BSc Engineering Sciences – A. Y. 2018/19

Written exam of the course Mathematical Analysis 2

February 14, 2019

1. Find the Taylor series expansion, with intial point $x_0 = 1$, of the function

$$f(x) = \frac{x}{(x-2)(x^2-2x+2)},$$

determine its radius of convergence r, and study the convergence for $x = 1 \pm r$.

Solution. Note first that

$$\begin{aligned} f(x) &= \frac{x}{(x-2)(x^2-2x+2)} = \frac{1}{x-2} - \frac{x-1}{x^2-2x+2} \\ &= \frac{1}{(x-1)-1} - \frac{x-1}{(x-1)^2+1} \end{aligned}$$

As $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ is a geometric series, we have

$$\frac{1}{(x-1)-1} = -\sum_{n=0}^{\infty}(x-1)^n.$$

Similarly, from $\frac{1}{y+1} = \sum_{n=0}^{\infty}(-y)^n$, we obtain

$$\frac{x-1}{(x-1)^2+1} = (x-1)\cdot\sum_{n=0}^{\infty}(-(x-1)^2)^n = \sum_{n=1}^{\infty}(-1)^n(x-1)^{2n+1}.$$

Altogether,

$$\begin{aligned} f(x) &= -\sum_{n=0}^{\infty}(x-1)^n - \sum_{n=0}^{\infty}(-1)^n(x-1)^{2n+1} \\ &= \sum_{n=0}^{\infty} a_n(x-1)^n, \end{aligned}$$

where

$$a_n = \begin{cases} -1 & \text{if } n = 2k, \\ -1 + (-1)^{k+1} & \text{if } n = 2k+1. \end{cases}$$

As both $\frac{1}{(x-1)-1} = -\sum_{n=0}^{\infty}(x-1)^n$ and $(-(x-1)^2)^n = \sum_{n=1}^{\infty}(-1)^n(x-1)^{2n+1}$ have radius of convergence 1, hence the radius of convergence of the sum is equal or larger than 1. If $|x-1| > 1$, the terms of the series are growing in absolute value, hence the series diverges. Therefore, the radius of convergence is 1.

At $x - 1 = 1$, the series becomes

$$\sum_{n=0}^{\infty} a_n = -1 - 2 - 1 + 0 - 1 - 2 - \cdots$$

which is divergent to $-\infty$. At $x-1=-1$, the series becomes

$$\sum_{n=0}^{+\infty}(-1)^n a_n = -1+2-1+0-1+2-\cdots$$

therefore, this is oscillating (neither convergent nor divergent).

Matriculation: ....

2.

(1) Find the extremal values of the function $f(x,y,z) = x+2y+2z$ on the surface S defined by $x^2+y^2+z^2 = 1$.

(2) Let $g(x,y)$ the function implicitly defined by $x^2+y^2+g(x,y)^2 = 1, g(x,y) > 0$ and $h(x,y) = x+2y+2g(x,y)$. Compute $\dfrac{\partial h}{\partial x}(x_0,y_0), \dfrac{\partial h}{\partial y}(x_0,y_0)$, where (x_0,y_0,z_0) is the maximum of (1).

Solution. (1) Put $G(x,y,z) = x^2+y^2+z^2-1$. By Lagrange's multiplier method, there is $\lambda \in \mathbb{R}$ such that $\lambda \nabla f(x,y,z) = \nabla G(x,y,z)$ at stationary points (x,y,z). Let us compute these gradients:

$$\nabla f(x,y,z) = (1,2,2),$$
$$\nabla G(x,y,z) = (2x,2y,2z).$$

From the equation of the multiplier method, for a stationary point (x,y,z), we have

$$(\lambda, 2\lambda, 2\lambda) = (2x,2y,2z),$$

or equivalently, $(x,y,z) = (\frac{\lambda}{2},\lambda,\lambda)$. As (x,y,z) must satisfy $G(x,y,z) = x^2+y^2+z^2-1 = 0$, we have $\frac{\lambda^2}{4}+\lambda^2+\lambda^2-1 = 0$. By solving this, $\lambda^2 = \frac{4}{9}, \quad \lambda = \pm\frac{2}{3}$. By substituting this to the equation above, we obtain

$$(x,y,z) = (\tfrac{1}{3},\tfrac{2}{3},\tfrac{2}{3}), \quad (-\tfrac{1}{3},-\tfrac{2}{3},-\tfrac{2}{3}).$$

