ or $P = Q = 0$ is similar. The statement is linear in $\boldsymbol{F}$, hence adding these cases completes the proof. □

Example 76. Let $\boldsymbol{F}(x, y, z) = (y, z, x)$ and $S : x^2 + y^2 + z^2 = 1, z \geq 0$.



- We compute the surface integral. With $\boldsymbol{r}(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)$, $T = \{(u, v) : 0 \leq u \leq 2\pi, 0 \leq v \leq \frac{\pi}{2}\}$. We know that $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = -\sin v \boldsymbol{r}$. We have curl $\boldsymbol{F}(x, y, z) = (-1, -1, -1)$ and hence curl $\boldsymbol{F}(\boldsymbol{r}(u, v)) = (-1, -1, -1)$.

$$\begin{aligned}\iint_S \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n}_1 dS &= \iint_T \sin v(\sin u \sin v + \sin u \sin v + \cos v) dudv \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin v \cos v dv du = \pi.\end{aligned}$$

  by performing the u-integral first.

- We compute the line integral. With t $\boldsymbol{r}(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$, the boundary of S is given by $v = \frac{\pi}{2}$. The other boundaries in T corresponds to the north pole and a quarter of the great circle in opposite directions, which do not contribute to the line integral. Hence we have to take $\boldsymbol{\alpha}(t) = (\cos t, -\sin t, 0), t \in [0, 2\pi]$, $\boldsymbol{\alpha}'(t) = (-\sin t, -\cos t, 0)$. $\boldsymbol{F}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha}'(t) = (-\sin t, 0, \cos t) \cdot (-\sin t, -\cos t, 0) = \sin^2 t$ and hence

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{\alpha} = \int_0^{2\pi} \sin^2 t \, dt = \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} dt = \pi.$$

Extension of Stokes' theorem

Recall that Green's theorem can be extended to regions with holes, namely,

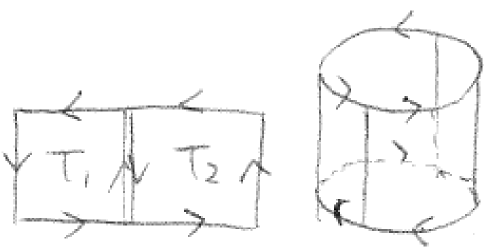
$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_{C_1} \boldsymbol{f} \cdot d\boldsymbol{\alpha} - \sum_k \int_{C_k} \boldsymbol{f} \cdot d\boldsymbol{\alpha},$$

wherer C_1 is the outermost boundary of S and C_k's are the boundaries of the holes.

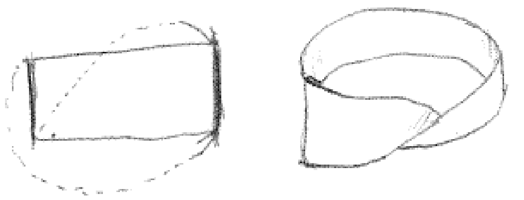
Similarly, Stokes's theorem can be extended to surfaces S with holes if S is the image of the one-to-one map $\boldsymbol{r}$ of a region T to which Green's theorem can be applied:

$$\iint_S \text{curl}\, \boldsymbol{F} \cdot \boldsymbol{n} dS = \int_{C_1} \boldsymbol{f} \cdot d\boldsymbol{\alpha} - \sum_k \int_{C_k} \boldsymbol{f} \cdot d\boldsymbol{\alpha},$$

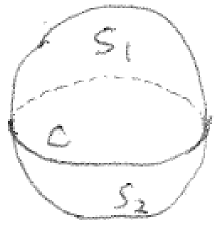
Stokes' theorem can be further extended to regions such as cylinder which has the boundary which is a union of two circles C_1, C_2. The surface integral $\iint \boldsymbol{F} \cdot \boldsymbol{n} dS$ can be devided into two pieces S_1, S_2, and by Stokes' theorem it is equal to $\int_{\tilde{C}_1 \cup \tilde{C}_2} \boldsymbol{f} \cdot d\boldsymbol{\alpha}$. The intersection of boundaries of S_1, S_2 are parametrized in different directions, hence they cancel and the integral $\int_{\tilde{C}_1 \cup \tilde{C}_2} \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ remains, where the parametrizations of C_1 and C_2 go in opposite directions. This is in accordance with the result on surfaces with holes, because a cylinder can be considered as a region with one hole.



But it cannot be extended to regions such as Möbius' band, because one cannot take an altas of parametrizations in such a way that the integrals on the boundaries cancel. In general, Stokes' theorem can be extended only to "orientable" surfaces.



A sphere S can be decomposed into two hemispheres S_1, S_2 with normal vectors going outside. If we add $\iint_{S_1 \cup S_2} \text{curl}\, \boldsymbol{F} \cdot \boldsymbol{n} dS$ and use Stokes' theorem, the boundary integrals go in the opposite directions, hence cance. Namely, we have $\iint_S \text{curl}\, \boldsymbol{F} \cdot \boldsymbol{n} dS = 0$. This result can be extened to closed surfaces, namely, surfaces without boundary.

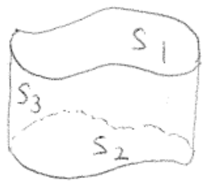


Dec. 19. Gauss' theorem

Gauss' theorem

Definition 77. Let $V \subset \mathbb{R}^3$ be a solid. We say that V is xy**-projectable** if there are functions f, g on a region $A \subset \mathbb{R}^2$ such that $V = \{(x, y, z) : (x, y) \in A, f(x, y) \leq z \leq g(x, y)\}$. For a xy-projectable region V, one can define the outside and inside direction on the boundary which is a surface.

Similarly, we introduce yz- and zx-projectable solids.



Example 78. (projectable solids)

- Cube $V = \{(x, y, z) : 0 \leq x, y, z \leq a\}$. V is xy-projectable, indeed, with the region $A = \{(x, y) : 0 \leq x, y \leq a\}$, $V = \{(x, y, z) : (x, y) \in A, 0 \leq z \leq a\}$, Similarly, it is yz- and zx-projectable.
- Ball $V = \{(x, y, z) : x^2 + y^2 + z^2 \leq a^2\}$. V is xy-projectable, indeed, with the region $A = \{(x, y) : x^2 + y^2 \leq a^2\}$, $V = \{(x, y, z) : (x, y) \in A, -\sqrt{a^2 - x^2 - y^2} \leq z \leq \sqrt{a^2 - x^2 - y^2}\}$. Similarly, it is yz- and zx-projectable.

Gauss' theorem connects a surface integral and a volume integral of $\operatorname{div} \boldsymbol{F}$.

Theorem 79 (Gauss)**.** *Let V be xy-, yz-, zx-projectable solid, or V is a finite disjoint union of such solids with surfaces which are either disjoint or the intersection which is a surface whose outside directions are opposite. Let S be the boundary which is a surface of V and $\boldsymbol{n}$ be the vector. For a continuously differentiable vector field $\boldsymbol{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$, it holds that*

$$\iiint_V \operatorname{div} \boldsymbol{F} dxdydv = \iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS.$$

Proof. Since V is xy-projectable, we can take A such that $V = \{(x, y, z) : (x, y) \in A, f(x, y) \leq z \leq g(x, y)\}$. The surface of S consists of

$$\begin{aligned} S_1 &= \{(x, y, z) : (x, y) \in A, z = g(x, y)\}, \\ S_2 &= \{(x, y, z) : (x, y) \in A, z = f(x, y)\}, \\ S_3 &= \{(x, y, z) : (x, y) \in \partial A, f(x, y) \leq z \leq g(x, y)\}, \end{aligned}$$

where ∂A is the boundary of A.

Let us first assume that $P = Q = 0$. Then $\operatorname{div} \boldsymbol{F} = \frac{\partial R}{\partial z}$ and

$$\iiint_V \operatorname{div} \boldsymbol{F} dxdydz = \iint_A \left[\int_{f(x,y)}^{g(x,y)} \frac{\partial R}{\partial z} dz\right] dxdy = \iint_A \left[R(x,y,g(x,y)) - R(x,y,f(x,y))\right] dxdy.$$

On the other hand, on S_1 we can take the parametrization $\boldsymbol{r}(u,v) = (u,v,g(u,v))$, and then $\frac{\partial \boldsymbol{r}}{\partial u} = (1, 0, \frac{\partial g}{\partial u})$, $\frac{\partial \boldsymbol{r}}{\partial v} = (0, 1, \frac{\partial g}{\partial v})$ and hence $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = \left(-\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1\right)$, and hence

$$\iint_{S_1} \boldsymbol{F} \cdot \boldsymbol{n} dS = \iint_A \boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \left(-\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1\right) dudv = \iint_A R(u,v,g(u,v))dudv$$

On S_2, we can take the parametrization $\boldsymbol{r}(u,v) = (u,v,f(u,v))$, and then we have $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right)$, but we should take an outgoing normal vector, hence

$$\iint_{S_2} \boldsymbol{F} \cdot \boldsymbol{n} dS = \iint_A \boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot (-\boldsymbol{n}) \cdot = -\iint_A R(u,v,f(u,v))dudv$$

Finally on S_3, the normal vector $\boldsymbol{n}$ is in the xy-plane, therefore $\boldsymbol{F} \cdot \boldsymbol{n} = 0$ and

$$\iint_{S_3} \boldsymbol{F} \cdot \boldsymbol{n} dS = 0.$$

Altogether,

$$\begin{aligned}\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS &= \iint_{S_1 \cup S_2 \cup S_3} \boldsymbol{F} \cdot \boldsymbol{n} dS = \iint_A (R(u,v,g(u,v)) - R(u,v,f(u,v)))dudv \\ &= \iiint_V \operatorname{div} \boldsymbol{F} dxdydz\end{aligned}$$

if $R = 0$.

By assuming $Q = R = 0$ and using yz-projectability, or $R = P = 0$ and using zx-projectability, we also have $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS = \iiint_V \operatorname{div} \boldsymbol{F} dxdydz$ in these cases. The general case is $\boldsymbol{F} = (P,0,0) + (0,Q,0) + (0,0,R)$ and the statement of the theorem is linear in $\boldsymbol{F}$, so adding these cases completes the proof if V is xy-, yz-, zx-projectable.

