BSc Engineering Sciences – A. Y. 2018/19

Written exam of the course Mathematical Analysis 2

July 10, 2019

1. Find a power series solution $y(x)$ of the differential equation

$$(1+x^2)y''(x) - xy'(x) - 3y(x) = x$$

subject to the intial conditions $y(0) = y'(0) = 1$, and determine its radius of convergence.

Solution.

Let us put $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then we have $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. If $y(x)$ satisfies the above equation, then

$$\begin{aligned}
x &= (1+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - 3\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n - 3\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^n - 3\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} + n(n-1)a_n - n a_n - 3a_n\right)x^n \\
&= \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} + (n-3)(n+1)a_n\right)x^n
\end{aligned}$$

where in the 3rd equality we shifted the index by $n \to n+2$ in the first summation while we used that the terms $n = 0, 1$ are 0 in the 2nd summation and the tern $n = 0$ is 0 in the 3rd summation. If this equality holds as power series, then it follows that $(n+2)(n+1)a_{n+2} + (n-3)(n+1)a_n = 0$ for $n \neq 1$ and $3 \cdot 2a_3 + (-2) \cdot 2a_1 = 1$. The former relation is equivalent to

$$a_{n+2} = -\frac{(n-3)(n+1)}{(n+2)(n+1)}a_n = -\frac{n-3}{n+2}a_n$$

From the initial conditions $y(0) = 1, y'(0) = 1$, we have $a_0 = 1, a_1 = 1$. By the recursion relation above, we have

$$a_3 = -\frac{1-3}{3}a_1 = \frac{2}{3}, \quad a_5 = -\frac{0}{5}a_3 = 0$$

and from this it follows that $a_{2n+1} = 0, n \geq 2$. For even numbers, the recursion relation implies, for $n \geq 3$,

$$a_{2n} = (-1)\frac{2n-5}{2n}a_{2n-2} = \cdots = (-1)^n\frac{(2n-5)!! \cdot (-1)(-3)}{(2n)!!}a_0 = (-1)^n 3\frac{(2n-5)!!}{(2n)!!},$$

where $m!! = m(m-2)(m-4)\cdots$ and $a_2 = \frac{3}{2}a_0 = \frac{3}{2}$, $a_4 = -\frac{1}{4}a_2 = -\frac{3}{8}$. Altogether, we obtain

$$y(x) = 1 + x + \frac{3}{2}x^2 + \frac{5}{6}x^3 - \frac{3}{8}x^4 + \sum_{n=3}^{\infty}(-1)^n 3\frac{(2n-5)!!}{(2n)!!}x^{2n}.$$

To see the convergence of the infinite sum, we apply the ratio test:

$$\left|\frac{(-1)^{n+1}3\frac{(2(n+1)-5)!!}{(2(n+1))!!}x^{2(n+1)}}{(-1)^n 3\frac{(2n-5)!!}{(2n)!!}x^{2n}}\right| = \frac{(2n-3)(2n-1)|x|^2}{4(n+1)n} \underset{n\to\infty}{\to} |x|^2$$

Therefore, the series is convergent if $|x| < 1$ and divergent if $|x| > 1$, hence the radius of convergence is 1.

2. Find all the stationary points of the following scalar field, defined on $\mathbb{R}^2$,

$$f(x,y) = e^{x^2+y^2}\left(x+y-\frac{5}{2}\right)$$

and classify them into relative minima, maxima and saddle points.

Solution.

For the f given above, it holds that

$$\nabla f(x,y) = \left(e^{x^2+y^2}\left(2x\left(x+y-\frac{5}{2}\right)+1\right), e^{x^2+y^2}\left(2y\left(x+y-\frac{5}{2}\right)+1\right)\right).$$

At stationary points, $\nabla f(x,y) =$ holds. Namely,

$$e^{x^2+y^2}\left(2x\left(x+y-\frac{5}{2}\right)+1\right) = 0, \quad e^{x^2+y^2}\left(2y\left(x+y-\frac{5}{2}\right)+1\right) = 0.$$

As $e^{x^2+y^2}$ takes never 0, this is equivalent to

$$2x\left(x+y-\frac{5}{2}\right)+1 = 0, \quad 2y\left(x+y-\frac{5}{2}\right)+1 = 0.$$

and by subtracting the both sides, one obtains $2(x-y)(x+y-\frac{5}{2}) = 0$.

