BSc Engineering Sciences – A. Y. 2018/19
Written exam of the course Mathematical Analysis 2
August 29, 2019

1. (1) Compute the derivative, with respect to t, of the function

$$f(t) = \int_{t^2}^{t^3} \frac{\sin u}{u}\, du.$$

(2) Let $f \in C^2(\mathbb{R}^2)$ be a solution of the first order linear partial differential equation

$$3\frac{\partial f}{\partial t} + 2\frac{\partial f}{\partial x} = 0.$$

Find $c \in \mathbb{R}$ such that f is also a solution of the one dimensional wave equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}.$$

Solution.

(1) Fix $t > 0$. Take $0 < a < t$ and let us put $F(s) = \int_a^s \frac{\sin u}{u} du$, then $F'(s) = \frac{\sin s}{s}$. Note that $F'(s)$ can be extended to a continuous function by setting $F'(0) = 1$. Then we have $f(t) = F(t^3) - F(t^2)$ and, by the chain rule,

$$f'(t) = 3t^2 F'(t^3) - 2tF'(t^2) = 3t^2 \cdot \frac{\sin t^3}{t^3} - 2t \cdot \frac{\sin t^2}{t^2} = \frac{3 \sin t^3}{t} - \frac{2 \sin t^2}{t}.$$

Similarly, the same formula is valid also for $t < 0$. For $t = 0$, we have $f'(0) = 0$.

(2) We know that a general solution of the first differential equation can be written as $f(t, x) = g(2t - 3x)$, where $g(s)$ is a differentiable function. As $f \in C^2(\mathbb{R}^2)$, $g \in C^2(\mathbb{R})$ as well.

By the chain rule, we have

$$\frac{\partial^2 f}{\partial t^2} = 2^2 g''(2t - 3x) = 4g''(2t - 3x), \quad \frac{\partial^2 f}{\partial x^2} = (-3)^2 g''(2t - 3x) = 9g''(2t - 3x).$$

From this we see that $\frac{\partial^2 f}{\partial t^2} = \frac{9}{4}\frac{\partial^2 f}{\partial x^2}$. In other words, $c = \pm\frac{3}{2}$.

2. Find the extremal values of the function $f(x, y, z) = x^2 + y^2 + z^2$ on the line L defined by two equations $x + y + z = 1$ and $x - z = 2$.

Solution.

Put $G(x, y, z) = x+y+z-1$ and $H(x, y, z) = x-z-2$. By Lagrange's multiplier method, there is $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\nabla f(x, y, z) = \lambda_1 \nabla G(x, y, z) + \lambda_2 H(x, y, z)$ at stationary points (x, y, z) of $f(x, y, z)$. Let us compute these gradients:

$$\begin{aligned} \nabla f(x, y, z) &= (2x, 2y, 2z), \\ \nabla G(x, y, z) &= (1, 1, 1), \\ \nabla G(x, y, z) &= (1, 0, -1). \end{aligned}$$

From the equation of the multiplier method, for a stationary point (x, y, z), we have

$$(2x, 2y, 2z) = (\lambda_1, \lambda_1, \lambda_1) + (\lambda_2, 0, -\lambda_2).$$

Or equivalently, $2x = \lambda_1 + \lambda_2, 2y = \lambda_1, 2z = \lambda_1 - \lambda_2$.

In addition (x, y, z) must satisfy $x + y + z = 1, x - z = 2$. From this we have that $2 = 2(x + y + z) = 3\lambda_1, 2 = x - z = \lambda_2$. By solving these equations, we have $\lambda_1 = \frac{2}{3}, \lambda_2 = 2$. By putting them in the equations above, we obtain $x = \frac{4}{3}, y = \frac{1}{3}, z = -\frac{2}{3}$.

At this point $(x, y, z) = (\frac{4}{3}, \frac{1}{3}, -\frac{2}{3})$, we have $f(\frac{4}{3}, \frac{1}{3}, -\frac{2}{3}) = \dfrac{7}{3}$. As L is a line and $f(x, y, z) = x^2 + y^2 + z^2$ can be arbitrarily large when (x, y, z) is far from the origin, the point $(\frac{4}{3}, \frac{1}{3}, -\frac{2}{3})$ is the minimum.

3. Let C be the curve $\{(x,y) : xy = 1, 1 \le x \le 3\}$ in $\mathbb{R}^2$. Find a parametrization $\boldsymbol{\alpha}(t)$ of C starting at $(1,1)$ and ending at $(3, \frac{1}{3})$, and compute the line integral

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha},$$

where $\boldsymbol{f}(x,y) = (y, -x^4)$ is a vector field in $\mathbb{R}^2$.

Solution.

The equation $xy = 1$ can be written as $y = \frac{1}{x}$. By taking $t = x$ as a parameter, we have $\boldsymbol{\alpha}(t) = (t, \frac{1}{t})$, for $t \in [1,3]$. This indeed starts at $(1,1)$ when $t = 1$ and ends at $(3, \frac{1}{3})$ when $t = 3$.

To compute the line integral, we need $\boldsymbol{\alpha}'(t) = \left(1, -\frac{1}{t^2}\right)$, $\boldsymbol{f}(\boldsymbol{\alpha}(t)) = \left(\frac{1}{t}, -t^4\right)$. Now by the definition of line integral, we have

$$\begin{aligned}
\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} &= \int_1^3 \left(\frac{1}{t}, -t^4\right) \cdot \left(1, -\frac{1}{t^2}\right) dt \\
&= \int_1^3 \frac{1}{t} + t^2 dt \\
&= \left[\log t + \frac{t^3}{3}\right]_1^3 \\
&= (\log 3 + 9) - \left(0 + \frac{1}{3}\right) = \log 3 + \frac{26}{3}.
\end{aligned}$$

4. Compute the integral

$$\iiint_T dxdydz\,(z+1)\sqrt{\frac{x^2+y^2}{4-x^2-y^2}}$$

where

$$T := \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2+y^2-2y \le 0,\, x^2+y^2+z^2 \le 4\}.$$

Solution.