If V is a disjoint union, the volume integral is a sum and the surface integral is also a sum if we note that the integral on the intersection cancel. □

Example 80. $\boldsymbol{F}(x,y,z) = (x,y,z), V = \{(x,y,z) : x^2 + y^2 + z^2 \leq a^2\}$.

- Compute $\iiint_V \operatorname{div} \boldsymbol{F} dxdydz$. $\operatorname{div} \boldsymbol{F} = 1+1+1 = 3$, $\iiint_V \operatorname{div} \boldsymbol{F} dxdydz = 3\iiint_V dxdydz = 4\pi a^3$.

- Compute $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$. We can take $x = a\cos u \sin v, y = a \sin u \sin v, z = a\cos v$, then $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = -a \sin \boldsymbol{r}$ and this is incoming, therefore, by noting that $\boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{r}$,

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS = -\iint_S (-a\sin v)\boldsymbol{r} \cdot \boldsymbol{r} dS = a^3 \iint_0^{2\pi} \iint_0^{\pi} \sin v dudv = 4\pi a^3.$$

An interpretation of divergence

Theorem 81. *Let $V(t)$ be the ball of radius t centered at $\boldsymbol{a}$ and $S(t)$ be its boundary with outgoing unit vector $\boldsymbol{n}$, $\boldsymbol{F}$ be a continuously differentiable vector field. Then* $\operatorname{div} \boldsymbol{F} = \lim_{t\to 0} \frac{1}{v(V(t))} \iint_{S(t)} \boldsymbol{F} \cdot \boldsymbol{n} dS$.

Proof. By Gauss' theorem, $\iint_{S(t)} \boldsymbol{F} \cdot \boldsymbol{n} = \iint \text{div } \boldsymbol{F} dxdydz$. As $\boldsymbol{F}$ is continuously differentiable, $|\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{a})| \le E(\boldsymbol{x}, \boldsymbol{a})$ and $E(\boldsymbol{x}, \boldsymbol{a}) \to 0$ as $\boldsymbol{x} \to \boldsymbol{a}$. Therefore,

$$\begin{aligned}\frac{1}{v(V(t))} \iint_{S(t)} \boldsymbol{F} \cdot \boldsymbol{n} dS &= \frac{1}{v(V(t))} \iint_{V(t)} \text{div } \boldsymbol{F} dxdydz \\ &= \text{div } \boldsymbol{F}(\boldsymbol{a}) + \frac{1}{v(V(t))} \iint (\text{div } \boldsymbol{F} - \boldsymbol{F}(\boldsymbol{a})) dxdydz \to \text{div } \boldsymbol{F}(\boldsymbol{a}).\end{aligned}$$

□

This theorem shows that div $\boldsymbol{F}$ represents how much $\boldsymbol{F}$ is "diverging out" at a given point.

Maxwell's equations

In physics, Gauss' law of electric charge says that $\iint_S \boldsymbol{E} \cdot \boldsymbol{n} dS = \iiint_V \rho dxdydz$, where $\boldsymbol{E}$ is the electric field, ρ is the electric charge density, V is a certain volume and S is its surface. By Gauss' law, this is equivalent to $\iiint_V \text{div } \boldsymbol{E} dxdydz = \iiint_V \rho dxdydz$, and as this holds for any volume V, we have div $\boldsymbol{E} = \rho$.

Similarly, Faraday's law says that $\int_C \boldsymbol{E} \cdot \boldsymbol{\alpha} = -\iint_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{n} dS$, where $\boldsymbol{B}$ is the magnetic field, t is time, S is a simply connected surface and C is its boundary. By Stokes' theorem, this is equivalent to $\iint_S \text{curl } \boldsymbol{E} \cdot \boldsymbol{n} dS = -\iint_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{n} dS$, and as this holds for any surface S, we have curl $\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$.

Together with other equations, with the current density $\boldsymbol{j}$ (and setting physical constants to 1),

$$\text{div } \boldsymbol{E} = \rho, \quad \text{div } \boldsymbol{B} = 0, \quad \text{curl } \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \text{curl } \boldsymbol{B} = \boldsymbol{j} + \frac{\partial \boldsymbol{E}}{\partial t},$$

are called Maxwell's equations and are fundamental in electrodynamics.

Jan. 7. Review of series of functions

Criteria of convergence

Let $\{a_n\}$ be a sequence of numbers. We say that it is **convergent to** a if for any $\epsilon > 0$ there is N such that for $n > N$ $|a_n - a| < \epsilon$. A **series** is a sequence $\{\sum_{k=0}^n a_k\}$ (or k may start with any other integer). If a series is convergent, we denote its **limit** by $\sum_{k=0}^\infty a_k$.
Some criteria:

- (Ratio test) Let $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to L$. If $L < 1$, $\sum_{n=0}^\infty a_n$ converges. If $L > 1$, $\sum_{n=0}^\infty a_n$ diverges.
- (Root test) Let $a_n > 0$ and $(a_n)^{\frac{1}{n}} \to R$. If $R < 1$, $\sum_{n=0}^\infty a_n$ converges. If $R > 1$, $\sum_{n=0}^\infty a_n$ diverges.
- (Comparison test) Let $a_n, b_n > 0, c > 0$ such that $a_n < cb_n$. If $\sum_{n=0}^\infty b_n$ converges, so does $\sum_{n=0}^\infty a_n$.

Pointwise and uniform convergences of series of functions

Let $\{f_n(x)\}$ be a sequence of functions on $S \subset \mathbb{R}$, namely, for any $x \in S$, $\{f_n(x)\}$ is a sequence of numbers.

Definition 82. We say that $\{f_n(x)\}$ is **pointwise convergent** if for each x, $\{f_n(x)\}$ is convergent to $f(x)$: namely, for each x and $\epsilon > 0$ there is N such that $|f_n(x) - f(x)| < \epsilon$. We say that $\{f_n(x)\}$ is **uniformly convergent** if for each $\epsilon > 0$ there is N such that for each x $|f_n(x) - f(x)| < \epsilon$.

- (Weierstrass' M-test) Let $\{\sum_{k=0}^{n} f_n(x)\}$ be a series of functions. If there is $a_n > 0$, such that $\sum_{k=0}^{\infty} a_n$ is convergent and $|f_n(x)| \le a_n$, then $\{\sum_{k=0}^{n} f_n(x)\}$ is uniformly convergent.

Theorem 83. *If $\{f_n\}$ is a sequence of continuous functions, uniformly convergent to f, then f is continuous and $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.*

A series of functions $\{f_n(x)\}$ is the sequence $\{\sum_{k=0}^{n} f_k(x)\}$.

Exercises

- Show that $\sum_{k=1}^{n} \frac{\sin x}{e^{k^2(x^2+1)}}$ is uniformly convergent for $x \in \mathbb{R}$.

  Solution. Note that $\left|\frac{\sin x}{e^{k^2(x^2+1)}}\right| < \frac{1}{e^{k^2}}$ and $\left(\frac{1}{e^{k^2}}\right)^{\frac{1}{k}} = \frac{1}{e^k} \to 0$. By root test, $\frac{1}{e^{k^2}}$ is convergent, hence by M-test, $\sum_{k=1}^{\infty} \frac{\sin x}{e^{k^2(x^2+1)}}$ is convergent.

- Study the uniform/pointwise convergence of $\sum_{n=1}^{\infty} \frac{x^n}{1-x^n}$ for $x \in [0,1)$.

  Solution. Let $R \in [0,1)$. Then for $x \in [0,R]$, $\frac{x^n}{1-x^n} \le \frac{R^n}{1-R}$. The latter is a geometric series with $|R| < 1$, hence is convergent. By M-test, $\sum_{n=1}^{\infty} \frac{x^n}{1-x^n}$ is uniformly convergent in $x \in [0,R]$. On the other hand, for a given n and for each $1 > \epsilon > 0$, there is x which is sufficiently close to 1 such that $\frac{x^n}{1-x^n} > \epsilon$: because we may assume that $x^n > \epsilon$ and then we can take $\frac{x^n}{1-x^n} > x^n > \epsilon$. Therefore, the series is not uniformly convergent in $[0,1)$.

- Study the conditional, absolute and uniform convergence of the series $\sum_{n=0}^{\infty} \frac{1}{3^n}(\sqrt{n+1} - \sqrt{n})(2x^2-5)^n$.
  Hint. $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, hence $\frac{1}{2\sqrt{n+1}} \le \sqrt{n+1} - \sqrt{n} \le \frac{1}{2\sqrt{n}}$.

- Show that $\sum_{k=1}^{n} \frac{x}{e^{k^2(x^2+1)}}$ is uniformly convergent for $-R \le x \le R$ for any $R > 0$.

- Show that $\sum_{k=1}^{n} \frac{1}{\sqrt{x^2+k^4}}$ is uniformly convergent in $\mathbb{R}$.

- Show that $\sum_{k=1}^{n} \frac{1}{\cosh kx}$ is uniformly convergent for $(-\infty,-a] \cup [a,\infty)$ for any $a > 0$, but it is not uniform in $(-\infty,0) \cup (0,\infty)$.

- Show that $\sum_{k=1}^{n} \frac{2^n}{x^n}$ is uniformly convergent for $-R \le x \le R$ for any $0 < R < \frac{1}{2}$, but uniform in $(-\frac{1}{2},\frac{1}{2})$.

Power series, applications to differential equations

A particular example is a **power series**, where $\{a_n\}$ is a sequence of numbers and one considers $\{\sum_{n=0}^{\infty} a_n(x-a)^n\}$.

Theorem 84. *For a power series $\{\sum_n a_n(x-a)^n\}$, there is $R \ge 0$ or $R = \infty$ (called the radius of convergence) such that for $r < R$, $\sum_{n=0}^{\infty} a_n(x-a)^n$ is uniformly convergent and absolutely convergent for $|x-a| < r$ and divergent for $|x-a| > R$.*

Furthermore, if $x \in (a-R, a+R)$, then with $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, it holds that $\int_a^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$ and $f'(x) = \sum_{n=1} na_n(x-a)^{n-1}$. In particular, $f^{(k)}(a) = k!a_k$.