Case 1. Assume that $x-y=0$. Substituting $y=x$ in one of these equations, one obtains $2x(2x-\frac{5}{2})+1=0$, or equivalently, $4x^2-5x+1=0$. By solving this, we obtain $(x,y) = (\frac{1}{4},\frac{1}{4}), (1,1)$.

Case 2. If $(x+y-\frac{5}{2}) = 0$, then we would have $e^{x^2+y^2}\cdot 1 = 0$, which is impossible.

To classify these points, let us compute the Hessian matrix:

$$\begin{pmatrix} e^{x^2+y^2}(2x(2x\left(x+y-\frac{5}{2}\right)+1)+(4x+2y-5)) & e^{x^2+y^2}(2y(2x\left(x+y-\frac{5}{2}\right)+1)+2x) \\ e^{x^2+y^2}(2x(2y\left(x+y-\frac{5}{2}\right)+1)+2y) & e^{x^2+y^2}(2y(2y\left(x+y-\frac{5}{2}\right)+1)+(2x+4y-5)) \end{pmatrix}.$$

At the point $(x,y) = (1,1)$, this becomes

$$\begin{pmatrix} e^2 & 2e^2 \\ 2e^2 & e^2 \end{pmatrix}.$$

Its determinant is $-3e^4 < 0$, therefore, it has both negative and positive eigenvalues, and the point $(1,1)$ is a saddle point.

At the point $(x,y) = (\frac{1}{4},\frac{1}{4})$, this becomes

$$\begin{pmatrix} -\frac{7}{2}e^{\frac{1}{8}} & \frac{1}{2}e^{\frac{1}{8}} \\ \frac{1}{2}e^{\frac{1}{8}} & -\frac{7}{2}e^{\frac{1}{8}} \end{pmatrix}.$$

Its determinant is $12e^{\frac{1}{4}} > 0$, and its trace is $-7e^{\frac{1}{8}}$, therefore, its eigenvalues are negative and the point $(\frac{1}{4},\frac{1}{4})$ is a relative maximum.

3. Let C be the curve $\{(x,y) : x^2 + (y-1)^2 = 1, x \geq 0\}$ in $\mathbb{R}^2$. Find a parametrization $\boldsymbol{\alpha}(t)$ of C starting at $(0,0)$ and ending at $(0,2)$, and compute the line integral

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha},$$

where $\boldsymbol{f}(x,y) = (y, x^2)$ is a vector field in $\mathbb{R}^2$.

Solution.

The equation $x^2 + (y-1)^2 = 1$ represents the circle centered at $(0,1)$ with radius 1. The condition $x \geq 0$ takes the right half of it. As $\boldsymbol{\alpha}(t)$ shoul starts at $(0,0)$ and end at $(0,2)$, such a parametrization is given by

$$\boldsymbol{\alpha}(t) = (\cos t, \sin t + 1), \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

We have $\boldsymbol{f}(\boldsymbol{\alpha}(t)) = (\sin t + 1, \cos^2 t)$ and $\boldsymbol{\alpha}'(t) = (-\sin t, \cos t)$. The line integral is then computed as

$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \boldsymbol{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t + 1, \cos^2 t) \cdot (-\sin t, \cos t)dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-\sin^2 t - \sin t + \cos^3 t)dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\cos 2t - 1}{2} - \sin t + (1 - \sin^2 t)\cos t\right) dt \\
&= \left[\frac{\cos 2t - 1}{2} - \sin t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-1}^{1} (1 - s^2)ds \\
&= -\frac{\pi}{2} + \left[s - \frac{s^3}{3}\right]_{-1}^{1} = \frac{4}{3} - \frac{\pi}{2}.
\end{aligned}$$

4. Find the volume of the set $D \subset \mathbb{R}^3$ which is contained inside the cylinder of equation $x^2 + y^2 = 1$ and bounded by the surfaces of equation $z = x^2 + y^2 - 2$ and $x + y + z = 4$.

