

Minus class groups of the fields of the l -th roots of unity.

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Abstract. We show that for any prime number $l > 2$ the minus class group of the field of the l -th roots of unity $\mathbf{Q}(\zeta_l)$ admits a finite free resolution of length 1 as a module over the ring $\widehat{\mathbf{Z}}[G]/(1 + \iota)$. Here ι denotes complex conjugation in $G = \text{Gal}(\mathbf{Q}(\zeta_l)/\mathbf{Q}) \cong (\mathbf{Z}/l\mathbf{Z})^*$. Moreover, for the primes $l \leq 509$ we show that the minus class group is cyclic as a module over this ring. For these primes we also determine the structure of the minus class group.

Introduction.

Let l be an odd prime and let ζ_l denote a primitive l -th root of unity. In this paper we study the cyclotomic fields $\mathbf{Q}(\zeta_l)$ and the class groups Cl_l of their rings of integers $\mathbf{Z}[\zeta_l]$. The class group Cl_l splits in a natural way into two parts: the natural map from the class group Cl_l^+ of the ring of integers of the subfield $\mathbf{Q}(\zeta_l + \zeta_l^{-1})$ to Cl_l is injective [24, p.40]. Its cokernel, the *minus class group of $\mathbf{Q}(\zeta_l)$* , is denoted by Cl_l^- . There is an exact sequence

$$0 \longrightarrow Cl_l^+ \longrightarrow Cl_l \longrightarrow Cl_l^- \longrightarrow 0.$$

About the groups Cl_l^+ little is known. For small primes l they are trivial [23]. See [3, 21] for a numerical study of these groups. In this paper we consider the other groups, the minus class groups Cl_l^- , which are easier to handle. There is, first of all, an explicit and easily computable formula for their cardinalities h_l^- . See [24, p.42]:

$$h_l^- = 2l \prod_{\chi \text{ odd}} -\frac{1}{2} B_{1,\chi},$$

where the product runs over the characters $\chi : (\mathbf{Z}/l\mathbf{Z})^* \longrightarrow \mathbf{C}^*$ which are odd, i.e. which satisfy $\chi(-1) = -1$. The numbers $B_{1,\chi}$ are generalized Bernoulli numbers; they are defined in section 1.

Around 1850, E.E. Kummer [9, 10] used this formula to compute the minus class numbers h_l^- for the primes $l < 100$. These calculations were extended by D.H. Lehmer and J.M. Masley [15] in 1978 to the primes $l \leq 509$. The numbers h_l^- grow very rapidly with l . For instance, h_{491}^- has already 138 decimal digits.

The class number h_l^- alone does, of course, not determine the structure of the group Cl_l^- . If h_l^- is squarefree, the group Cl_l^- is cyclic, but in general h_l^- has multiple factors. It is a natural problem to try and determine the *structure* of the minus class groups. Kummer [12] addressed this problem in 1853. He showed, for instance, that for $l = 29$ the minus class group is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. He claimed moreover that the minus class group of $\mathbf{Q}(\zeta_{31})$ is cyclic of order 9. Only in 1870 he gave a rigorous proof of this fact [11]. It involves a lengthy calculation in the field $\mathbf{Q}(\zeta_{31})$. His claim that the group Cl_{71}^- is cyclic of order $7^2 \cdot 79241$ is correct, but has, as far as I know, never been justified previously [6].

In this paper we study the structure of the minus class groups Cl_l^- as Galois modules. Since complex conjugation ι acts as -1 on Cl_l^- , it is natural to study Cl_l^- as a module over the ring $\widehat{\mathbf{Z}}[G]/(1 + \iota)$ where $\widehat{\mathbf{Z}}$ denotes the profinite ring $\varprojlim \mathbf{Z}/n\mathbf{Z}$ and $G = \text{Gal}(\mathbf{Q}(\zeta_l)/\mathbf{Q}) \cong (\mathbf{Z}/l\mathbf{Z})^*$. We prove the following:

Theorem I. *Let l be an odd prime. Then there exist an exact sequence of $\widehat{\mathbf{Z}}[G]/(1 + \iota)$ -modules*

$$0 \longrightarrow L \xrightarrow{\Theta} L \longrightarrow Cl_l^- \longrightarrow 0$$

where L is free of finite rank over $\widehat{\mathbf{Z}}[G]/(1 + \iota)$.