Since $x^2+y^2-2y = x^2+(y-1)^2-1$, the region T can be written as

$$T = \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2+(y-1)^2 \le 1,\, x^2+y^2+z^2 \le 4\}.$$

Namely, it is the intersection of the cylinder based on the disk $x^2+(y-1)^2 \le 1$ and the sphere of radius 2 with the center at the origin. Note that, if $x^2+y^2+z^2 \le 4$, then $x^2+y^2 \le 4$ and it also holds that $x^2+(y-1)^2 \le 1$. Therefore, T is xy-projectable with $S = \{(x,y) \in \mathbb{R} : x^2+(y-1)^2 \le 1\}$ and

$$T = \{(x,y,z) : (x,y) \in S, -\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2}\}.$$

With this expression, we have

$$\begin{aligned}
&\iiint_T dxdydz\,(z+1)\sqrt{\frac{x^2+y^2}{4-x^2-y^2}} \\
&= \iint_S dxdy \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} (z+1)\sqrt{\frac{x^2+y^2}{4-x^2-y^2}}dz \\
&= \iint_S dxdy\sqrt{\frac{x^2+y^2}{4-x^2-y^2}} \cdot \left[\frac{z^2}{2}+z\right]_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} \\
&= \iint_S dxdy\sqrt{\frac{x^2+y^2}{4-x^2-y^2}} \cdot 2\sqrt{4-x^2-y^2} = 2\iint_S \sqrt{x^2+y^2}dxdy
\end{aligned}$$

To carry out the xy-integral, we go to the polar coordinate: $x = r\cos\theta, y = r\sin\theta$. Note that $(y-1)^2 = r^2\sin^2\theta - 2r\sin\theta + 1$ and hence $x^2+(y-1)^2 = r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + 1 = r^2 - 2r\sin\theta + 1$. In the polar coordinate, S corresponds to

$$\tilde{S} = \{(r,\theta) \in [0,\infty) \times [0,2\pi) : r^2 \le 2r\sin\theta\} = \{(r,\theta) \in [0,\infty) \times [0,2\pi) : r \le 2\sin\theta\}.$$

For any $\theta \in [0,\pi]$, there is some $r \in [0,\infty)$.

Finally, with the Jacobian determinant $J(r,\theta) = r$,

$$\begin{aligned}
&2\iint_S \sqrt{x^2+y^2}dxdy = 2\iint_{\tilde{S}} r \cdot r drd\theta = 2\int_0^\pi \int_0^{2\sin\theta} r^2 drd\theta \\
&= 2\int_0^\pi \left[\frac{r^3}{3}\right]_0^{2\sin\theta} d\theta = \frac{16}{3}\int_0^\pi \sin\theta(1-\cos^2\theta)d\theta = \frac{16}{3}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^\pi = \frac{64}{9}.
\end{aligned}$$

5. Let $\boldsymbol{F}(x,y,z) = (xy, e^{-y^2}, yz)$ be a vector field on $\mathbb{R}^3$ and

$$S = \{(x,y,z) : x^2+z^2 = 9,\ 0 \le x,\ 0 \le y \le 2\}$$

be a surface in $\mathbb{R}^3$. Compute the surface integral

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\, dS,$$

where $\boldsymbol{n}$ is a unit normal vector on S with positive x-component.

Solution.

S is the surface of the cylinder based on the disk $x^2+z^2 \le 9$. As $0 \le x$, We can parametrize it by

$$\boldsymbol{r}(y,\theta) = (X(y,\theta), Y(y,\theta), Z(y,\theta)) = (3\cos\theta, y, 3\sin\theta), \qquad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \in [0,2]$$

In order to compute the surface integral, we need

$$\begin{aligned} \frac{\partial \boldsymbol{r}}{\partial y} &= (0,1,0) \\ \frac{\partial \boldsymbol{r}}{\partial \theta} &= (-3\sin\theta, 0, 3\cos\theta) \\ \frac{\partial \boldsymbol{r}}{\partial y} \times \frac{\partial \boldsymbol{r}}{\partial \theta} &= (3\cos\theta, 0, 3\sin\theta) \end{aligned}$$

$\frac{\partial \boldsymbol{r}}{\partial y} \times \frac{\partial \boldsymbol{r}}{\partial \theta}$ has positive x-component. From this, we have $\boldsymbol{F}(\boldsymbol{r}(y,\theta)) = (3y\cos\theta, e^{-y^2}, 3y\sin\theta)$.

Now by a formula for surface integral,

$$\begin{aligned} \iint_S \boldsymbol{F}\cdot\boldsymbol{n}\, dS &= \int_0^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \boldsymbol{F}(\boldsymbol{r}(y,\theta)) \cdot \frac{\partial \boldsymbol{r}}{\partial y} \times \frac{\partial \boldsymbol{r}}{\partial \theta}(y,\theta) d\theta dy \\ &= \int_0^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 9y(\cos^2\theta + \sin^2\theta) d\theta dy \\ &= 9\pi \cdot \frac{1}{2}[y^2]_0^2 = 18\pi. \end{aligned}$$