BSc Engineering Sciences – A. Y. 2018/19

Written exam of the course Mathematical Analysis 2

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Last name: .... First name: ....
Matriculation: ....

Solve the following problems, motivating in detail the answers.

1. Find the Taylor series expansion, with initial point $x_0 = 1$, of the function

$$f(x) := (x-1)\log(x^2-2x+2),$$

find its radius of convergence r, and study the convergence for $x = 1 \pm r$.

Solution.

Note that $\log(x^2-2x+2) = \log(1+(x-1)^2)$.

Recall that $\log(1+y) = \sum_{n=1}^{\infty}(-1)^n\frac{y^n}{n}$ with the radius of convergence 1, namely it is convergent for $|y| < 1$ and divergent for $|y| > 1$. By putting $y = (x-1)^2$, It follows that $\log(1+(x-1)^2) = \sum_{n=1}^{\infty}(-1)^n\frac{(x-1)^{2n}}{n}$ converges for $|x-1| < 1$ and diverges for $|x-1| > 1$. Multiplying by $x-1$ does not change the radius of convergence, therefore,

$$f(x) = (x-1)\sum_{n=1}^{\infty}(-1)^n\frac{(x-1)^{2n}}{n} = \sum_{n=1}^{\infty}(-1)^n\frac{(x-1)^{2n+1}}{n}$$

and the radius of convergence is 1.

For $x = 1+1 = 2$, we have

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

which is an alternating series, and as $\left|(-1)^n\frac{1}{n}\right| \to 0$ monotonically, the series if convergent by the Leibniz criterion. The same hold for $x = 1-1 = 0$.

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2. Find all the stationary points of the following scalar field, defined on $\mathbb{R}^2$,

$$f(x,y) = 2x^3 - 2x^2y - x + y^2$$

and classify them into relative minima, maxima and saddle points.

Solution. For the f given above, it holds that

$$\nabla f(x,y) = (6x^2 - 4xy - 1, -2x^2 + 2y).$$

At stationary points, $\nabla f(x,y) = \mathbf{0}$ holds. Namely,

$$6x^2 - 4xy - 1 = 0, -2x^2 + 2y = 0.$$

The latter is equivalent to $x^2 = y$. By putting this to the first equation, we obtain $6x^2 - 4x^3 - 1 = 0$. This has a solution $x = \frac{1}{2}$, hence the left-hand side has the factor $2x - 1$, and hence by dividing by it, we can factorize the left-hand side: $(2x-1)(-2x^2 + 2x + 1) = 0$. From this, the solutions are $x = \frac{1}{2}, \frac{1}{2} \pm \frac{\sqrt{3}}{2}$, and correspondingly, there are three stationary points: $(x,y) = (\frac{1}{2}, \frac{1}{4}), (\frac{1}{2} + \frac{\sqrt{3}}{2}, 1 + \frac{\sqrt{3}}{2}), (\frac{1}{2} - \frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2})$.

To classify these points, let us compute the Hessian matrix:

$$\begin{pmatrix} 12x - 4y & -4x \\ -4x & 2 \end{pmatrix}.$$

- At the point $(x,y) = (\frac{1}{2}, \frac{1}{4})$, this becomes

  $$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}.$$

  Its determinant is 6 and its trace is 7, therefore, its eivenvalues are positive, and this point is a relative minumum.

- At the point $(x,y) = (\frac{1}{2} + \frac{\sqrt{3}}{2}, 1 + \frac{\sqrt{3}}{2})$, this becomes

  $$\begin{pmatrix} 2 + 4\sqrt{3} & -2 - 2\sqrt{3} \\ -2 - 2\sqrt{3} & 2 \end{pmatrix}.$$

  Its determinant is -12, therefore, it has both positive and negative eivenvalues, and this point is a saddle.

- At the point $(x,y) = (\frac{1}{2} - \frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2})$, this becomes

  $$\begin{pmatrix} 2 - 4\sqrt{3} & -2 + 2\sqrt{3} \\ -2 + 2\sqrt{3} & 2 \end{pmatrix}.$$

  Its determinant is -12, therefore, it has both positive and negative eivenvalues, and this point is a saddle.

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3. Let C be the curve $\{(x, y) : x^2 + 4y^2 = 4, 0 \leq x\}$ in $\mathbb{R}^2$. Find a parametrization $\boldsymbol{\alpha}(t)$ of C starting at $(0, -1)$ and ending at $(0, 1)$, and compute the line integral

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha},$$

where $\boldsymbol{f}(x, y) = (y + 1, x)$ is a vector field in $\mathbb{R}^2$.

Solution.

On C, $x^2 + (2y)^2 = 4$, hence we can put $\boldsymbol{\alpha}(t) = (2\cos t, \sin t)$. By taking $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, this starts at $(0, -1)$ and ends at $(0, 1)$.

In order to compute the line integral, we need $\boldsymbol{\alpha}(t) = (-2\sin t, \cos t)$ and $\boldsymbol{f}(\boldsymbol{a}(t)) = (\sin t + 1, 2\cos t)$.