Theorem I is an immediate consequence of Theorems 2.2(i) and 3.2(i). For small l we can be more precise:

Theorem II. *For $l \leq 509$ one can take L of rank 1 in Theorem I. In other words, the minus class group, is isomorphic to $\widehat{\mathbf{Z}}[G]/(1 + \iota, \Theta)$ as a $\widehat{\mathbf{Z}}[G]/(1 + \iota)$ -module. Moreover, for Θ one can take the modified Stickelberger element introduced in section 1.*

Theorem II is proved in section 4. In the course of the proof we determine completely the structure of the minus class groups Cl_l^- as abelian groups for $l \leq 509$. As an example we mention Cl_{491}^- , which we show to be isomorphic to a product of six cyclic groups:

$$\begin{aligned} & \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/982\mathbf{Z} \times \mathbf{Z}/10802\mathbf{Z} \times \mathbf{Z}/1868018926266582415566481720580405499878668/ \\ & /116196370441793818260257581579588321194122827298258625221939971178506931727800584004906\mathbf{Z}. \end{aligned}$$

Theorem II probably holds for several other primes l , but is definitely not true in general. It does, for instance, not hold for $l = 3299$. This follows from the fact that, when $l \equiv 3 \pmod{4}$, the minus class group Cl_l^- is cyclic over $\widehat{\mathbf{Z}}[G]/(1 + \iota)$ if and only if the class group of the quadratic subfield $\mathbf{Q}(\sqrt{-l}) \subset \mathbf{Q}(\zeta_l)$ is a cyclic group. Since the class group of $\mathbf{Q}(\sqrt{-3299})$ is isomorphic to $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/9\mathbf{Z}$, the group Cl_{3299}^- is *not* cyclic as a $\widehat{\mathbf{Z}}[G]/(1 + \iota)$ -module [13, p.80].

Finally, we single out a particularly simple consequence of our results. Roughly speaking, it says that for prime divisors p of $l - 1$, the p -part of Cl_l^- is cyclic whenever it is small.

Theorem III. *Let l and p be odd primes and let M denote the p -part of the minus class group of $\mathbf{Q}(\zeta_l)$. If $\#M$ divides $(l - 1)^2$, then M is a cyclic group.*

Theorem III is proved in section 2. Applying it with $l = 31$, $p = 3$ and $l = 71$, $p = 7$ respectively we obtain a proof of Kummer's claims. The condition that $\#M$ divide $(l - 1)^2$ cannot be relaxed further: in section 4 we show that the 5-part of the minus classgroup of $\mathbf{Q}(\zeta_{101})$ is isomorphic to $\mathbf{Z}/125\mathbf{Z} \times \mathbf{Z}/25\mathbf{Z}$.

Our method is, in some sense, a finite version of Iwasawa theory. It is closely related to V.A. Kolyvagin's work [7]. In order to obtain information about the structure of a certain χ -eigenspace of the p -part of a minus class group, we "deform" the Dirichlet character χ and study

the extension L corresponding to $\chi\psi$, where ψ is some character of p -power order. The generalized Bernoulli numbers $B_{1,\chi\psi}$ contain information about the χ -eigenspace of the class group of this extension. This information is obtained by viewing the field L as a “truncated” \mathbf{Z}_p -extension and by studying the χ -part of the minus class group of L by mimicking techniques from Iwasawa theory. The main results are Theorem III and the two criteria for cyclicity, Theorems 2.3 and 3.3.

The main difficulty in extending Theorem II to primes $l > 509$ is the size of the class numbers. For larger l one is bound to encounter composite numbers that cannot be factored within reasonable time. Sooner or later one will also encounter χ -parts that are *not* cyclic Galois modules. In these cases the methods of this paper do not apply.