Exercises

Determine the radius of convergence (and compute the sum if possible).

- $\sum_{n=0}^{\infty}(n+1)x^{2n}$.

  Solution. We put $y = x^2$. We have $\frac{(n+2)y^{n+1}}{(n+1)y^n} = \frac{(n+2)y}{n+1} \to x$ as $n \to \infty$, hence by ratio test, the series is convergent if $|y| < 1$ and divergent if $|y| > 1$, or equivalently, convergent if $|x| < 1$ and divergent if $|x| > 1$, Namely, the radius of convergence is 1. To compute the sum, note that with $f(y) = \sum_{n=0}^{\infty}(n+1)y^n$, we have $F(y) = \sum_{n=0}^{\infty} y^{n+1} = \frac{1}{1-y} - 1$, where $F(y)$ is a primitive function of $f(y)$. Therefore, $f(y) = \frac{d}{dy}(\frac{1}{1-y} - 1) = \frac{1}{(1-y)^2}$ and $\sum_{n=0}^{\infty}(n+1)x^{2n} = \frac{1}{1-x^2}$.

- $\sum_{n=0}^{\infty} \frac{x^n}{(n+3)!}$.

  Solution. We put $a_n = \frac{x^n}{(n+3)!}$. By ratio test, $\frac{a_{n+1}}{a_n} = \frac{x}{n+4} \to 0$ for any $x \in \mathbb{R}$, hence the radius of convergence is ∞. Put $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n+3)!}$, then

  $$x^3 f(x) = \sum_{n=0}^{\infty} \frac{x^{n+3}}{(n+3)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} - \left(1 + x + \frac{x^2}{2}\right) = e^x - 1 - x - \frac{x^2}{2},$$

  therefore, $f(x) = \frac{1}{x^3}\left(e^x - 1 - x - \frac{x^2}{2}\right)$.

- (without computing the sum) $\sum_{n=1}^{\infty} \frac{1}{n}\left(1 + \frac{1}{n}\right)^{n^2} x^n$.

  Hint. Use root test.

- $\sum_{n=0}^{\infty} \frac{3^n x^n}{n}$
- $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$
- $\sum_{n=0}^{\infty}(3^n + 2^n)x^n$
- $\sum_{n=0}^{\infty} \frac{x^n}{2^{n^2}}$

Theorem 85. *In the radius of convergence, if $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$, then $a_n = b_n$ for all n.*

We can use this to solve differential equations by assuming that the solution can be represented by a power series: $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and by comparing the coefficients.

Exercises

Solve the following differential equations and determine the radius of convergence.

- $f'(x) = 2xf(x)$.

  Solution. Put $f(x) = \sum_{n=0}^{\infty} a_n x^n$. As $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, a_n must satisfy

  $$\begin{aligned}\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \\ &= 2x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2a_n x^{n+1} = \sum_{n=1}^{\infty} 2a_{n-1}x^n,\end{aligned}$$

  therefore, $a_1 = 0$, $(n+1)a_{n+1} = 2a_{n-1}$, or $a_{2n} = \frac{a_{2(n-1)}}{n}$, hence $a_{2n} = \frac{a_0}{n!}$ (and one can also show that $f(x) = a_0 e^{x^2}$).

- $f'(x) = f(x) + 1$.

  Solution. Put $f(x) = \sum_{n=0}^{\infty} a_n x^n$. As $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, a_n must satisfy

$$\begin{aligned}\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\ &= \sum_{n=0}^{\infty} a_n x^n + 1 = a_0 + 1 + \sum_{n=1}^{\infty} a_n x^n\end{aligned}$$

  therefore, $a_0 + 1 = a_1$, $(n+1)a_{n+1} = a_n$ for $n \geq 1$, or $a_n = \frac{a_0+1}{n!}$ (and one can also show that $f(x) = (a_0+1)e^x - 1$).

- $f(x) = f'(x)$. (**Result**: $a_n = n! a_0$)

- $xf(x)' = 2f(x)$. (**Result**: $a_n = 0$ except $n = 2$)

- $xf''(x) + (1+x)f(x)' + 2f(x) = 0$.

  Hint. Use $f''(x) = \sum_{n=2} n(n-1) a_n x^{n-2}$.

- $f''(x) = -f(x)$. **Note**. The solutions are parametrized by two numbers, say a_0, a_1.

Jan. 9. Review of scalar field and partial derivatives

Taylor series

Let f be a function on $(a-r, a+r)$, infinitely many times differentiable, and denote the n-th derivative by $f^{(n)}(x)$.

Theorem 86. *Assume that* $|f^{(n)}(x)| \leq \frac{Cn!}{r^n}$ *for some* $C > 0$. *Then it holds that* $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Proof. Let $E_n(x)$ be the error term in the n-th order Taylor formula, namely, $E_n(x) = f(x) - \sum_{k=0}^{k} \frac{f^{(k)}(a)}{k!}(x-a)^k = \frac{f^{(n+1)}(y)}{(n+1)!}(x-a)^{n+1}$, where y is between x and a. From the assumption, it follows that $|E_n(x)| \leq \frac{C(n+1)!}{r^{n+1}(n+1)!}|x-a|^{n+1} = \frac{C|x-a|^{n+1}}{r^{n+1}} \to 0$ as $n \to \infty$. □

Example 87. (Taylor expansion)

- $\frac{d^n}{dx^n}(e^x) = e^x$. $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$
- $\frac{d^{2n}}{dx^{2n}}(\sin x) = (-1)^n \sin x$, $\frac{d^{2n+1}}{dx^{2n+1}}(\sin x) = (-1)^n \cos x$. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots$
- $\frac{d^{2n}}{dx^{2n}}(\cos x) = (-1)^n \cos x$, $\frac{d^{2n+1}}{dx^{2n+1}}(\cos x) = (-1)^n \sin x$. $\sin x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots$
- $\frac{d^n}{dx^n}(\log(x+1)) = \frac{(n-1)!(-1)^{n-1}}{(x+1)^n}$, hence $|\frac{d^n}{dx^n}(\log(x+1))| \leq \frac{n!}{2^n}$ for $x \in (-\frac{1}{2}, \frac{1}{2})$. $\log(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$

Exercises

Find the Taylor expansion of the following functions around $x = 0$.

- $f(x) = \cosh x (= \frac{e^x + e^{-x}}{2})$.

  Solution. $f^{(n)}(x) = \frac{e^x + (-1)^n e^{-x}}{2}$, $f^{(2n)}(0) = 1$, $f^{(2n+1)}(0) = 0. f(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$.

- $f(x) = \cosh x^2$.

  Solution. With $y = x^2$, $\cosh x^2 = \cosh y = \sum_{n=0}^{\infty} \frac{1}{(2n)!} y^{2n} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{4n}$.

- $f(x) = \log\sqrt{\frac{1+x}{1-x}}$.

  Solution. Note that $f(x) = \frac{1}{2}(\log(1+x) - \log(1-x))$, and $\frac{d^n}{dx^n}(\log(x+1)) = \frac{(n-1)!(-1)^{n-1}}{(x+1)^n}$, $\log(x+1) = \sum_{n=1} \frac{(-1)^{n-1}}{n} x^n$, $\log(-x+1) = \sum_{n=1} -\frac{1}{n} x^n$, therefore, by summing them, $f(x) = \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1}$.

- $f(x) = \frac{2x}{2x^2-3x+1}$.

  Hint. $f(x) = \frac{-2}{2x-1} + \frac{2}{x-1}$.

- $f(x) = \frac{1}{x-3}$.

- $f(x) = x\sin x$. (use the expansion of $\sin x$ and multiply it with x).

Definition and computation of partial derivatives

Let $f(x,y)$ be a function of two varibles x, y (some times denoted by $f(\boldsymbol{x})$ with $\boldsymbol{x} = (x,y)$). We call it a **scalar field**.

Partial derivatives of f are defined by

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \qquad \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Practically, they can be computed by treating "the other variables as constants". Similar definitions are given for functions with more variables (x,y,z) or $(x_1, x_2, x_3, \cdots, x_n)$.

If the partial derivatives are still differentiable, we also introduce the higher partial derivatives

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^3 f}{\partial x^3}, \cdots$$

If $\frac{\partial^2 f}{\partial x \partial y}$ is continuous, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Exercises

Compute the partial derivatives $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$.

- $f(x,y) = xy^2$.

  Solution. $\frac{\partial f}{\partial x} = y^2$, $\frac{\partial f}{\partial y} = 2xy$, $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial x \partial y} = 2y = \frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial y^2} = 2$.

- $f(x,y) = \sin(xe^y)$.

  Solution. $\frac{\partial f}{\partial x} = e^y \cos(xe^y)$, $\frac{\partial f}{\partial y} = xe^y \cos(xe^y)$, $\frac{\partial^2 f}{\partial x^2} = -e^{2y} \sin(xe^y)$, $\frac{\partial^2 f}{\partial x \partial y} = e^y \cos(xe^y) - xe^y \sin(xe^y) = \frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial y^2} = xe^y \cos(xe^y) - xe^{2y} \sin(xe^y)$.

Chain rule

Let $f(x,y)$ be a scalar field, $X(t), Y(t)$ be functions on $\mathbb{R}$. The composition $F(t) = f(X(t), Y(t))$ is a function of t. One can consider the pair $(X(t), Y(t))$ as a curve $\boldsymbol{\alpha}(t)$ in $\mathbb{R}^2$, and $F(t) = f(\boldsymbol{\alpha}(t))$. Recall that the vector field $\nabla f(x,y) = (\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y))$ is called the **gradient** of f.

Chain rule: $\frac{d}{dt}F(t) = \frac{d}{dt} f(\boldsymbol{\alpha}(t)) = \nabla f(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)$.

The same formula holds for scalar fields on $\mathbb{R}^n$.

Exercises

- Compute $\frac{d}{dt}f(\boldsymbol{\alpha}(t))$, where $f(x,y)=x^2+y^2, \boldsymbol{\alpha}(t)=(t,t^2)$.

