The Principle of Least Action in Hamiltonian dynamics, Analysis and Symplectic geometry

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Introduction

Study of the dynamics of Hamiltonian systems

Order (stability) versus Chaos (instability)

- Methods from classical mechanics
- Perturbative methods (KAM theory,...)

- Variational methods (Aubry-Mather theory)
- Geometric methods (Symplectic geometry, Floer Homology,...)
- PDE methods (Hamilton-Jacobi, weak KAM theory, ...)
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Aubry - Mather theory

Variational methods based on the *Principle of Least Lagrangian Action* ("Nature is thrifty in all its actions", Pierre Louis Moreau de Maupertuis, 1744).

- Serge Aubry & John Mather ’80s: twist maps of the annulus;
- John Mather ’90s: Hamiltonian flows of *Tonelli* type.
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Tonelli Hamiltonian

Let $M$ be finite dimensional closed Riemannian manifold. $H \in C^2(T^* M, \mathbb{R})$ is said to be Tonelli if:

- $H$ is strictly convex in each fibre: $\partial^2_{pp} H(x, p) > 0$;
- $H$ is super-linear in each fibre:

$$\lim_{\|p\| \to +\infty} \frac{H(x, p)}{\|p\|} = +\infty \quad \text{uniformly in } x.$$
Examples of Tonelli Hamiltonians

- **Geodesic Flow**
  Let $g$ be a Riemannian metric on $M$. The Hamiltonian (or Kinetic energy)
  \[ H(x, p) = \frac{1}{2} \| p \|_x^2 := \frac{1}{2} g_x(p, p) \]
  corresponds to the geodesic flow on $M$.

- **Hamiltonians from classical mechanics** (Kinetic Energy + Potential Energy):
  \[ H(x, p) = \frac{1}{2} \| p \|_x^2 + U(x) \]
  where $U : M \to \mathbb{R}$ represents the potential energy.
Examples of Tonelli Hamiltonians

- **Geodesic Flow**
  Let \( g \) be a Riemannian metric on \( M \). The Hamiltonian (or Kinetic energy)

  \[
  H(x, p) = \frac{1}{2} ||p||_x^2 := \frac{1}{2} g_x(p, p)
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- **Hamiltonians from classical mechanics** (Kinetic Energy + Potential Energy):

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  H(x, p) = \frac{1}{2} ||p||_x^2 + U(x)
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  where \( U : M \to \mathbb{R} \) represents the potential energy.

The dynamics associated to any vector field \( X \) on \( M \) can be embedded into the flow of a Tonelli Hamiltonian \( H : T^*M \to \mathbb{R} \):

\[
H(x, p) = \frac{1}{2} ||p||_x^2 + p \cdot X(x).
\]
Let $H : T^* M \to \mathbb{R}$ a Tonelli Hamiltonian. We can associate to it the so-called Lagrangian function $L : TM \to \mathbb{R}$, where

$$L(x, v) := \sup_{p \in T_x^* M} (p \cdot v - H(x, p))$$

Euler-Lagrange equations: $\frac{d}{dt} \frac{\partial L}{\partial v} = \frac{\partial L}{\partial x} \quad \rightarrow \quad$ Euler-Lagrange flow.
Lagrangian formalism

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Euler-Lagrange equations: \[ \frac{d}{dt} \frac{\partial L}{\partial v} = \frac{\partial L}{\partial x} \rightarrow \text{Euler-Lagrange flow.} \]

The Hamiltonian flow and the Euler-Lagrange flow are equivalent from a dynamical system point of view:

\[ TM \xrightarrow{\Phi^L_t} TM \]
\[ T^* M \xrightarrow{\Phi^H_t} T^* M \]
Lagrangian formalism

The Euler-Lagrange flow has an interesting variational characterization in terms of the Lagrangian Action Functional.

