BSc Engineering Sciences – A. Y. 2018/19

Written exam of the course Mathematical Analysis 2

June 24, 2019

Last name: .... First name: ....

Matriculation: ....

Solve the following problems, motivating in detail the answers.

1.

(1) Find a function $g : \mathbb{R} \to \mathbb{R}$ such that the solution $u(x,t)$ of the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = \dfrac{1}{1+x^2} \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$$

is given by $u(x,t) = \frac{1}{1+(x-ct)^2}$. (Note after the exam: in the original paper there was a typo in the last line: $\frac{\partial u}{\partial x}$ should have been $\frac{\partial u}{\partial t}$).

(2) Find a function $g : (0, +\infty) \to \mathbb{R}$ such that $f(x,y) = g(\sqrt{x^2+y^2})$ is a solution of the 2-dimensional Poisson equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \sqrt{x^2+y^2}$$

(Hint: express the Laplacian $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ in polar coordinates.)

Solution.

(1) As $u(x,t) = \frac{1}{1+(x-ct)^2}$ is given, we can compute

$$\frac{\partial u}{\partial x}(x,t) = \frac{-2(x-ct)}{(1+(x-ct)^2)^2}$$

and hence $g(x) = \frac{\partial u}{\partial x}(x,0) = \frac{-2x}{(1+x^2)^2}$.

(2) We know that, with the polar coordinates $x = r\cos\theta, y = r\sin\theta$,

$$\frac{\partial^2 f}{\partial x^2}(r,\theta) + \frac{\partial^2 f}{\partial y^2}(r,\theta) = \frac{\partial^2 f}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial f}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}(r,\theta).$$

The function $f(x,y) = g(\sqrt{x^2+y^2})$ depends only on $r = \sqrt{x^2+y^2}$, therefore, $f(r,\theta) = g(r)$. The given equation is equivalent to

$$r = \frac{\partial^2 f}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial f}{\partial r}(r,\theta) = \frac{d^2 g}{dr^2}(r) + \frac{1}{r}\frac{dg}{dr}(r),$$

which, in terms of $h(r) := g'(r)$ becomes the first order linear ordinary differential equation

$$h'(r) + \frac{1}{r}h(r) = r.$$

This has the general solution

$$h(r) = e^{-\int \frac{dr}{r}} \left(\int dr\, e^{\int \frac{dr}{r}} r + c_1 \right) = \frac{r^2}{3} + \frac{c_1}{r}$$

and therefore $g(r) = \int dr\, h(r) = \frac{r^3}{9} + c_1 \log r + c_2$, with $c_1, c_2 \in \mathbb{R}$ arbitrary integration consants. Therefore the required solution is

$$f(x,y) = \frac{1}{9}(x^2+y^2)^{\frac{3}{2}} + \frac{c_1}{2}\log(x^2+y^2) + c_2.$$

Matriculation: ....

2.

(1) Let us set $(x_1, y_1) = (1, 2), (x_2, y_2) = (2, 3), (x_3, y_3) = (-1, 1)$. Find and classify all the stationary points $(a, b) \in \mathbb{R}^2$ of the function $f(a, b) = \sum_{n=1}^{3}(ax_n + b - y_n)^2$.

(2) Compute the derivative of the following function $g(x)$ of x:

$$g(x) = \int_{-\sin x}^{1} \cos(t^3)dt.$$

Solution.
(1) For the f given above, it holds that

$$\nabla f(a, b) = \left(\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}\right) = \left(\sum_{n=1}^{3} 2x_n(ax_n + b - y_n), \sum_{n=1}^{3} 2(ax_n + b - y_n)\right).$$

By putting the concrete coordinates $(x_1, y_1) = (1, 2), (x_2, y_2) = (2, 3), (x_3, y_3) = (-1, 1)$, this is

$\nabla f(a, b)$
$= (2(a + b - 2) + 4(2a + b - 3) - 2(-a + b - 1),\ 2(a + b - 2) + 2(2a + b - 3) + 2(-a + b - 1))$
$= (2(6a + 2b - 6),\ 2(2a + 3b - 6))$

At stationary points, $\nabla f(a, b) = \mathbf{0}$ holds. Namely,

$$2(6a + 2b - 6) = 0,\ 2(2a + 3b - 6) = 0.$$

By solving these equations, we obtain $a = \frac{9}{14}, b = \frac{11}{7}$.
To classify this point, let us compute the Hessian matrix:

$$\begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}.$$

Its determinant is $6 \times 3 - 2 \times 2 = 14 > 0$, its trace is $6 + 3 = 9 > 0$, therefore, the eigenvalues are positive, and the point $(\frac{9}{14}, \frac{11}{7})$ is a minimum.
(2)
Let us put $F(t)$ a primitive function of $\cos(t^3)$, namely $F'(t) = \cos(t^3)$, then $g(x) = F(1) - F(-\sin x)$. By the chain rule, $g'(x) = -(-\cos x) \cdot F'(-\sin x) = \cos x \cos((-\sin x)^3) = \cos x \cos(-\sin^3 x)$.