  Solution 1. $\frac{d}{dt}f(\boldsymbol{\alpha}(t))=\frac{d}{dt}(t^2+(t^2)^2)=\frac{d}{dt}(t^2+t^4)=2t+4t^3$.

  Solution 2. $\nabla f(x,y)=(2x,2y), \nabla f(\boldsymbol{\alpha}(t))=(2t,2t^2), \boldsymbol{\alpha}'(t)=(1,2t), \nabla f(\boldsymbol{\alpha})\cdot\boldsymbol{\alpha}'(t)=2t+4t^3$.

- Compute $\frac{dF}{dt}$, where $F(t)=t^t$.

  Solution. Take $f(x,y)=x^y, \boldsymbol{\alpha}(t)=(t,t)$, then $f(\boldsymbol{\alpha})=t^t=F(t)$. Therefore, $\nabla f(x,y)=(yx^{y-1}, \log x x^y)$, $\nabla f(\boldsymbol{\alpha}(t))=(t^t,\log t\cdot t^t), \boldsymbol{\alpha}'(t)=(1,1), \nabla f(\boldsymbol{\alpha})\cdot\boldsymbol{\alpha}'(t)=t^t+\log t\cdot t^t$.

- Compute $\frac{dF}{dt}$, where $F(t)=\int_0^{t^2} e^{st}ds$.

  Solution 1. We have $F(t)=[\frac{e^{st}}{t}]_0^{t^2}=\frac{e^{t^3}}{t}-\frac{1}{t}$ and take its derivative directly.

  Solution 2. Take $f(x,y)=\int_0^x e^{ys}ds, \boldsymbol{\alpha}(t)=(t^2,t)$, then $f(\boldsymbol{\alpha}(t))=F(t)$. $\nabla f(x,y)=(e^{yx},\int_0^x se^{ys}ds), \nabla f(\boldsymbol{\alpha}(t))=(e^{t^3},\int_0^{t^2} se^{ts}ds)=(e^{t^3},te^{t^3}-\frac{1}{t^2}e^{t^3}+\frac{1}{t^2})$ because $\int se^{ts}ds=\frac{s}{t}e^{ts}-\frac{1}{t^2}e^{ts}$, $\boldsymbol{\alpha}'(t)=(2t,1), \nabla f(\boldsymbol{\alpha})\cdot\boldsymbol{\alpha}'(t)=2te^{t^3}+te^{t^3}-\frac{1}{t^2}e^{t^3}+\frac{1}{t^2}=3te^{t^3}-\frac{1}{t^2}e^{t^3}+\frac{1}{t^2}$.

- Compute $\frac{dF}{dt}$, where $F(t)=t^{t^t}$.

  Hint. Take $f(x,y,z)=x^{y^z}, \boldsymbol{\alpha}(t)=(t,t,t)$.

PDE with constant coefficients, wave equation

A **partial differential equation** (PDE) is an equaiton about a scalar (vector) field which involves partial derivatives.

We learned two types of PDEs:

- Linear PDE with constant coefficients for $f(x,y)$: $a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial x}=0$. A general solution is $f(x,y)=g(bx-ay)$, where g is continuously differentiable.

- One-dimensional wave equation for $f(x,t)$: $\frac{\partial^2 f}{\partial t^2}=c^2\frac{\partial^2 f}{\partial x^2}$. A general solution is

$$f(x,t)=\frac{F(x+ct)+F(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}G(s)ds,$$

  where F is twice continuously differentiable and G is continuously differentiable. Furthermore, it holds that $f(x,0)=F(x)$ and $\frac{\partial f}{\partial t}(x,0)=G(x)$.

Exercises

- Find the solution $f(x,y)$ of the partial differential equation

$$3\frac{\partial f}{\partial x}+5\frac{\partial f}{\partial y}=0$$

  with the initial condition $f(x,0)=\cos(x^2)$.

  Solution. $f(x,y)=g(5x-3y)$. With the given initial condition, it should hold that $f(x,0)=g(5x)=\cos(x^2)$, namely, $g(t)=\cos\dfrac{t^2}{25}$. Again by the general formula, $f(x,y)=g(5x-3y)=\cos\dfrac{(5x-3y)^2}{25}$.

- Let $c > 0$. Find the solution $f(x,t)$ of the partial differential equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

with the initial condition $f(x,0) = e^{-x^2}, \frac{\partial f}{\partial t}(x,0) = \frac{x}{(x^2+1)^2}$.

Solution. $f(x,t) = \frac{1}{2}\left(F(x-ct) + F(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds$.

We are given the initial conditions $f(x,0) = e^{-x^2}, \frac{\partial f}{\partial t}(x,0) = \frac{x}{(x^2+1)^2}$, hence we can take $F(s) = e^{-s^2}, G(s) = \frac{s}{(s^2+1)^2}$. Note that $\int G(s)ds = -\frac{1}{2(s^2+1)} + \text{Const}$. Altogether, we have

$$f(x,t) = \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right) + \frac{1}{4c}\left(\frac{1}{(x-ct)^2+1} - \frac{1}{(x+ct)^2+1}\right).$$

- Solve $5\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} = 0$ with $f(0,0) = 0, \frac{\partial f}{\partial x}(x,0) = e^x$. **Result**: $f(x,y) = e^{\frac{2x+5y}{2}}$.

- Solve $\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2}$ with $f(x,0) = 0, \frac{\partial f}{\partial t}(x,0) = x^3 e^{-x^2}$. **Result**:
$f(x,y) = \frac{1}{2c}\left(\frac{1}{2}(x-ct)^2 e^{-(x-ct)^2} + \frac{1}{2}e^{-(x-ct)^2} - \frac{1}{2}(x+ct)^2 e^{-(x+ct)^2} - \frac{1}{2}e^{-(x+ct)^2}\right)$

Jan. 11. Review of extremal values

Minima, maxima and suddle points

Let $f(\boldsymbol{x})$ be a scalar field on an open region $S \subset \mathbb{R}^n$. We say that $\boldsymbol{a} \in S$ is a **relative minumum** (respectively maximum) if there is a ball $B(\boldsymbol{a}, r) \subset S$ such that $f(\boldsymbol{a}) \le f(\boldsymbol{a})$ (respectively $f(\boldsymbol{a}) \ge f(\boldsymbol{x})$) for all $\boldsymbol{x} \in B(\boldsymbol{a}, r)$.

Let $f(\boldsymbol{x})$ be differentiable. If $\nabla f(\boldsymbol{a}) = \mathbf{0}$, $\boldsymbol{a}$ is said to be a **stationary point**. A stationary point which is neither a relative minimum nor a relative maximum is a **saddle**. The **Hessian matrix** of f is

$$H(\boldsymbol{x}) = \begin{pmatrix} D_{11}f(\boldsymbol{x}) & D_{12}f(\boldsymbol{x}) & \cdots & D_{1n}f(\boldsymbol{x}) \\ \vdots & & & \vdots \\ D_{n1}f(\boldsymbol{x}) & D_{n2}f(\boldsymbol{x}) & \cdots & D_{nn}f(\boldsymbol{x}) \end{pmatrix}$$

Theorem 88. *Let f be a differentiable scalar field with continuous second derivatives. Let $\boldsymbol{a}$ be a stationary point.*

- *If all eigenvalues of $H(\boldsymbol{a})$ are positive, then $\boldsymbol{a}$ is a relative minumum.*
- *If all eigenvalues of $H(\boldsymbol{a})$ are negative, then $\boldsymbol{a}$ is a relative maximum.*
- *If $H(\boldsymbol{a})$ has both positive and negative eigenvalues of, then $\boldsymbol{a}$ is a saddle.*

In $\mathbb{R}^2$, it is easy to determine the signs of a symmetric matrix: for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $\det A = ad - bc < 0$, it has both positive and negative eigenvalues. If $ad - bc > 0$ and $\operatorname{tr} A = a + d > 0$ (respectively < 0, then both of the eigenvalues are positive (respectively negative).

Exercises

Find all stationary points and classify them.

- $f(x, y) = x^3 + y^3 - 3xy$.

  Solution. $\nabla f(x, y) = (3x^2 - 3y, 3y^2 - 3x)$. $\nabla f(x, y) = (0, 0) \Leftrightarrow 3x^2 - 3y = 3y^2 - 3x = 0 \Rightarrow x^4 - x = 0, x^2 = y \Rightarrow (x, y) = (0, 0), (1, 1)$. $H(x, y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$. $H(0, 0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$, $\det H(0, 0) = -9 < 0$, hence $(0, 0)$ is a saddle. $H(1, 1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$, $\det H(1, 1) = 27 > 0$ and $\operatorname{tr} H(1, 1) = 12 > 0$, hence $(1, 1)$ is a relative minimum.

- $f(x, y) = e^y(x^2 - 2xy + 3)$.

  Solution. $\nabla f(x, y) = (e^y(2x - 2y), e^y(x^2 - 2xy + 3 - 2x))$. $\nabla f(x, y) = (0, 0) \Rightarrow e^y(2x - 2y) = e^y(x^2 - 2xy + 3 - 2x) = 0 \Rightarrow x = y, x^2 - 2x^2 + 3 - x = 0 \Rightarrow (x, y) = (1, 1), (-3, -3)$.

  $$H(x, y) = \begin{pmatrix} e^y \cdot 2 & e^y(2x - 2y - 2) \\ e^y(2x - 2y - 2) & e^y(x^2 - 2xy + 3 - 4x) \end{pmatrix}.$$

  $H(1, 1) = \begin{pmatrix} 2e & -2e \\ -2e & -2e \end{pmatrix}$ and $\det H(1, 1) = -8e^2 < 0$, hence $(1, 1)$ is a saddle.

  $H(-3, -3) = \begin{pmatrix} 2e^{-3} & -2e^{-3} \\ -2e^{-3} & 6e^{-3} \end{pmatrix}$ and $\det H(-3, -3) = 8e^{-6} > 0, \operatorname{tr} H(-3, -3) = 8e^{-6} > 0$, hence $(-3, -3)$ is a relative minimum.

- $f(x, y) = e^{2x+3y}(8x^2 - 6xy + 3y^2)$.
- $f(x, y) = e^{x+y}(x^2 + xy)$.