The line integral can be computed as

$$\begin{aligned}
\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t + 1, 2\cos t) \cdot (-2\sin t, \cos t) dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2(\cos^2 t - \sin^2 t - \sin t) dt \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(2t) - \sin t) dt \\
&= [\sin(2t) + 2\cos t]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 0
\end{aligned}$$

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4. Compute the integral

$$\iiint_T x^4 dxdydz\,,$$

with

$$T := \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2+z^2 \leq 1,\, 1 \leq y \leq 2-x^2-z^2\}.$$

Solution.

Let us put $S = \{(x,z) \in \mathbb{R}^2 : x^2+z^2 \leq 1\}$. We can rewrite

$$\begin{aligned} T &= \{(x,y,z) \in \mathbb{R}^3 : x^2+z^2 \leq 1,\, 1 \leq y \leq 2-x^2-z^2\} \\ &= \{(x,y,z) \in \mathbb{R}^3 : (x,z) \in S,\, 1 \leq y \leq 2-x^2-z^2\}. \end{aligned}$$

From the last expression we see that T is zx-projectable, so the triple integral in question can be written as an iterated integral.

Let us compute:

$$\begin{aligned} \iiint_T x^4 dxdydz &= \iint_S \left[\int_1^{2-x^2-z^2} x^4 dy\right] dzdx \\ &= \iint_S [x^4 y]_1^{2-x^2-z^2} dzdx \\ &= \iint_S x^4(1-x^2-z^2)dzdx. \end{aligned}$$

Then we use the polar coordinate $x = r\cos\theta, z = r\sin\theta$. The Jacobian determinant is $J(r,\theta) = r$, therefore, using $\cos^4\theta = (\frac{1+\cos 2\theta}{2})^2 = \frac{1}{4}(1+2\cos 2\theta + \frac{1+\cos 4\theta}{2})$

$$\begin{aligned} \iint_S x^4(1-x^2-z^2)dzdx &= \iint_S r^4\cos^4\theta(1-r^2)rdrd\theta \\ &= \int_0^{2\pi}\int_0^1 \frac{1}{4}\left(1+2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)(r^5-r^7)drd\theta \\ &= \int_0^{2\pi} \frac{1}{4}\left(1+2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)\left[\frac{1}{6}r^6 - \frac{1}{8}r^8\right]_0^1 d\theta \\ &= \frac{1}{96}\int_0^{2\pi}\left(1+2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta \\ &= \frac{1}{96}\left[\theta + \sin 2\theta + \frac{4\theta + \sin 4\theta}{8}\right]_0^{2\pi} \\ &= \frac{\pi}{32}. \end{aligned}$$

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5. Let $\boldsymbol{F}(x, y, z) = (0, xyz, x)$ be a vector field on $\mathbb{R}^3$ and

$$S = \{(x, y, z) : x^2 + y^2 + z^2 \leq 4,\ y = z,\ x \geq 0\}$$

be a surface in $\mathbb{R}^3$. Compute the surface integral

$$\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS,$$

where $\boldsymbol{n}$ is a unit normal vector on S with positive z-component.

Solution. S can be parametrized by

$$\boldsymbol{r}(u, v) = (u, v, v), \qquad (u, v) \in \tilde{S} = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 + v^2 \leq 4, u \geq 0\}.$$

and we can rewrite

$$\tilde{S} = \{(u, v) \in \mathbb{R}^2 : u^2 + 2v^2 \leq 4, u \geq 0\} = \{(u, v) \in \mathbb{R}^2 : -\sqrt{2} \leq v \leq \sqrt{2}, 0 \leq u \leq \sqrt{4 - 2v^2}\}.$$

The fundamental vector product is

$$\frac{\partial \boldsymbol{r}}{\partial u}(u, v) = (1, 0, 0), \quad \frac{\partial \boldsymbol{r}}{\partial u}(u, v) = (0, 1, 1), \quad \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial u}(u, v) = (0, -1, 1)$$

and this last vector has positive z-component. We also have $\boldsymbol{F}(\boldsymbol{r}(u, v)) = (0, uv^2, u)$.

By the formula for surface integral, we can compute:

$$\begin{aligned}
\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS &= \iint_{\tilde{S}} \boldsymbol{F}(\boldsymbol{r}(u, v)) \cdot \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial u}(u, v) du dv \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \left[\int_0^{\sqrt{4-2v^2}} (-uv^2 + u) du \right] dv \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{2}(1 - v^2)[u^2]_0^{\sqrt{4-2v^2}} dv \\
&= \frac{1}{2} \int_{-\sqrt{2}}^{\sqrt{2}} (1 - v^2)(4 - 2v^2) dv \\
&= \frac{1}{2} \int_{-\sqrt{2}}^{\sqrt{2}} (2v^4 - 6v^2 + 4) dv \\
&= \frac{1}{2} \left[\frac{2}{5}v^5 - 2v^3 + 4v \right]_{-\sqrt{2}}^{\sqrt{2}} \\
&= \frac{8\sqrt{2}}{5}.
\end{aligned}$$