The paper is organized as follows. In section 1 we briefly recall some well known facts concerning $\mathbf{Z}[G]$ -modules when G is a finite abelian group. In this section we also discuss some elementary properties of Stickelberger elements and generalized Bernoulli numbers. Even though there are similarities between the structure of the odd and even parts of the minus class groups, the differences are sufficiently big to merit separate treatment. In section 2 we consider the p -parts of minus class groups for odd primes p . In section 3 we do the same for $p = 2$. Finally, in section 4, we present the numerical results and prove Theorem II.

We need to know the complete prime decomposition of the class numbers h_l^- for $l \leq 509$. In the appendix a table of the prime factorizations of these numbers is given. This table is complete and supersedes the one computed by Lehmer and Masley [15]. The present table contains also the factorizations of the unfactored composite numbers in their table. I thank Arjen Lenstra, Peter Montgomery, Bob Silverman and Herman te Riele and for computing the unknown prime factors, François Morain for several primality proofs and Pietro Cornacchia for catching an error in Table 4.4.

1. Preliminaries.

In this section we recall some elementary facts concerning modules over group rings $\mathbf{Z}[G]$ when G is a finite abelian group. In addition we recall some basic properties of Stickelberger elements and generalized Bernoulli numbers.

Let G be a finite abelian group. For a G -module M , we denote by M^G the subgroup of G -invariant elements of M . Now fix a prime p and let

$$G \cong \pi \times \Delta,$$

where π is the p -part of G and Δ is the maximal subgroup of G of order prime to p . We write the group ring $\mathbf{Z}_p[G]$ as $\mathbf{Z}_p[\Delta][\pi]$. By the orthogonality relations there is an isomorphism of rings

$$\mathbf{Z}_p[\Delta] \cong \prod_{\chi} O_{\chi}.$$

Here χ runs over the characters $\chi : \Delta \rightarrow \overline{\mathbf{Q}}_p^*$ up to $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ -conjugacy. The rings O_{χ} are unramified extensions of \mathbf{Z}_p generated by the values of χ . They are $\mathbf{Z}_p[\Delta]$ -algebras via the rule $\sigma \cdot x = \chi(\sigma)x$ for $x \in O_{\chi}$ and $\sigma \in \Delta$. The ring isomorphism is given by mapping $\sigma \in \Delta$ to $\chi(\sigma)$ in each component O_{χ} . The residue field of O_{χ} is $\mathbf{F}_p(\zeta_d)$ where d is the order of χ .

Definition. Let M be a $\mathbf{Z}_p[G]$ -module and let $\chi : \Delta \rightarrow \overline{\mathbf{Q}}_p^*$ be a character. Equivalently, χ is a character of G of order prime to p . The χ -eigenspace $M(\chi)$ or χ -part of M is defined by

$$M(\chi) = M \otimes_{\mathbf{Z}_p[\Delta]} O_{\chi}.$$

We have a decomposition into eigenspaces of M :

$$M \cong \prod_{\chi} M(\chi),$$

where χ runs over the characters $\chi : \Delta \rightarrow \overline{\mathbf{Q}}_p^*$ up to $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ -conjugacy. Each eigenspace $M(\chi)$ is a module over the local ring $O_{\chi}[\pi]$. The residue field of this ring is equal to the residue field of O_{χ} which is $\mathbf{F}_p(\zeta_d)$, where d is the order of χ .

We frequently use the following properties of the Tate cohomology groups [2]. Let M be a G -module and let $P \subset \pi$. The natural action of P on the Tate cohomology groups $\widehat{H}^q(P, M)$ is trivial, but Δ acts, in general, in a non-trivial way. Note that the groups $\widehat{H}^q(P, M)$ are $\mathbf{Z}_p[\Delta]$ -modules, because they are killed by $\#P$.