Lagrange's multiplier method

If a scalar field $f(\boldsymbol{x})$ is restricted to the subset of $\mathbb{R}^n$ defined by $g_1(\boldsymbol{x}) = \cdots = g_m(\boldsymbol{x}) = 0$, we cannot use the condition $\nabla f(\boldsymbol{x}) = \boldsymbol{0}$ in order to find extremal values of f on that subset. Instead, if $\nabla g_k(\boldsymbol{x})$ are independent, then we can use **Lagrange's method**: if f takes an extremal value at $\boldsymbol{a}$, then there are $\lambda_1, \cdots, \lambda_m \in \mathbb{R}$ such that

$$\lambda f(\boldsymbol{a}) = \lambda_1 \nabla g_1(\boldsymbol{a}) + \cdots + \lambda_m \nabla g_m(\boldsymbol{a}).$$

We have $(m + n)$ equations for $(m + n)$ variables $\boldsymbol{a} = (a_1, \cdots, a_n), \lambda_1, \cdots, \lambda_m$.

Exercises

- Find extremal values of $f(x, y, z) = x - 2y + 2z$ on the sphere $x^2 + y^2 + z^2 = 1$.

  Solution. We must find extremal point $\boldsymbol{x}$ of f under the constraint $g(x, y, z) = x^2+y^2+z^2-1 = 0$. By Lagrange's method, there is λ such that $\nabla f(\boldsymbol{x}) = \lambda \nabla g(\boldsymbol{x})$. $\nabla f(\boldsymbol{x}) = (1, -2, 2)$ and $\nabla g(\boldsymbol{x}) = (2x, 2y, 2z)$. We may assume that $\lambda \neq= 0$ and we need to solve $(1, -2, 2) = \lambda(2x, 2y, 2z), x^2 + y^2 + z^2 - 1 = 0$. From the first equation, $(x, y, z) = (\frac{1}{2\lambda}, -\frac{1}{\lambda}, \frac{1}{\lambda})$, and from the second equation, $\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = 1$, namely, $\lambda = \pm\frac{3}{2}$.

  With $\lambda = \frac{3}{2}$, $(x, y, z) = (\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}), f(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}) = \frac{1}{3}$ (maximum), and $\lambda = -\frac{3}{2}$, $(x, y, z) = (-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}), f(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}) = -\frac{1}{3}$ (maximum).

- Find the points on the curve of the intersection of two surfaces $x^2 - xy + y^2 - z^2 = 1$ and $x + y = 0$ which are nearest to the origin.

  Solution. We need to minimize function $f(x, y, z) = x^2 + y^2 + z^2$ under the conditions $g_1(x, y, z) = x^2 - xy + y^2 - z^2 - 1$ and $g_2(x, y, z) = x + y - 1$. By Lagrange's method, there are λ_1, λ_2 such that $\nabla f(x, y, z) = \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z)$. We have $\nabla f(x, y, z) = (2x, 2y, 2z), \nabla g_1(x, y, z) = (2x - y, -x + 2y, -2z), \nabla g_2(x, y, z) = (1, 1, 0)$.

Therefore, $(2x, 2y, 2z) = (\lambda_1(2x - y) + \lambda_2, \lambda_1(-x + 2y) + \lambda_2, \lambda_1(-2z))$. From $2z = -2\lambda_1 z$, there are two possibilities:

- $z = 0$. As $x+y = 0$, $y = -x$ and $x^2 - x(-x) + (-x)^2 - 0^2 = 1$, hence $3x^2 = 1$. $(x, y, z) = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0)$ and $f(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0) = \frac{2}{3}$, $(x, y, z) = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0)$ and $f(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0) = \frac{2}{3}$.
- $z \neq 0$ and $\lambda_1 = -1$. Then, again with $y = -x$, $2x = -3x$ hence $x = 0$, but then $z^2 = -1$ which is impossible, so there is no solution in this case.

So the minumum is $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0)$ and $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0)$

- Find the nearest point from the origin of the surface $z^2 - xy = 1$.
- Find the extremal values of the function $f(x, y) = 3x - 4y$ on the curve C defined by $3x^2 + 2y^2 = 1$.

Implicit functions

If $F(x, y, z)$ is a function, the equation $F(x, y, z) = 0$ **may** define a function $f(x, y)$ such that $F(x, y, f(x, y)) = 0$. Then,

$$\frac{\partial f}{\partial x}(x, y) = -\frac{\frac{\partial F}{\partial x}(x, y, f(x, y))}{\frac{\partial F}{\partial z}(x, y, f(x, y))}, \qquad \frac{\partial f}{\partial y}(x, y) = -\frac{\frac{\partial F}{\partial y}(x, y, f(x, y))}{\frac{\partial F}{\partial z}(x, y, f(x, y))}.$$

Exercises

- Compute the partial derivatives of $g(x, y) = x - 2y + 2f(x, y)$, at $(\frac{1}{3}, -\frac{2}{3})$ where $f(x, y)$ is defined by $F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$.

  Solution. By the formula above, with $\frac{\partial F}{\partial x} = 2x$, $\frac{\partial F}{\partial y} = 2y$, $\frac{\partial F}{\partial z} = 2z$, we have $\frac{\partial f}{\partial x}(x, y) = -\frac{2x}{2f(x,y)} = -\frac{x}{f(x,y)}$. At $(\frac{1}{3}, -\frac{2}{3})$ we have $f(\frac{1}{3}, -\frac{2}{3}) = \frac{2}{3}$ and hence $\frac{\partial f}{\partial x}(\frac{1}{3}, -\frac{2}{3}) = -\frac{1}{2}$. Therefore, $\frac{\partial g}{\partial x}(\frac{1}{3}, -\frac{2}{3}) = 1 + 2(-\frac{1}{2}) = 0$.

  Similarly, $\frac{\partial f}{\partial y}(x, y) = -\frac{2y}{2f(x,y)} = -\frac{y}{f(x,y)}$, and at $(\frac{1}{3}, -\frac{2}{3})$, $\frac{\partial f}{\partial y}(\frac{1}{3}, -\frac{2}{3}) = -1$ Therefore, $\frac{\partial g}{\partial y}(\frac{1}{3}, -\frac{2}{3}) = -2 + 2(-1) = 0$.

- Compute the partial derivatives of the function $x^2 + y^2 + f(x, y)^2$ at $(x, y) = (1, 1)$, where $f(x, y)$ defined by $F(x, y, z) = z^2 - xy - 1 = 0, z = f(x, y)$.

Jan. 14. Review of line integrals

Definition and computations

Let $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \cdots, f_n(\boldsymbol{x}))$ be a continuous vector field on an open region $S \subset \mathbb{R}^n$, and $\boldsymbol{\alpha}(t) = (\alpha_1(t), \cdots, \alpha_n(t))$ be a continuously differentiable parametrized curve C in $\mathbb{R}^n$ on $[a, \tilde{a}]$. The **line integral** of $\boldsymbol{f}$ along $\boldsymbol{\alpha}$ is defined by

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} := \int_a^{\tilde{a}} \boldsymbol{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) dt,$$

where $\boldsymbol{\alpha}'(t) = (\alpha_1'(t), \cdots, \alpha_n'(t))$. In other textbooks, this is also denoted by $\int_C f_1 dx_1 + \cdots f_n dx_n$. The line integral depends on $\boldsymbol{f}$ and the curve C, but not on the parametrization $\boldsymbol{\alpha}$ (up to the direction).

Exercises

Compute the line integrals $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha}$.

- $\boldsymbol{f}(x,y,z) = (x, xz, -y)$, $\boldsymbol{\alpha}(t) = (t^2, t, t^3), t \in [0,1]$.

  Solution. $\boldsymbol{f}(\boldsymbol{\alpha}(t)) = (t^2, t^5, -t), \boldsymbol{\alpha}'(t) = (2t, 1, 3t^2)$. $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int_0^1 (2t^3 + t^5 - 3t^3)dt = [\frac{t^4}{2} + \frac{t^6}{6} - \frac{3t^4}{4}]_0^1 = \frac{1}{2} + \frac{1}{6} - \frac{3}{4} = -\frac{1}{12}$.

- $\boldsymbol{f}(x,y,z) = (2xy, x^2+z, y)$, $\boldsymbol{\alpha}(t)$ is the line segment from $(1,0,2)$ to $(3,4,1)$.

  Solution. We can take $\boldsymbol{\alpha}(t) = (2t+1, 4t, 2-t), \boldsymbol{\alpha}'(t) = (2,4,-1), t \in [0,1]$. We have $\boldsymbol{f}(\boldsymbol{\alpha}(t)) = (2(2t+1)\cdot 4t, (2t+1)^2 + 2 - t, 4t)$ and $\int \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \int_0^1 (32t^2 + 16t + 16t^2 + 16t + 4 + 8 - 4t + 8t - 4t)dt = \int_0^1 (48t^2 + 24t + 12)dt = [16t^3 + 12t^2 + 12t]_0^1 = 40$.

- $\boldsymbol{f}(x,y) = (x^2, -2xy)$, C is the parabola $y = x^2$, from $x = -1$ to $x = 1$.

- $\boldsymbol{f}(x,y) = (\frac{x+y}{x^2+y^2}, \frac{-(x-y)}{x^2+y^2})$, C is the circle $x^2 + y^2 = a^2$, going counterclockwise, starting at $(a,0)$.

We also defined the length of a curve, although it is not a line integral. If C is parametrized by $\boldsymbol{\alpha}(t)$ on $[a, \tilde{a}]$, then its length is

$$\int_a^{\tilde{a}} \|\boldsymbol{\alpha}'(t)\| dt.$$

Exercises

Compute the length of the curve.

- $\boldsymbol{\alpha}(t) = (t, \frac{1}{2}t^2), t \in [0,1]$.