Matriculation: ....

3. Determine whether the following vector field on $\mathbb{R}^2$

$$\boldsymbol{f}(x,y) = \big(\cos xy - xy\sin xy,\ -x^2\sin xy + x^3\big)$$

is a gradient of some scalar field. Depending on this result,

- If $\boldsymbol{f}(x,y)$ is a gradient, find one of these scalar fields φ such that $\boldsymbol{f}(x,y) = \nabla\varphi(x,y)$.
- If $\boldsymbol{f}(x,y)$ is not a gradient, compute $\int_C \boldsymbol{f}\cdot d\boldsymbol{\alpha}$, where

$$\boldsymbol{\alpha}(t) = \begin{cases} (t,0) & 0\le t\le 1 \\ (1-(t-1),(t-1)) & 1\le t\le 2 \\ (0,1-(t-2)) & 2\le t\le 3 \end{cases}$$

Solution.
Let us call $\boldsymbol{f}(x,y) = (P(x,y),Q(x,y))$, where $P(x,y) = \cos xy - xy\sin xy$, $Q(x,y) = -x^2\sin xy + x^3$. We compute:

$$\frac{\partial P}{\partial y}(x,y) = -x\sin xy - x\sin xy - x^2y\cos xy = -2x\sin xy - x^2y\cos xy,$$
$$\frac{\partial Q}{\partial x}(x,y) = -2x\sin xy - x^2y\cos xy + 3x^2,$$

and we see that they are different. This implies that $\boldsymbol{f}$ is not a gradient.

To compute the line integral $\int_C \boldsymbol{f}\cdot d\boldsymbol{\alpha}$, this is equal to $\iint_S \frac{\partial Q}{\partial y}(x,y) - \frac{\partial P}{\partial x}(x,y)dxdy$ by Green' theorem, where S is the region surrounded by the curve C, namely S is the triangle $\{(x,y) : 0\le x\le 1, 0\le y\le 1-x\}$. We carry out the integral:

$$\begin{aligned}\int_C \boldsymbol{f}\cdot d\boldsymbol{\alpha} &= \iint_S \frac{\partial Q}{\partial y}(x,y) - \frac{\partial P}{\partial x}(x,y)dxdy \\ &= \iint_S 3x^2dxdy = 3\int_0^1 x^2\left[\int_0^{1-x} dy\right]dx \\ &= 3\int_0^1 x^2(1-x)dx = 3\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{4}.\end{aligned}$$

Matriculation: ....

4. Compute the integral

$$\iiint_D (\sin x + y^2) z \, dxdydz$$

where

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 \, : \, \sqrt{\frac{x^2}{9} + \frac{y^2}{4}} \le z \le \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \right\}.$$

Solution.

There is $z \in \mathbb{R}$ such that $\sqrt{\frac{x^2}{9} + \frac{y^2}{4}} \le z \le \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$ for some $x, y \in \mathbb{R}$ if and only if $\sqrt{\frac{x^2}{9} + \frac{y^2}{4}} \le \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$, or equivalently $\frac{x^2}{9} + \frac{y^2}{4} \le \frac{1}{2}$. Therefore, the region D is xy-projectable and can be written as

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 \, : (x,y) \in S, \sqrt{\frac{x^2}{9} + \frac{y^2}{4}} \le z \le \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \right\},$$

where $S = \left\{ (x,y,z) \in \mathbb{R}^2 \, : \, 0 \le \frac{x^2}{9} + \frac{y^2}{4} \le \frac{1}{2} \right\}$.

The integral can be reduced to

$$\begin{aligned}
\iiint_D (\sin x + y^2) z \, dxdydz &= \iint_S (\sin x + y^2) \left[\int_{\sqrt{\frac{x^2}{9} + \frac{y^2}{4}}}^{\sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}} z dz \right] dxdy \\
&= \frac{1}{2} \iint_S (\sin x + y^2) \left[z^2 \right]_{\sqrt{\frac{x^2}{9} + \frac{y^2}{4}}}^{\sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}} dxdy \\
&= \iint_S (\sin x + y^2) \left(\frac{1}{2} - \frac{x^2}{9} - \frac{y^2}{4} \right) dxdy
\end{aligned}$$

As $\sin x$ is an antisymmetric function ($\sin(-x) = -\sin x$) and $\left(\frac{1}{2} - \frac{x^2}{9} - \frac{y^2}{4} \right)$ is symmetric in x, and the integral region is symmetric in x, the term with $\sin x$ vanishes.