Lemma 1.1. *Let p be a prime and let G be a finite abelian group. Let π and Δ be as above and let P be a subgroup of π .*

(i) *For every $\mathbf{Z}[G]$ -module M we have that $\widehat{H}^q(P, M^{\Delta}) \cong \widehat{H}^q(P, M)^{\Delta}$ for all $q \in \mathbf{Z}$.*

(ii) *For every $\mathbf{Z}_p[G]$ -module M and every character $\chi : \Delta \rightarrow \overline{\mathbf{Q}}_p^*$ we have that*

$$\widehat{H}^q(P, M(\chi)) \cong \widehat{H}^q(P, M)(\chi) \quad \text{for all } q \in \mathbf{Z}.$$

Proof. (i) Since the actions of Δ and P commute, the inclusion $i : M^{\Delta} \hookrightarrow M$ and the Δ -norm map $N : M \rightarrow M^{\Delta}$ are P -morphisms. The maps $i \cdot N$ and $N \cdot i$ induce multiplication by $\#\Delta$ on $\widehat{H}^q(P, M)^{\Delta}$ and $\widehat{H}^q(P, M^{\Delta})$ respectively. Since $\#\Delta$ and $\#P$ are coprime, multiplication by $\#\Delta$ is an isomorphism and (i) follows.

(ii) Since the actions of Δ and P commute, the eigenspaces $M(\chi)$ are P -modules. Taking the sum over the characters $\chi : \Delta \rightarrow \overline{\mathbf{Q}}_p^*$, up to $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ -conjugacy, of the natural maps $\widehat{H}^q(P, M(\chi)) \rightarrow \widehat{H}^q(P, M)(\chi)$, we obtain precisely the map $\bigoplus_{\chi} \widehat{H}^q(P, M(\chi)) \rightarrow \widehat{H}^q(P, M)$ induced by the isomorphism $\bigoplus_{\chi} M(\chi) \rightarrow M$. This proves (ii).

The remainder of this section is devoted to properties of Stickelberger elements and generalized Bernoulli numbers. Let $f \not\equiv 2 \pmod{4}$ be a conductor and let $G = (\mathbf{Z}/f\mathbf{Z})^*$. The *Stickelberger element* θ_f of conductor f is given by

$$\theta_f = \sum_{\substack{a=1 \\ \gcd(a,f)=1}}^f \left(\frac{a}{f} - \frac{1}{2} \right) [a]^{-1} \in \mathbf{Q}[G].$$

For any prime number p we write $G = \pi \times \Delta$ as above. We have $\mathbf{Q}_p[G] \cong \bigoplus_{\chi} K_{\chi}[\pi]$ where the sum runs over the characters $\chi : \Delta \rightarrow \overline{\mathbf{Q}}_p^*$ up to $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ -conjugacy and K_{χ} is the quotient field of O_{χ} . We denote the algebra homomorphism $\mathbf{Q}_p[G] \rightarrow K_{\chi}[\pi]$ induced by χ again by χ . For every character $\chi \neq \omega$, the image $\frac{1}{2}\chi(\theta_f)$ of $\frac{1}{2}\theta_f$ in $K_{\chi}[\pi]$ is an element of the subring $O_{\chi}[\pi]$. Here $\omega : (\mathbf{Z}/p\mathbf{Z})^* \rightarrow \overline{\mathbf{Q}}_p^*$ denotes the Teichmüller character. It is the character that gives the action of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the group μ_p of p -th roots of unity. Note that $\omega = 1$ when $p = 2$. For odd p the element $\frac{1}{2}\theta_f$ annihilates the χ -part of the p -part of the ideal class group of $\mathbf{Q}(\zeta_f)$. This is Stickelberger's Theorem [24, Chpt.6]. For $p = 2$, C. Greither [4] has shown the same when π is cyclic and the conductor f is odd.