  Solution. $\boldsymbol{\alpha}'(t) = (1,t)$, $\|\boldsymbol{\alpha}'(t)\| = \sqrt{1+t^2}$.
  $\int_0^1 \sqrt{1+t^2}dt = [\frac{1}{2}(t\sqrt{1+t^2} + \sinh^{-1} t)]_0^1 = \frac{1}{2} + s$ (by substitution $t = \sinh t'$ and $1 + \sinh^2 t' = \cosh t'$, where s is the positive solution of $\frac{e^s - e^{-s}}{2} = 1$, or with $x = e^s$ we have $x^2 - 2x - 1 = 0$, hence $x = 1 + \sqrt{2}$ and $s = \log(1+\sqrt{2})$.

- $\boldsymbol{\alpha}(t) = (\cos t, \sin t), t \in [0, \frac{\pi}{4}]$.

Potential and Green's theorem

Let $\boldsymbol{f}$ be a vector field, $\boldsymbol{\alpha}$ a parametrized curve on $[a, \tilde{a}]$.

Theorem 89. *If $\boldsymbol{f} = \nabla\varphi$ for some scalar field φ, then $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \varphi(\boldsymbol{\alpha}(\tilde{a})) - \varphi(\boldsymbol{\alpha}(a))$.*
If $\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha}$ depends only on the end points of C but not on $\boldsymbol{\alpha}$, then $\boldsymbol{f}$ is a gradient.

Theorem 90. *Let $\boldsymbol{f}(x,y)$ be a vector field with continuous second derivatives on a convex region $S \subset \mathbb{R}^n$. $\boldsymbol{f} = \nabla\varphi$ for some scalar field φ if and only if $\frac{\partial f_k}{\partial x_l} = \frac{\partial f_l}{\partial x_k}$.*

Let $\boldsymbol{f}$ be a vector field with continuous second derivatives on an open, **simply connected** region $S \subset \mathbb{R}^2$, with the boundary C parametrized by a curve $\boldsymbol{\alpha}$ going counterclockwise.

Theorem 91. *For any $\boldsymbol{f}(x,y) = (P(x,y), Q(x,y))$, it holds that*

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

(We will review double integrals in the next lecture)

If $\boldsymbol{f}$ is a gradient, the **potential** φ can be constructed by line integral. In $\mathbb{R}^2$, we can take

$$\varphi(x,y) = \int_0^x f_1(t,0)dt + \int_0^y f_2(x,t)dt = \int_0^y f_2(0,t)dt + \int_0^x f_1(t,x)dt.$$

In $\mathbb{R}^3$, we can take

$$\varphi(x,y,z) = \int_0^x f_1(t,0,0)dt + \int_0^y f_2(x,t,0)dt + \int_0^z f_3(x,y,t)dt.$$

Exercises

Determine whether the following vector field is a gradient, and if so, find a potential.

- $\boldsymbol{f}(x,y) = (x, 2y)$. (yes, $\varphi(x,y) = \frac{x^2+2y^2}{2}$)
- $\boldsymbol{f}(x,y,z) = (2xy^3, x^2z^2, 3x^2yz^2)$. (no)
- $\boldsymbol{f}(x,y) = (e^y\cos(xe^y), xe^y\cos(xe^y))$. (yes, $\varphi(x,y) = \sin(xe^y)$).

  Solution.

$$\begin{aligned}\varphi(x,y) &= \int_0^y f_2(0,t)dt + \int_0^x f_1(t,y)dt \\ &= 0 + [\sin(te^y)]_0^x \\ &= \sin(xe^y).\end{aligned}$$

- $\boldsymbol{f}(x,y) = (2xe^y + y, x^2e^y + x)$. (yes, $\varphi(x,y) = x^2e^y + xy$)
- $\boldsymbol{f}(x,y) = (y^2\cos x + z^3, -4 + 2y\sin x, 3xz^2 + 2)$. (yes, $\varphi(x,y,z) = y^2\sin x + xz^3 - 4y + 2z$)

Applications to PDE

Let $\boldsymbol{f}(x,y) = (P(x,y), Q(x,y))$ be a vector field and assume that $\boldsymbol{f} = \nabla\varphi$. The differential equation $P(x, f(x)) + Q(x, f(x))f'(x) = 0$ is said to be exact. Exact equations can be solved implicitly by $\varphi(x, f(x)) = C$ for some $C \in \mathbb{R}$.

Even if $P(x, f(x)) + Q(x, f(x))f'(x) = 0$ is not exact, one might find $\mu(x,y)$ such that $\mu(x, f(x))P(x, f(x)) + \mu(x, f(x))Q(x, f(x))f'(x) = 0$ is exact.

Exercises

- $(x + 2f(x)) + (2x + f(x))f(x)' = 0$.

  Solution. Take $P(x,y) = x + 2y, Q(x,y) = 2x + y$. This is a gradient, indeed, one can take $\varphi(x,y) = \frac{x^2}{2} + 2xy + \frac{y^2}{2}$. So f should satisfy $\frac{x^2}{2} + 2xf(x) + \frac{f(x)^2}{2} = C$, or $\frac{f(x)}{2} = -2x \pm \sqrt{2C - 5x^2}$.

- $2x^2f(x) + x^3f'(x) = 0$.

  Solution. Dividing by x, we take $P(x,y) = 2xy, Q(x,y) = x^2$. This is a gradient, indeed, one can take $\varphi(x,y) = x^2y$. So f should satisfy $x^2f(x) = C$, or $f(x) = \frac{C}{x^2}$.

- $f(x) - (2x + f(x))f'(x) = 0$.

  Solution. Dividing by y^3, we take $P(x,y) = \frac{1}{y^2}, Q(x,y) = \frac{-2x-y}{y^3}$. This is exact, and $\varphi(x,y) = \frac{x}{y^2} + \frac{1}{y}$, hence $\frac{x}{f(x)^2} + \frac{1}{f(x)} = C$.

Jan. 16. Review of multiple integrals

Double and triple integrals

Recall that a region $S \subset \mathbb{R}^2$ is said to be of type I if it can be written as $S = \{(x,y) : a \leq x \leq \tilde{a}, \varphi_1(x) \leq y \leq \varphi_2(x)\}$ with continuous funcitons φ_1, φ_2. Similarly, S is of type II if $S = \{(x,y) : b \leq y \leq \tilde{b}, \psi_1(y) \leq x \leq \psi_2(y)\}$ with continuous funcitons ψ_1, ψ_2.

Let f be a continuous function on a type I region S. We learned that f is **integrable** on S, and it satisfies

$$\iint_S f(x,y)dxdy = \int_a^{\tilde{a}} \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right] dx.$$

Recall that integrability is defined through approximation by step functions, but we showed it for continuous functions and it can be computed by iterated integrals. Similarly, if S is of type II, then

$$\iint_S f(x,y)dxdy = \int_b^{\tilde{b}} \left[\int_{\psi_1(x)}^{\psi_2(x)} f(x,y)dx\right] dy.$$

In $\mathbb{R}^3$, we say that a region V is said to be xy-projectable if there is a region $S \subset \mathbb{R}^2$ and continuous functions φ_1, φ_2 on S such that $V = \{(x,y,z) : (x,y) \in S, \varphi_1(x,y) \leq z \leq \varphi_2(x,y)\}$. Similarly, we defined yz- and zx-projectable regions. If V is xy-projectable, then for a continuous function f, we have

$$\iiint_V f(x,y,z)dxdy = \iint_S \left[\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)dz\right] dxdy.$$

Exercises

Compute the following integrals.

- $\iint_S e^x \sin y dxdy$, $S = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq \pi\}$.

  Solution.

$$\iint_S e^x \sin y dxdy = \int_0^1 e^x \int_0^\pi \sin y dydx = \int_0^1 e^x[-\cos y]_0^\pi dx = 2[e^x]_0^1 = 2(e-1).$$

- $\iint_S (x+2y)dxdy$, $S = \{(x,y) : 4x^2 + y^2 \leq 4, x \geq 0, y \geq 0\}$.

  Solution. $S = \{(x,y) : 4x^2 + y^2 \leq 4, 0 \leq x \leq 1, 0 \leq y \leq 2\sqrt{1-x^2}\}$.

$$\begin{aligned}
\iint_S (x+2y)dxdy &= \int_0^1 \left[\int_0^{2\sqrt{1-x^2}} (x+2y)dy\right] dx = \int_0^1 \left[xy + y^2\right]_0^{2\sqrt{1-x^2}} dx \\
&= \int_0^1 (2x\sqrt{1-x^2} + 4(1-x^2))dx = \left[-\frac{2}{3}(1-x^2)^{\frac{3}{2}} + 4x - \frac{4}{3}x^3\right]_0^1 \\
&= -\frac{2}{3} + 4 - \frac{4}{3} - (-1) = 3
\end{aligned}$$

- $\iint_V \frac{1}{(1+x+y+z)^2} dxdydz$, $V = \{(x,y,z) : x,y,z \geq 0, x+y+z \leq 1\}$.

Solution. $V = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le 1-x, 0 \le z \le 1-x-y\}$.

$$\iiint_V \frac{1}{(1+x+y+z)^2} dxdydz = \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} \frac{1}{(1+x+y+z)^2} dz\right] dydx$$
$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{1+x+y+z}\right]_0^{1-x-y} dydx = \int_0^1 \int_0^{1-x} \left(\frac{1}{1+x+y} - \frac{1}{2}\right) dydx$$
$$= \int_0^1 \left[\log(1+x+y) - \frac{y}{2}\right]_0^{1-x} dx = \int_0^1 \left(\log 2 - \frac{1-x}{2} - \log(1+x)\right)_0^{1-x} dx$$
$$= \left[x \log 2 - \frac{2x - x^2}{4} - (1+x)\log(1+x) + (1+x)\right]_0^1$$
$$= \log 2 - \frac{1}{4} - 2\log 2 + 2 - 1 = 1 - \log 2 - \frac{1}{4}$$

- $\iint_V \frac{1}{(1+x+y+z)^3} dxdydz$, $V = \{(x,y,z) : x,y,z \ge 0, x+y+z \le 1\}$.
- $\iint_S \frac{y}{(x^2+y^2)^2} dxdy$, where S is the quadrilateral with vertices $(1,0)$, $(1,\sqrt{3})$, $(3,3\sqrt{3})$, $(3,0)$.
- $\iint_S x^2 dxdy$, $S : \{(x,y) : 0 \le x \le 1, 0 \le y \le x\}$.
- $\iint_S \frac{y}{\cos^2 xy} dxdy$, $S = \{(x,y) : 0 \le x \le \frac{1}{3}, 0 \le y \le \frac{\pi}{2}\}$.