At $(\frac{1}{3},\frac{2}{3},\frac{2}{3})$, we have $f(\frac{1}{3},\frac{2}{3},\frac{2}{3}) = 3$, and at $(-\frac{1}{3},-\frac{2}{3},-\frac{2}{3})$ we have $f(-\frac{1}{3},-\frac{2}{3},-\frac{2}{3}) = -3$. Therefore, $(\frac{1}{3},\frac{2}{3},\frac{2}{3})$ is the maximum and $(-\frac{1}{3},-\frac{2}{3},-\frac{2}{3})$ is the minimum.
(2) For the function $g(x,y)$ implicity defined by $G(x,y,z) = 0$, at (x_0,y_0,z_0) we have

$$\frac{\partial g}{\partial x}(x_0,y_0) = -\frac{\frac{\partial G}{\partial x}(x_0,y_0,z_0)}{\frac{\partial G}{\partial z}(x_0,y_0,z_0)}, \qquad \frac{\partial g}{\partial y}(x_0,y_0) = -\frac{\frac{\partial G}{\partial y}(x_0,y_0,z_0)}{\frac{\partial G}{\partial z}(x_0,y_0,z_0)}.$$

As $\frac{\partial G}{\partial x} = 2x, \frac{\partial G}{\partial y} = 2y, \frac{\partial G}{\partial z} = 2z$, it holds that $\frac{\partial g}{\partial x}(x_0,y_0) = -\frac{x_0}{z_0}, \frac{\partial g}{\partial y}(x_0,y_0) = -\frac{y_0}{z_0}$.
To compute the derivatives of h:

$$\frac{\partial h}{\partial x}(x_0,y_0) = 1+2\frac{\partial g}{\partial x}(x_0,y_0) = 1-2\cdot\frac{1}{2} = 0, \quad \frac{\partial h}{\partial y}(x_0,y_0) = 2+2\frac{\partial g}{\partial x}(x_0,y_0) = 2-2\cdot 1 = 0.$$

Note: this can be understood that in (1) we found the extremal points of $h(x,y)$, hence $\nabla(x_0,y_0) = \mathbf{0}$.

Matriculation: ....

3.

(1) Let $c > 0$. Find the solution $f(x,t)$ of the partial differential equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

with the initial condition $f(x,0) = \dfrac{\sin x}{x^2+1}$, $\dfrac{\partial f}{\partial t}(x,0) = xe^{-x^2}$.

(2) Find $\alpha > 0$ for which the function $g(x,y,z) = \dfrac{1}{(x^2+y^2+z^2)^\alpha}$ satisfies the partial differential equation on $\mathbb{R}^3 \setminus \{(0,0,0)\}$:

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = 0.$$

Solution.

(1) A general solution of this equation (1d wave equation) is given by

$$f(x,t) = \frac{1}{2}\left(F(x-ct) + F(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds,$$

where $F(s)$ is a twice continuously differentiable function, $G(s)$ is a once continuously differentiable function. Furthermore, it holds that $f(x,0) = F(x)$ and $\frac{\partial f}{\partial t}(x,0) = G(x)$. We are given the initial conditions $f(x,0) = \frac{\sin x}{x^2+1}$, $\frac{\partial f}{\partial t}(x,0) = xe^{-x^2}$, hence we can take $F(s) = \frac{\sin s}{s^2+1}$, $G(s) = se^{-s^2}$. Note that $\int G(s)ds = -\frac{1}{2}e^{-s^2} + \text{Const}$. Altogether, we have

$$f(x,t) = \frac{1}{2}\left(\frac{\sin(x-ct)}{(x-ct)^2+1} + \frac{\sin(x+ct)}{(x+ct)^2+1}\right) + \frac{1}{4c}\left(e^{-(x-ct)^2} - e^{-(x+ct)^2}\right).$$