If $\gamma : [a, b] \rightarrow M$ is an abs. cont. curve, we define its action as:

$$A_L(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, dt.$$ 

$\gamma$ is a solution of the Euler-Lagrange flow if and only if it is an extremal for the fixed-end variational problem.
These extremals are not necessarily minimisers, although they are *local* minimisers, *i.e.* for very short times.

**Example:** In the geodesic flow case not all geodesics are length minimising! But (global) minimising geodesics do exist.

**Questions:**

- Do *global* minimisers exist?
- What are their *dynamical*/*geometric* properties?
- Does this minimising property translate into some *rigid structure*?
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**Example:** In the geodesic flow case not all geodesics are length minimising! But (global) minimising geodesics do exist.

**Questions:**

- Do global minimisers exist? **YES** (Tonelli Theorem)
- What are their *dynamical/geometric* properties?
- Does this minimising property translate into some *rigid structure*?
Aubry-Mather theory

Idea:

Study **orbits** and **invariant probability measures** that minimise the Lagrangian action of $L$.
Aubry-Mather theory

Idea:

Study orbits and invariant probability measures that minimise the Lagrangian action of $L - \eta_c(x) \cdot v$, where $\eta_c$ is any smooth closed 1-form on $M$ with cohomology class $c$.

Observation:

- $\eta_c$ closed $\implies$ $L$ and $L - \eta_c$ have the same Euler-Lagrange flow.
- The corresponding Hamiltonian is $H(x, \eta_c(x) + p)$.
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Idea:
Study orbits and invariant probability measures that minimise the Lagrangian action of $L - \eta (x) \cdot v$, where $\eta$ is any smooth closed 1-form on $M$ with cohomology class $c$.

Observation:
• $\eta$ closed $\Rightarrow L$ and $L - \eta$ have the same Euler-Lagrange flow.
• The corresponding Hamiltonian is $H(x, \eta(x) + p)$.

Tonelli Hamiltonian systems

(Aubry-Mather theory)

(Invariant sets)

$\{\mathcal{M}_c\}_{c \in H^1(M; \mathbb{R})}$ & $\{\mathcal{A}_c\}_{c \in H^1(M; \mathbb{R})}$

Minim. Lagrangian action

Aubry-Mather sets
Aubry-Mather sets

The Aubry-Mather sets are:

- non-empty and compact;
- invariant under the Hamiltonian flow;
- supported on Lipschitz graphs
  (Mather’s graph theorem);

(Credits to Dr. Oliver Knill, Harvard)
Aubry-Mather sets

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In particular:

- Symplectic invariant ↔ Invariant under the action of simplectomorphisms.
- Lagrangian Structure ↔ The are supported on Lipschitz Lagrangian, graphs of cohomology class $c$. 
Proposition

If $\Lambda$ is an invariant Lagrangian graph in $(T^*M, \omega_{\text{stand}})$ with cohomology class $c$, then all orbits on $\Lambda$ (resp. invariant prob. measures supported on $\Lambda$), minimises the action of $L - \eta_c(x) \cdot v$, where $\eta_c$ is any smooth closed 1-form on $M$, with cohomology class $c$.

Therefore:

Aubry-Mather sets $\overset{\leftrightarrow}{\implies}$ Invariant Lagrangian graphs (when they exist)

Invariant Lagrangian graphs are very rare. What if they do not exist?
Proposition

If $\Lambda$ is an invariant Lagrangian graph in $(T^* M, \omega_{\text{stand}})$ with cohomology class $c$, then $\Lambda = \text{Graph}(c + du)$ and

$$H(x, c + du(x)) = \alpha(c).$$

Therefore, one can study viscosity solutions and subsolutions of Hamilton-Jacobi equations:

- **Aubry-Mather sets** $\longleftrightarrow$ supported on the “graphs” of the differentials of these weak solutions
  (Uniqueness set)

- **Weak KAM theory** $\longleftrightarrow$ Homogenization of Hamilton-Jacobi equation
  (Albert Fathi '90s)
  (à la Lions-Papanicolaou-Varadhan and Evans)
A more geometric characterization

For a fixed $c$, the sets $\mathcal{M}_c$ and $\mathcal{A}_c$ lie in an energy level $\{H(x, p) = \alpha(c)\}$. The function

$$\alpha : H^1(M; \mathbb{R}) \longrightarrow \mathbb{R}$$

is what we call Minimal average action or Effective Hamiltonian.