  Hint: $(\tan t)' = \frac{1}{\cos^2 t}$.

If we set $f = 1$, a two-dimensional integral $\iint_S dxdy$ gives the area of S and a three-dimensional integral $\iiint_V dxdydz$ gives the volume of V.

Exercises

Compute the volume.

- V is the region bounded by the cylinder $x^2 + y^2 = 1$, the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 0$.

  Solution. $V = \{(x,y,z) : x^2 + y^2 \le 1, 0 \le z \le \sqrt{4 - x^2 - y^2}\}$.
- V is the solid bounded by $z = 0, z = x^2 - y^2, x = 1, x = 3$.

  Solution. $V = \{(x,y,z) : 1 \le x \le 3, 0 \le x^2 - y^2, 0 \le x^2 - y^2 \le z\}$.

Change of variables

Let S be a region in the xy-plane. Assume that there is another region T in uv-plane and the map $\boldsymbol{r}(u,v) = (X(u,v), Y(u,v))$ is one-to-one from T to S. With the Jacobian determinant $J(u,v) = \begin{vmatrix} \frac{\partial X}{\partial u}(u,v) & \frac{\partial Y}{\partial u}(u,v) \\ \frac{\partial X}{\partial v}(u,v) & \frac{\partial Y}{\partial v}(u,v) \end{vmatrix}$, it holds that

$$\iint_S f(x,y) dxdy = \iint_T f(X(u,v), Y(u,v))|J(u,v)| dudv.$$

Similarly, if V is a region in the xyz-space and if there is another region Q in uvw-space and a map $\boldsymbol{r}(u,v,w) = (X(u,v,w), Y(u,v,w), Z(u,v,w)$ from Q to V, then

$$\iint_V f(x,y,z) dxdy = \iint_Q f(X(u,v,w), Y(u,v,w), Z(u,v,w)|J(u,v,w)| dudv,$$

where $J(u,v,w) = \begin{vmatrix} \frac{\partial X}{\partial u}(u,v,w) & \frac{\partial Y}{\partial u}(u,v,w) & \frac{\partial Z}{\partial u}(u,v,w) \\ \frac{\partial X}{\partial v}(u,v,w) & \frac{\partial Y}{\partial v}(u,v,w) & \frac{\partial Z}{\partial v}(u,v,w) \\ \frac{\partial X}{\partial w}(u,v,w) & \frac{\partial Y}{\partial w}(u,v,w) & \frac{\partial Z}{\partial w}(u,v,w) \end{vmatrix}$

Example 92. (various coordinates)

- Linear transform $x = Au + Bx, y = Cu + Dv$. $J(u, v) = AD - BC$.
- Polar coordinates $x = r\cos\theta, y = r\sin\theta$. $J(r, \theta) = r$.
- Cylindrical coordinates $x = r\cos\theta, y = r\sin\theta, z = z$. $J(r, \theta, z) = r$.
- Spherical coordinates $x = r\cos\theta\sin\varphi, y = r\sin\theta\sin\varphi, z = r\cos\varphi$. $J(r, \theta, \varphi) = -r^2\sin\varphi$.

Exercises

Compute the integrals.

- $\iiint_V (x^2+y^2+z^2)dxdydz$, $V = \{(x, y, z) : x^2+y^2+z^2 \le a^2\}$. In the spherical coordinates, V corresponds to $Q = \{(r, \theta, \varphi) : 0 \le r \le a,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$.

$$\iiint_V (x^2 + y^2 + z^2)dxdydz = \iiint_Q r^2 \cdot r^2 \sin\varphi\, drd\theta d\varphi$$
$$= \int_0^{\pi}\int_0^{2\pi}\int_0^a r^4 \sin\varphi\, drd\theta d\varphi = 2 \cdot 2\pi \cdot \frac{a^5}{5} = \frac{4\pi a^5}{5}.$$

- $\iint_S \frac{y}{(x^2+y^2)^2}dxdy$ where S is the quadrilateral with vertices $(1, 0)$, $(1, \sqrt{3})$, $(3, 3\sqrt{3})$, $(3, 0)$.
  Solution. $S = \{(x, y) : 1 \le x \le 3, 0 \le y \le \sqrt{3}x\}$.
  In the polar coordinates it corresponds to

$$\tilde{T} = \left\{(r, \theta) : 0 \le \theta \le \frac{\pi}{3}, 1 \le r\cos\theta \le 3\right\} = \left\{(r, \theta) : 0 \le \theta \le \frac{\pi}{3}, \frac{1}{\cos\theta} \le r \le \frac{3}{\cos\theta}\right\}.$$

$$\iint_T \frac{y}{(x^2 + y^2)^2}dxdy = \iint_{\tilde{T}} \frac{r\sin\theta}{r^4} rdrd\theta = \int_0^{\frac{\pi}{3}}\int_{\frac{1}{\cos\theta}}^{\frac{3}{\cos\theta}} \frac{\sin\theta}{r^2}\, drd\theta$$
$$= \int_0^{\frac{\pi}{3}} \sin\theta \left[-\frac{1}{r}\right]_{\frac{1}{\cos\theta}}^{\frac{3}{\cos\theta}} d\theta = \frac{1}{4}.$$

- $\iint_S x^2 dxdy$, $S = \{(x, y) : 0 \le x^2 + y^2 \varepsilon 1\}$.
- $\iint_S \sqrt{x^2 + y^2}dxdy$, $S = \{(x, y) : 0 \le x \le a, 0 \le y \le x\}$.
- $\iiint_V \sqrt{x^2 + y^2}dxdy$, $V = \{(x, y, z) : x^2 + y^2 \le z^2 \le 1\}$.
- $\iint_S (5x^2 + 6xy + 5y^2)dxdy$, $S = \{(x, y) : 5x^2 + 6xy + 5y^2 \le 4\}$.
  Hint. $5x^2 + 6xy + 5y^2 = (x - y)^2 + 4(x + y)^2$. Use the coordinates $u = x - y, v = 2(x + y)$.

Compute the area/volume.

- $S = \{(x, y) : 2x^2 + 2xy + 5y^2 \le 1\}$.
  Hint: $2x^2 + 2xy + 4y^2 = (x - y)^2 + (x + 2y)^2$.
- $S = \{(x, y) : (x^2 + y^2)^3 \le 4x^2y^2, x, y \ge 0\}$.
- $V = \{(x, y, z) : x^2 + \frac{y^2}{4} + \frac{z^2}{9} \le 1\}$.
- $V = \{(x, y, z) : x^2 + y^2 \le z^2, 0 \le z \le 1\}$.

Jan. 18. Review of Stokes' and Gauss' theorems

Surface integrals

Let S be a two-dimensional surface in $\mathbb{R}^3$ and assume that it is parametrized by $\boldsymbol{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$ on a region $T \subset \mathbb{R}^2$.

Consider the upper hemisphere $x^2+y^2+z^2=a^2, z \geq 0$. It admits at least two parametrizations:

(1) $x = X(u,v) = u, y = Y(u,v) = v, z = Z(u,v) = \sqrt{a^2-u^2-v^2}$, on the region $T_1 = \{(u,v) : u^2+v^2 \leq a^2\}$.

(2) $x = X(\theta,\varphi) = a\cos\theta\sin\varphi, y = Y(\theta,\varphi) = a\sin\theta\sin\varphi, z = Z(\theta,\varphi) = a\cos\varphi$, on $T_2 = \{(\theta,\varphi) : 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}$.

The fundamental vector product of the parametrization is defined by $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}$, where $\frac{\partial \boldsymbol{r}}{\partial u} = (\frac{\partial X}{\partial u}, \frac{\partial Y}{\partial u}, \frac{\partial Z}{\partial u})$ and $\frac{\partial \boldsymbol{r}}{\partial v} = (\frac{\partial X}{\partial v}, \frac{\partial Y}{\partial v}, \frac{\partial Z}{\partial v})$.

- For example (1), we have $\frac{\partial \boldsymbol{r}}{\partial u} = (1, 0, \frac{u}{\sqrt{a^2-u^2-v^2}})$, $\frac{\partial \boldsymbol{r}}{\partial v} = (0, 1, \frac{v}{\sqrt{a^2-u^2-v^2}})$, hence $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = (-\frac{u}{\sqrt{a^2-u^2-v^2}}, -\frac{v}{\sqrt{a^2-u^2-v^2}}, 1)$.
- For (2), $\frac{\partial \boldsymbol{r}}{\partial \theta} = (-a\sin\theta\sin\varphi, a\cos\theta\sin\varphi, 0)$, $\frac{\partial \boldsymbol{r}}{\partial \varphi} = (a\cos\theta\cos\varphi, a\sin\theta\cos\varphi, -\sin\varphi)$ and hence $\frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial \varphi} = (-a^2\cos\theta\sin^2\varphi, -a^2\sin\theta\sin^2\varphi, -a^2\sin\varphi\cos\varphi) = -a\sin\varphi\boldsymbol{r}$.

The **surface integral** of a scalar field f on S is defined by

$$\iint_S f dS := \iint_T f(\boldsymbol{r}(u,v)) \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| dudv$$

We showed that the surface integral does not depend on the parametrization. If we integrate the function $f = 1$, then we get the area $a(S)$ of S.

Let $\boldsymbol{f}$ be a vector field, $\boldsymbol{N} = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}$ and $\boldsymbol{n} = \frac{\boldsymbol{N}}{\|\boldsymbol{N}\|}$. The product $\boldsymbol{f} \cdot \boldsymbol{n}$ is a scalar field on S, and it holds that

$$\iint_S \boldsymbol{f} \cdot \boldsymbol{n}\, dS = \iint_T \boldsymbol{f}(\boldsymbol{r}(u,v)) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right) dudv$$

Exercises

- Compute the surface integral $\iint_S \boldsymbol{f} \cdot \boldsymbol{n} dS$, where $\boldsymbol{f}(x,y,z) = (xz, yz, 0)$ and $S = \{(x,y,z) : x^2+y^2-z^2 = a^2, 0 \leq z \leq 1\}$, where $\boldsymbol{n}$ is outgoing (from $\mathbf{0}$).