If we change to the coordinate $x = 3r\cos\theta, y = 2r\sin\theta$, then the Jacobian is

$$\det \begin{pmatrix} 3\cos\theta & 2\sin\theta \\ -3r\sin\theta & 2r\cos\theta \end{pmatrix} = 6r,$$

and S corresponds to $\tilde{S} = \{(r,\theta) : 0 \le r \le \frac{1}{\sqrt{2}}, 0 \le \theta \le 2\pi\}$. The remaining integral becomes

$$\begin{aligned}
&\iint_S y^2 \left(\frac{1}{2} - \frac{x^2}{9} - \frac{y^2}{4} \right) dxdy = \iint_{\tilde{S}} 4r^2 \sin^2\theta \left(\frac{1}{2} - r^2 \right) 6r dr d\theta \\
&= \int_0^{2\pi} \left[\int_0^{\frac{1}{\sqrt{2}}} \sin^2\theta \left(12r^3 - 24r^5 \right) dr \right] d\theta = \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{2\pi} \left[3r^4 - 4r^6 \right]_0^{\frac{1}{\sqrt{2}}} \\
&= \pi \left(\frac{3}{4} - \frac{1}{2} \right) = \frac{\pi}{4}.
\end{aligned}$$

Matriculation: ....

5. Let $\boldsymbol{F}(x,y,z) = (xz, yz, 0)$ be a vector field on $\mathbb{R}^3$ and

$$S = \{(x,y,z) : x^2 + y^2 - z^2 = -4,\ 0 \le z \le 3\}$$

be a surface in $\mathbb{R}^3$. Compute the surface integral

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS,$$

where $\boldsymbol{n}$ is a unit normal vector on S with positive z-component.

Solution.

$x^2 + y^2 - z^2 = -4 \iff x^2 + y^2 = z^2 - 4$, hence there is such a triple (x,y,z) of real numbers if and only if $z^2 \ge 4$. Together with the condition $0 \le z \le 3$, this means that $2 \le z \le 3$. Correspondingly, $0 \le x^2 + y^2 \le 5$, and for such (x,y), there is a unique z with $x^2 + y^2 - z^2 = -4$, namely, $z = \sqrt{x^2 + y^2 + 4}$.

Now we can paramezrize S by $\boldsymbol{r}(u,v) = (u, v, f(u,v))$, where $f(u,v) = \sqrt{u^2 + v^2 + 4}$. Then $\frac{\partial \boldsymbol{r}}{\partial u}(u,v) = (1, 0, \frac{u}{\sqrt{u^2+v^2+4}})$, $\frac{\partial \boldsymbol{r}}{\partial v}(u,v) = (0, 1, \frac{v}{\sqrt{u^2+v^2+4}})$ and

$$\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}(u,v) = \left(\frac{-u}{\sqrt{u^2+v^2+4}}, \frac{-v}{\sqrt{u^2+v^2+4}}, 1 \right).$$

Therefore, by the formula for the surface integral of a vector field,

$$\begin{aligned}
&\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS \\
&= \iint_{0 \le u^2+v^2 \le 5} (u\sqrt{u^2+v^2+4}, v\sqrt{u^2+v^2+4}, 0) \cdot \left(\frac{-u}{\sqrt{u^2+v^2+4}}, \frac{-v}{\sqrt{u^2+v^2+4}}, 1 \right) dudv \\
&= -\iint_{0 \le u^2+v^2 \le 5} (u^2 + v^2) dudv = -\int_0^{2\pi} \left[\int_0^{\sqrt{5}} r^2 r dr \right] d\theta = -\frac{1}{4} \cdot 2\pi \cdot 25 = -\frac{25\pi}{2}.
\end{aligned}$$

Note: as the surface S is not the boundary of a solid, one cannot use directly Gauss' theorem. One way to use it is to add the upper surface $\tilde{S} = \{(x,y,z) : x^2 + y^2 = 5\}$. Then $S \cup \tilde{S}$ is the boundary of the solid $V = \{(x,y,z) : x^2 + y^2 - z^2 \le -4,\ 2 \le z \le 3\}$. Then one can apply Gauss' theorem, but now the vector field in the question is incoming for V, hence one has to reverse the sign. Now the volume integral reduces to the integral of the function $2z$, which is not diffucult.