Solution. Note that, under the condition that $x^2 + y^2 < 1$, $z = x^2 + y^2 - 2 < -1$ and $z = 4 - x - y > 4 - \sqrt{2}$, hence the surface defined by the former lies below that defined by the latter.

By definition, we can represent D as

$$D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 4 - x - y \leq z \leq x^2 + y^2 - 2\}.$$

With the region $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$, this is already xy-projectable. Therefore, to obtain the volume of D, we have to integral the function 1 in D:

$$\begin{aligned}\iiint_D dxdydz &= \iint_S dxdy \int_{x^2+y^2-2}^{4-x-y} dz \\ &= \iint_S (4 - x - y - (x^2 + y^2 - 2))dxdy.\end{aligned}$$

Here, the region S is symmetric in x while the function $f(x) = x$ is anti-symmetric ($f(-x) = -f(x)$), therefore, its integral is 0. Similarly, the integral of y over S is 0. For the rest, we use the polar coordinates $x = r\cos\theta, y = r\sin\theta$ with the Jacobian $J(r, \theta) = r$, and S corresponds to the region $\tilde{S} = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$:

$$\begin{aligned}&\iint_S (4 - (x^2 + y^2 - 2))dxdy = \iint_S (6 - x^2 - y^2)dxdy = \int_0^{2\pi} \int_0^1 (6 - r^2)rdrd\theta \\ &= \int_0^{2\pi} \left[3r^2 - \frac{r^4}{4}\right]_0^1 = 2\pi\left(3 - \frac{1}{4}\right) = \frac{11\pi}{2}.\end{aligned}$$

5. Let $\boldsymbol{F}(x,y,z) = (x^2 + x - 2xy^2, y^3 + 4yz^2, x^2z)$ be a vector field on $\mathbb{R}^3$, S be the surface of the ellipsoid:

$$S := \{(x,y,z) : (x+1)^2 + y^2 + 4z^2 = 4\},$$

and $\boldsymbol{n}$ the outgoing normal unit vector on S at each point of S.

Compute the surface integral

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS.$$

Solution.

As S is the surface of the closed ellipsoid, we can apply Gauss' theorem and obtain

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iiint_V \operatorname{div} \boldsymbol{F}\, dxdydz$$

where $V = \{(x,y,z) : (x+1)^2 + y^2 + 4z^2 \leq 4\}$.

Let us compute:

$$\operatorname{div} \boldsymbol{F} = 2x + 1 - 2y^2 + 3y^2 + 4z^2 + x^2 = (x+1)^2 + y^2 + 4z^2.$$

To perform the volume integral, we first make the change of coordinate: $u = (x+1), y = v, w = 2z$, and the Jacobian determinant of this transformation is $\frac{1}{2}$, and V corresponds to $\tilde{V} = \{(u,v,w) : u^2 + v^2 + w^2 \leq 4\}$: therefore,

$$\iiint_V \operatorname{div} \boldsymbol{F}\, dxdydz = \iiint_V ((x+1)^2 + y^2 + 4z^2)\, dxdydz = \iiint_{\tilde{V}} \frac{1}{2}(u^2 + v^2 + w^2)\, dudvdw$$

Now we use the spherical coordinate $u = r\cos\theta\sin\varphi, v = r\sin\theta\sin\varphi, w = r\cos\varphi$. The region $\tilde{V}$ correponding to Q in this change of coordinate is $Q = \{(r,\theta,\varphi) : 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \pi\}$, and the Jacobian determinant is $J(r,\theta,\varphi) = -r^2\sin\varphi$. Theorefore,

$$\begin{aligned}
&\iiint_{\tilde{V}} \frac{1}{2}(u^2 + v^2 + w^2)\, dudvdw = \frac{1}{2}\iiint_Q r^2 r^2 \sin\varphi dr d\theta d\varphi \\
&= \frac{1}{2}\int_0^{\pi}\int_0^{2\pi}\int_0^2 r^4 \sin\varphi dr d\theta d\varphi = \frac{1}{2}\int_0^{\pi}\int_0^{2\pi}\left[\frac{r^5}{5}\right]_0^2 \sin\varphi dr d\theta d\varphi \\
&= \frac{64}{5}\pi.
\end{aligned}$$