(2) By chain rule, we have

$$\frac{\partial g}{\partial x} = \frac{-2\alpha x}{(x^2+y^2+z^2)^{\alpha+1}}, \quad \frac{\partial g}{\partial y} = \frac{-2\alpha y}{(x^2+y^2+z^2)^{\alpha+1}}, \quad \frac{\partial g}{\partial z} = \frac{-2\alpha z}{(x^2+y^2+z^2)^{\alpha+1}}.$$

Continuing to the second derivative,

$$\begin{aligned}
\frac{\partial^2 g}{\partial x^2} &= \frac{-2\alpha}{(x^2+y^2+z^2)^{\alpha+1}} + \frac{4\alpha(\alpha+1)x^2}{(x^2+y^2+z^2)^{\alpha+2}}, \\
\frac{\partial^2 g}{\partial y^2} &= \frac{-2\alpha}{(x^2+y^2+z^2)^{\alpha+1}} + \frac{4\alpha(\alpha+1)y^2}{(x^2+y^2+z^2)^{\alpha+2}}, \\
\frac{\partial^2 g}{\partial z^2} &= \frac{-2\alpha}{(x^2+y^2+z^2)^{\alpha+1}} + \frac{4\alpha(\alpha+1)z^2}{(x^2+y^2+z^2)^{\alpha+2}}.
\end{aligned}$$

Therefore, $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = \frac{-6\alpha}{(x^2+y^2+z^2)^{\alpha+1}} + \frac{4\alpha(\alpha+1)(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\alpha+2}} = \frac{-6\alpha+4\alpha(\alpha+1)}{(x^2+y^2+z^2)^{\alpha+1}}$. For this to be 0, $\alpha = \frac{1}{2}$ is necessary and sufficient.

Matriculation: ....

4. Compute the integral

$$\iiint_D z(x^2+y^2+z^2)e^{-(x^2+y^2)}\,dxdydz,$$

with $D = \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2+y^2+z^2 \le 1,\, z \ge \sqrt{x^2+y^2}\}$.

Solution. The condition $x^2+y^2+z^2 \le 1$ is equivalent to $-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$, hence, we can represent

$$\begin{aligned} D &= \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2+y^2+z^2 \le 1,\, z \ge \sqrt{x^2+y^2}\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \,:\, -\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}\, z \ge \sqrt{x^2+y^2}\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \,:\, \sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2}\} \end{aligned}$$

and from the last expression D is an xy-projectable solid. Moreover, the inequality in the last expression $\sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2}$ holds only if $\sqrt{x^2+y^2} \le \sqrt{1-x^2-y^2}$, which is equivalent to $x^2+y^2 \le \frac{1}{2}$. Hence we can further rewrite with $D_0 = \{(x,y) \in \mathbb{R}^2 \,:\, x^2+y^2 \le \frac{1}{2}\}$:

$$D = \{(x,y,z) \in \mathbb{R}^3 \,:\, (x,y) \in D_0,\ \sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2}\}.$$

Therefore, the integral can be performed first with z:

$$\begin{aligned} \iiint_D z(x^2+y^2+z^2)e^{-(x^2+y^2)}\,dxdydz &= \iint_{D_0} \left[\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z(x^2+y^2+z^2)e^{-(x^2+y^2)}dz\right] dxdy \\ = \iint_{D_0} e^{-(x^2+y^2)} \left[\frac{z^2}{2}(x^2+y^2) + \frac{z^4}{4}\right]_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dxdy \\ = \iint_{D_0} e^{-(x^2+y^2)} \left(\frac{x^2+y^2}{2} - \frac{5(x^2+y^2)^2}{4} + \frac{(1-x^2-y^2)^2}{4}\right) dxdy \\ = \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} e^{-r^2}\left(\frac{r^2}{2} - \frac{5r^2}{4} + \frac{(1-r^2)^2}{4}\right) rdrd\theta = \frac{\pi}{2}\int_0^{\frac{1}{\sqrt{2}}} e^{-r^2}(r-4r^5)dr, \end{aligned}$$