For any character φ of G of conductor f , the generalized Bernoulli number $B_{1,\varphi}$ is simply the value of the algebra homomorphism $\mathbf{Q}_p[G] \rightarrow \overline{\mathbf{Q}}_p$ induced by φ evaluated on the Stickelberger element:

$$B_{1,\varphi} = \varphi(\theta_f) = \sum_{\substack{a=1 \\ \gcd(a,f)=1}}^f \left(\frac{a}{f} - \frac{1}{2} \right) \varphi(a)^{-1} \in \overline{\mathbf{Q}}_p.$$

Finally we assume that $f = l$ is prime, so that $G = (\mathbf{Z}/l\mathbf{Z})^*$ and we introduce the modified Stickelberger element $\Theta \in \widehat{\mathbf{Z}}[G]/(1 + \iota)$ that occurs in Theorem II. We have that $\widehat{\mathbf{Z}}[G]/(1 + \iota) \cong \prod_p \mathbf{Z}_p[G]/(1 + \iota)$. Moreover, each factor $\mathbf{Z}_p[G]/(1 + \iota)$ is isomorphic to $\prod_\chi O_\chi[\pi_p]$, where the χ run over all odd characters of order prime to p when p is odd and all characters of odd order when $p = 2$ respectively. Here π_p denotes the p -part of G . Therefore it suffices to describe the various components $\chi(\Theta)$ of Θ : if $p = l$ and $\chi = \omega$ or if $p = 2$ and $\chi = 1$, we let $\chi(\Theta) = 1$. In all other cases $\chi(\Theta) = \frac{1}{2}\chi(\theta_l)$.

The modified Stickelberger element $\Theta \in \widehat{\mathbf{Z}}[G](1 + \iota)$ annihilates Cl_l^- . The order of $\widehat{\mathbf{Z}}[G](1 + \iota, \Theta)$ is equal to the minus class number h_l^- .

2. Odd primes p .

In this section we study the p -parts of the minus class groups of complex abelian number fields for odd primes p . We show that certain eigenspaces of these groups are cohomologically trivial Galois modules. This puts restraints on their structure. We derive an easily applicable criterion for these eigenspaces to be cyclic Galois modules.

In this section $p \neq 2$ is a prime. We fix a complex abelian number K field with $G = \text{Gal}(K/\mathbf{Q})$. Let π denote the p -part of G and $F = K^\pi$ its fixed field. We fix an odd character $\chi : G \rightarrow \overline{\mathbf{Q}}_p^*$ of order prime to p , which is not equal to the Teichmüller character ω . Since $p \neq 2$, we have that $Cl_K^-(\chi) = Cl_K(\chi)$. Therefore we work, in this section, with the class group Cl_K itself rather than the minus class group Cl_K^- .

Theorem 2.1. *Let $P \subset G$ be a subgroup of π with fixed field $E = K^P$. Suppose that for all primes r that are ramified in $E \subset K$ we have that $\chi(r) \neq 1$. Then*

- (i) *the eigenspace $Cl_K(\chi)$ is a cohomologically trivial $O_\chi[P]$ -module;*
- (ii) *the natural map $Cl_E(\chi) \rightarrow Cl_K(\chi)^P$ is bijective and the norm map $Cl_K(\chi) \rightarrow Cl_E(\chi)$ is surjective.*

Proof. (i) It suffices to show that $\widehat{H}^q(P, Cl_K(\chi)) = 0$ for all $q \in \mathbf{Z}$. Let O_K denote the ring of integers of K , let C_K denote the idèle class group of K and let U_K denote the group of unit idèles, i.e. the group of K -idèles that have trivial valuation at all finite primes. We have the exact sequence of G -modules [2]

$$0 \rightarrow O_K^* \rightarrow U_K \rightarrow C_K \rightarrow Cl_K \rightarrow 0.$$

We show that the χ -parts of the Tate P -cohomology groups of these modules are all zero. For the unit group O_K^* we have the following exact sequence [24, p.39]

$$0 \rightarrow \{1, -1\} \rightarrow \mu_K \times O_{K^+}^* \rightarrow O_K^* \rightarrow Q \rightarrow 0.$$

Here O_{K^+} is the ring of integers of the maximal real subfield K^+ of K and μ_K denotes the group of roots of unity in K . The group Q has order at most 2. Complex conjugation acts trivially on $\{1, -1\}$, on Q and on $O_{K^+}^*$. Since χ is an odd character, we have, by Lemma 1.1, that $\widehat{H}^q(P, O_K^*)(\chi) \cong \widehat{H}^q(K, \mu_K)(\chi)$ for all $q \in \mathbf{Z}$. Since χ is not the Teichmüller character, the χ -part of μ_K is zero so that, by Lemma 1.1, $\widehat{H}^q(P, O_K^*)(\chi) = 0$ for all $q \in \mathbf{Z}$.