- It corresponds to the Homogenized Hamiltonian.
A more geometric characterization

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is what we call **Minimal average action** or **Effective Hamiltonian**.

- It corresponds to the **Homogenized Hamiltonian**.

This function is also related to what is called **Symplectic shape**:

- $\alpha(c) = \inf\{k : \text{the sublevel } \{H(x, p) \leq k\} \text{ contains Lagrangian graphs of cohomology class } c\}$.
- $\alpha$ is related to **Hofer geometry** on the group of Hamiltonian diffeomorphisms (Sorrentino-Viterbo, *Geom&Top* 2010)
Hamiltonian dynamics

Variational meth.

Minim. Lagr. Action

Variational meth.

PDE meth.

Sympl. Geom.

Hamilton-Jacobi eq.

Weak KAM

Aubry-Mather theory

Aubry-Mather sets

Lagrangian graphs

Geometric methods
Hamiltonian dynamics

Variational meth.

Minim. Lagr. Action

Aubry-Mather theory

Dynamics

Stable & Unstable orb.

Hamilton-Jacobi eq.

Weak KAM

Analysis

Regularity (sub)sol.

Lagrangian graphs

Sympl. Geom.

PDE meth.
Research Interests

Structure of these Action-minimizing sets and H-J equation:
- Generic topological properties. Symplectic and contact properties.
- Implications to dynamics and Symplectic geometry;
- Implications to the regularity of viscosity solutions and subsolutions of Hamilton-Jacobi equation;
- Generalised forms of Homogenization of Hamilton-Jacobi equation.

Properties of the minimal average action:
- Symplectic properties and relation to Hofer geometry.
- Regularity, lack of regularity and geometric/dynamical implications.

Birkhoff Billiards (proposal for a SIR project 2014):
- Rigidity phenomena; (Length) Spectral properties.
- Integrability and Birkhoff conjecture.
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A. S. and Claude Viterbo *Action minimizing properties and distances on the group of Hamiltonian diffeomorphisms*,

Daniel Massart and A. S., *Differentiability of Mather’s average action and integrability on closed surfaces*,

A. S., *Computing Mathers beta-function for Birkhoff billiards*,
Relation to Hofer geometry

Consider the group of (compactly supported) Hamiltonian diffeomorphism $\text{Ham}(M, \omega)$:

Fundamental (unsolved) question

What is the relation between the geometry of this curve and the dynamics of the system?
Relation to Hofer geometry

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![Diagram showing dynamical and geometric perspectives of a Hamiltonian system]

**Fundamental (unsolved) question**

What is the relation between the geometry of this curve and the dynamics of the system?

Minimal average action $\leftrightarrow$ Asymptotic distance from Identity
### Research Projects

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What is a (mathematical) billiard?

The billiard ball moves on a rectilinear path: when it hits the boundary it reflects elastically according to the standard reflection law:

angle of reflection = angle of incidence.

This is a conceptually simple model, yet mathematically very complicated,
In collaborations with Vadim Kaloshin (University of Maryland, USA) we have an ongoing project aimed at studying two important (and related) questions:

- **Is it possible to hear the shape of a billiard?** Can a planar convex domain be characterized in terms of the lengths of its periodic orbits, i.e., its *Length spectrum* (or *Marked length spectrum*), as conjectured by Guillemin and Melrose?

- **Birkhoff conjecture** on the integrability of convex billiards. Namely: the only *integrable* billiards are billiards in circles and ellipses.
Thank you for your attention!