where in the last steps we used the polar coordinates. By first making the change of variables $t = r^2$ and then integrating by parts, we obtain

$$\begin{aligned} \int e^{-r^2}(r-4r^5)dr = \frac{1}{2}\int e^{-t}(1-4t^2)dt &= \frac{1}{2}\left[-e^{-t}(1-4t^2) - 4\int e^{-t}2tdt\right] \\ &= \frac{1}{2}\left[-e^{-t}(1-8t-4t^2) - 8\int e^{-t}dt\right] \\ &= \frac{1}{2}e^{-t}(7+8t+4t^2) + c = \frac{1}{2}e^{-r^2}(7+8r^2+4r^4) + c \end{aligned}$$

hence the last integral is equal to

$$\frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}(7+8r^2+4r^4)\right]_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{4}\left(e^{-\frac{1}{2}}(7+4+1) - 7\right) = \frac{\pi}{4}\left(15e^{-\frac{1}{2}} - 7\right).$$

Matriculation: ....

5. Let $\boldsymbol{F}(x,y,z) = (x^3 - xy^2z, -xz^3 - xy^2z, y^3 - yz^2)$ be a vector field on $\mathbb{R}^3$, C be the circle

$$C = \{(x,y,z) : y^2 + z^2 = 1, x = 2\}.$$

Compute the line integral

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{\alpha},$$

where $\boldsymbol{\alpha} = (2, \sin t, \cos t), t \in [0, 2\pi]$.

Solution.

The circle C is contained in the plane $x = 2$ the latter can be parametrized in the uv-plane as $\boldsymbol{r}(u,v) = (2, v, u)$. In this parametrization, the circle C corresponds to a circle Γ in the uv-plane, which is parametrized by $\boldsymbol{\beta}(t) = (\cos t, \sin t)$. Indeed, $\boldsymbol{r}(\boldsymbol{\beta}(t)) = (2, v, u) = \boldsymbol{\alpha}(t)$. Note that $\boldsymbol{\beta}(t)$ is going counterclockwise.

The circle C is the boundary of the disk $S = \{(x,y,z) : y^2 + z^2 \leq 1, x = 2\}$, and in the uv-plane this corresponds to $T = \{(u,v) : u^2 + v^2 = 1\}$. Let us compute the fundamental vector product: $\frac{\partial \boldsymbol{r}}{\partial u} = (0,0,1)$, $\frac{\partial \boldsymbol{r}}{\partial v} = (0,1,0)$, hence $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = (-1,0,0)$. This is already a unit vector.

By Stokes' theorem, $\int_C \boldsymbol{F} \cdot d\boldsymbol{\alpha} = \iint_S \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n} dS$, where $\boldsymbol{n} = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}$ (note that we have checked that $\beta(t)$ goes counterclockwise in the parametrization $\boldsymbol{r}$). As $\boldsymbol{n} = (-1,0,0)$, it is enough to compute the x-component of $\operatorname{curl} \boldsymbol{F}$, which is $(3y^2 - z^2) - (-3xz^2 - xy^2) = (3+x)y^2 + (3x-1)z^2$. Therefore, $\operatorname{curl} \boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \boldsymbol{n}(\boldsymbol{r}(u,v)) = -5(u^2+v^2)$.

Altogether,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{\alpha} = \iint_S \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n} dS$$

$$\iint_T (-5(u^2+v^2)) du dv = -5 \int_0^{2\pi} \int_0^1 r^2 r dr d\theta$$

$$= -10\pi \left[\frac{r^4}{4}\right]_0^1 = -\frac{5\pi}{2}$$