By global class field theory there are natural isomorphisms $\widehat{H}^q(P, C_K) \cong \widehat{H}^{q-2}(P, \mathbf{Z})$ for all $q \in \mathbf{Z}$. Since G acts trivially on \mathbf{Z} , it follows from Lemma 1.1 that $\widehat{H}^q(P, C_K)(\chi) = 0$ for all $q \in \mathbf{Z}$.

We use local class field theory to compute the cohomology of U_K . See also [20]. By Shapiro's lemma we have

$$\widehat{H}^q(P, U_K) \cong \bigoplus_v \widehat{H}^q(P_v, O_v^*) = \bigoplus_r \bigoplus_{v|r} \widehat{H}^q(P_v, O_v^*)$$

where v runs over the prime ideals of E and r runs over ordinary prime numbers. The ring O_w is the ring of integers of the completion K_w of K at a prime w of K over v . We have $\mathbf{Q}_r \subset E_v \subset K_w$ with Galois groups $G_r = \text{Gal}(K_w/\mathbf{Q}_r)$, $P_r = \text{Gal}(K_w/E_v)$ and $H_r = \text{Gal}(E_v/\mathbf{Q}_r)$. Since G is abelian, the decomposition groups P_r and H_r only depend on the prime r . Since $\widehat{H}^q(P_r, O_w^*)$ vanishes when v is unramified in K , it suffices to consider only primes r that are ramified in $E \subset K$. For each prime ideal v of F dividing a ramified prime r , there is an exact sequence of G_r -modules

$$0 \longrightarrow O_w^* \longrightarrow K_w^* \longrightarrow \mathbf{Z} \longrightarrow 0.$$

Consider the long exact sequence of Tate P_r -cohomology groups. By Lemma 1.1, the group H_r acts trivially on the cohomology groups $\widehat{H}^q(P_r, \mathbf{Z})$. By local class field theory there are natural isomorphisms $\widehat{H}^q(P_r, K_w^*) \cong \widehat{H}^{q-2}(P_r, \mathbf{Z})$ for all $q \in \mathbf{Z}$, so that H_r also acts trivially on the groups $\widehat{H}^q(P_r, K_w^*)$. Let Δ_r denote the maximal subgroup of H_r of order prime to p . Then Δ_r and P_r have coprime orders, so that the long cohomology sequence remains exact when we take Δ_r -invariants. It follows that $\widehat{H}^q(P_r, O_w^*)$ is Δ_r -invariant. Therefore Δ_r acts trivially on the sum $\bigoplus_{v|r} \widehat{H}^q(P_r, O_w^*)$. Since $\chi(r) \neq 1$ for all ramified primes r , we see that $\Delta_r \not\subset \ker(\chi)$. This implies that the χ -part of $\bigoplus_{v|r} \widehat{H}^q(P_r, O_w^*)$ is zero.

It follows that $\widehat{H}^q(G, U_K)(\chi) = 0$ for all $q \in \mathbf{Z}$. Combining all this and using Lemma 1.1 one more time, we deduce that $\widehat{H}^q(P, Cl_K(\chi)) = 0$ for all $q \in \mathbf{Z}$. This proves (i).

(ii) It is easy to see that the natural map $C_E/N(C_K) \rightarrow Cl_E/N(Cl_K)$ is surjective. Since $\chi \neq 1$, the group $C_E/N(C_K) = \widehat{H}^0(P, C_K) \cong \widehat{H}^{-2}(P, \mathbf{Z})$ has trivial χ -part, and it follows that the norm map $N : Cl_K(\chi) \rightarrow Cl_E(\chi)$ is surjective. Notice that in order to prove surjectivity of this norm map we have not really used the condition on χ , but merely the fact that χ is not trivial.