  Solution. By taking the parametrization $\boldsymbol{r}(\theta, z) = (\sqrt{a^2+z^2}\cos\theta, \sqrt{a^2+z^2}\sin\theta, z)$, we can compute $\frac{\partial \boldsymbol{r}}{\partial \theta} = (-\sqrt{a^2+z^2}\sin\theta, \sqrt{a^2+z^2}\cos\theta, 0)$, $\frac{\partial \boldsymbol{r}}{\partial z} = (\frac{z}{\sqrt{a^2+z^2}}\cos\theta, \frac{z}{\sqrt{a^2+z^2}}\sin\theta, 1)$, $\frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial z} = (\sqrt{a^2+z^2}\cos\theta, \sqrt{a^2+z^2}\sin\theta, -z)$. Correspondingly, we should take $T = \{(\theta, z) : 0 \leq \theta \leq 2\pi, 0 \leq z \leq 1\}$. $\boldsymbol{f}(\boldsymbol{r}(\theta,z)) = (z\sqrt{a^2+z^2}\cos\theta, z\sqrt{a^2+z^2}\sin\theta, 0)$.

  $\iint_S \boldsymbol{f} \cdot \boldsymbol{n}\, dS = \iint_T z(a^2+z^2) d\theta dz = 2\pi \cdot \frac{1}{2}[(a^2+z^2)^2]_0^1 = \pi((a^2+1)^2 - a^4)$.

- Compute the area $S = \{(x,y,z) : x^2+y^2 = z, z \leq 4\}$.

  Solution. Set $r = \sqrt{x^2+y^2}$, then $x = r\cos\theta, y = r\sin\theta, z = r^2$ and we need to take $T = \{(r,\theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$.

  We get $\frac{\partial \boldsymbol{r}}{\partial r} = (\cos\theta, \sin\theta, 2r)$, $\frac{\partial \boldsymbol{r}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$, $\frac{\partial \boldsymbol{r}}{\partial r} \times \frac{\partial \boldsymbol{r}}{\partial \theta} = (-2r^2\cos\theta, -2r^2\sin\theta, r)$, $\|\frac{\partial \boldsymbol{r}}{\partial r} \times \frac{\partial \boldsymbol{r}}{\partial \theta}\| = r\sqrt{4r^2+1}$. $\iint_S dS = \iint_T r\sqrt{4r^2+1} drd\theta = 2\pi[\frac{1}{12}(4r^2+1)^{\frac{3}{2}}]_0^2 = \frac{\pi}{6}(17\sqrt{17}-1)$.

- Let $S = \{(x,y,z) : x^2+z^2 = a^2, -1 \leq y \leq 1, z \leq 0\}$. Compute $\iint_S \boldsymbol{f} \cdot \boldsymbol{n}\, dS$, where $\boldsymbol{f}(x,y,z) = (0,0,1)$ and $\boldsymbol{n}$ has positive z-component.

Curl and divergence

Let $\boldsymbol{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$. We defined

$$\operatorname{div}\boldsymbol{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}, \quad \operatorname{curl}\boldsymbol{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

Exercises

Compute $\operatorname{div}\boldsymbol{F}$ and $\operatorname{curl}\boldsymbol{F}$.

- $\boldsymbol{F}(x,y,z) = (x^2yz, y^2xz, z^2xy)$.

  Solution. $\operatorname{div}\boldsymbol{F} = 6xyz$ and $\operatorname{curl}\boldsymbol{F} = (z^2x - y^2x, x^2y - z^2y, y^2z - x^2z)$.
- $\boldsymbol{F}(x,y,z) = (\cos(xy), \sin(yz), e^{zx})$.

Stokes' theorem

Assume that S is a piecewise continuously differentiable surface, with a piecewise continuously differentiable boundary C that is a single curve, parametrized by $\boldsymbol{\alpha}$. Let $\boldsymbol{r}$ be a parametrization of S from a simply connected region with the corresponding boundary parametrized counterclockwise, which is compatible with $\boldsymbol{\alpha}$. Let $\boldsymbol{n}$ parallel to $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}$. Then, for a continuously differentiable vector field $\boldsymbol{F}$,

$$\iint_S \operatorname{curl}\boldsymbol{F} \cdot \boldsymbol{n}\, dS = \int_C \boldsymbol{F} \cdot d\boldsymbol{\alpha}.$$

Exercises

Compute the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{\alpha}$.

- $\boldsymbol{F}(x,y,z) = (x-y, y-z, z-x)$, $C = \{(x,y,z) : x^2 + z^2 = 1, y = 0\}$, $\boldsymbol{\alpha}$ going going counterclockwise on the zx-plane.

  Solution. For the C above, we can take $S = \{(x,y,z) : x^2 + z^2 \le 1, z = 0\}$, and a parametrization $X(u,v) = v, Y(u,v) = 0, Z(u,v) = u$. The corresponding region in the uv-plane is $T = \{(u,v) : u^2 + v^2 \le 1\}$. It follows that $\frac{\partial \boldsymbol{r}}{\partial u} = (0,0,1), \frac{\partial \boldsymbol{r}}{\partial v} = (1,0,0)$ and hence $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = (0,1,0)$.
  $\operatorname{curl}\boldsymbol{F} = (1,1,1))$.

  $$\int_C \boldsymbol{F} \cdot d\boldsymbol{\alpha} = \int_S \operatorname{curl}\boldsymbol{F} \cdot \mathbb{n}\, dS = \iint_T 1\, dudv = \pi.$$

- $\boldsymbol{F}(x,y,z) = (e^{zy^2}, e^{yx^2}, e^{xz^2})$, C is the boundary of the square with the four vertices $(0,0,0), (1,0,0), (1,1,0), (0,1,0)$, $\boldsymbol{\alpha}$ going going counterclockwise on the xy-plane.

  Solution. For the C above, we can take $S = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le 1, z = 0\}$, and a parametrization $X(u,v) = u, Y(u,v) = v, Z(u,v) = 0$. The corresponding region in the uv-plane is $T = \{(u,v) : 0 \le u \le 1, 0 \le v \le 1\}$. It follows that $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = (0,0,1)$.

  $\operatorname{curl}\boldsymbol{F} = \left(0, y^2e^{zy^2} - z^2e^{xz^2}, 2xye^{yx^2} - 2yze^{zy^2})\right)$ Therefore, on the uv-plane, namely $x = u, y = v, z = 0$, $\operatorname{curl}\boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \boldsymbol{n}(u,v) = (0,0,2uve^{vu^2}) \cdot (0,0,1) = 2uve^{vu^2}$.

  $$\int_C \boldsymbol{F} \cdot d\boldsymbol{\alpha} = \int_S \operatorname{curl}\boldsymbol{F} \cdot \mathbb{n}\, dS = \iint_T 2uve^{vu^2}\, dudv = e - 2.$$

- $C = \{(x,y,z) : x^2 + y^2 = 1, z = 0\}$, $\boldsymbol{\alpha}$ going counterclockwise on the xy-plane, $\boldsymbol{F}(x,y,z) = ((x - y + z)e^{x^2+y^2+z^2},\ (x + y + z)e^{x^2+y^2+z^2},\ (-x + y - z)e^{x^2+y^2+z^2})$.
- C is the boundary of the square $\{(x,y,z) : 0 \le x, z \le 4\}$, $\boldsymbol{\alpha}$ going counterclockwise on the xz-plane, $\boldsymbol{F}(x,y,z) = (xy\cos(xyz), zy\cos(xyz), yz\cos(xyz))$.

Gauss' theorem

Let V be a solid in $\mathbb{R}^3$, xy-, yz-, zx-projectable, bounded by the surface S, $\boldsymbol{n}$ be the outgoing normal unit vector on S, $\boldsymbol{F}$ a continuously differentiable vector field. Then

$$\iiint_V \operatorname{div} \boldsymbol{F}\, dxdydz = \iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS.$$

Exercises

Compute the surface integral $\iiint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$, where $\boldsymbol{n}$ the outgoing normal unit vector on S.

- $\boldsymbol{F}(x, y, z) = (x^3, y^3, z^3)$, $S = \{(x, y, z) : x^2 + y^2 + z^2 = a^2\}$, where $a > 0$.

  Solution. We have $\operatorname{div}\boldsymbol{F} = 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2)$. To perform the volume integral, we use the spherical coordinate $x = r\cos\theta\sin\varphi, y = r\sin\theta\sin\varphi, z = r\cos\varphi$. With $V = \{(x, y, z) : x^2 + y^2 + z^2 \leq a^2\}$,

  $$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iiint_V \operatorname{div}\boldsymbol{F}\, dxdydz = 3\iiint_Q r^2 \cdot r^2 \sin\varphi\, drd\theta d\varphi = \frac{12\pi a^5}{5}.$$

- $\boldsymbol{F}(x, y, z) = (x(x^2 + y^2)z, y(x^2 + y^2)z, z)$, $S = \{(x, y, z) : x^2 + y^2 \leq 1, z = 0\} \cup \{(x, y, z) : x^2 + y^2 \leq 1, z = 1\} \cup \{(x, y, z) : x^2 + y^2 = 1, 0 \leq z \leq 1\}$.

  Solution. We have $\operatorname{div}\boldsymbol{F} = 4(x^2 + y^2)z + 1$. With $V = \{(x, y, z) : x^2 + y^2 \leq 1, 0 \leq z \leq 1\}$,

  $$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iiint_V \operatorname{div}\boldsymbol{F}\, dxdydz = \iiint_V (4(x^2 + y^2)z + 1)dxdydz = 2\pi.$$

- $\boldsymbol{F}(x, y, z) = (x^2y, xyz, xz^3)$, S is the surface of the cube $\{(x, y, z) : 0 \leq x, y, z \leq 1\}$.
- $\boldsymbol{F}(x, y, z) = (x^2, y^2, z^2)$, $S = \{(x, y, z) : x^2 + y^2 + z^2 \leq a^2\}$.