The P -cohomology groups of each module in the exact sequence $0 \rightarrow O_K^* \rightarrow U_K \rightarrow C_K \rightarrow Cl_K \rightarrow 0$ have trivial χ -parts. Since the natural maps $O_E^* \rightarrow O_K^{*P}$, $U_E \rightarrow U_K^P$ and $C_E \rightarrow C_K^P$ are all isomorphisms, so is $Cl_E(\chi) \rightarrow Cl_K(\chi)^P$. This proves (ii).

Theorem 2.2. *If for all primes r that are ramified in $F \subset K$ we have that $\chi(r) \neq 1$, then*

(i) *there is an exact sequence of $O_\chi[\pi]$ -modules*

$$0 \longrightarrow O_\chi[\pi]^d \xrightarrow{\Theta} O_\chi[\pi]^d \longrightarrow Cl_K(\chi) \longrightarrow 0$$

where d is the O_χ -rank of $Cl_F(\chi)$;

(ii) *we have*

$$\#Cl_K(\chi) = \#O_\chi / \left(\prod_{\psi} B_{1, \chi^{-1}\psi} \right)$$

where ψ runs over all characters $\psi : \pi \rightarrow \overline{\mathbf{Q}}_p^*$.

Proof. By Nakayama's lemma there is a surjective $O_\chi[\pi]$ morphism $O_\chi[\pi]^d \rightarrow Cl_K(\chi)$. By Theorem 2.1, the class group $Cl_K(\chi)$ and hence the kernel of this map are cohomologically trivial. Now one copies the proof of [2, p.113, Thm.8] with \mathbf{Z} replaced by the discrete valuation ring O_χ . It follows that the kernel is a projective $O_\chi[\pi]$ -module. Since $O_\chi[\pi]$ is local, the kernel is therefore free. It has rank d since it is of finite index in $O_\chi[\pi]^d$. This proves (i).

Part (ii) is a generalization of the Theorem of B. Mazur and A. Wiles [7, 16, 17, 18]. By D. Solomon's Theorem [22, p.472], we have for every subgroup $P \subset \pi$ with cyclic quotient π/P ,

$$\#Cl_{K^P}(\chi)[N_{P'}/N_P] = \#O_\chi / \left(\prod_{\ker \psi = P} B_{1, \chi^{-1}\psi} \right).$$

Here the ψ run over the characters of G for which $\ker \psi = P$. Here P' denotes the unique subgroup of π containing P as a subgroup of index p and N_P and $N_{P'}$ denote the norm maps $\sum_{\sigma \in P} \sigma$ and $\sum_{\sigma \in P'} \sigma$ respectively. In the exceptional case $P = \pi$ the group P' is not defined and we simply put $N_{P'} = 0$. By $Cl_K^P(\chi)$ we denote the kernel of the relative norm map $N_{P'}/N_P$ from the class group $Cl_K(\chi)$ to itself.

Put $S_\chi = \prod_P N_P O_\chi[\pi]/N_{P'} O_\chi[\pi]$. Here P runs over the subgroups of π with cyclic quotient π/P . The natural map

$$g : O_\chi[\pi] \longrightarrow S_\chi$$

becomes an isomorphism when we take the tensor product with the quotient field K_χ of O_χ . Therefore g is injective and has finite cokernel.

All modules occurring in the exact sequence of part (i) are cohomologically trivial. Therefore it remains exact when we apply the functor $\prod_P N_P(-)/N_{P'}(-)$ to it. We obtain the following diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & O_\chi[\pi]^d & \xrightarrow{\Theta} & O_\chi[\pi]^d & \longrightarrow & Cl_K(\chi) & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow g & & \downarrow & & \\ 0 & \longrightarrow & S_\chi^d & \longrightarrow & S_\chi^d & \longrightarrow & \prod_P N_P Cl_K(\chi)/N_{P'} Cl_K(\chi) & \longrightarrow & 0 \end{array}$$

Theorem 2.1(i) and (ii) and an application of the snake lemma gives then that

$$\#Cl_K(\chi) = \prod_P \#(N_P Cl_K(\chi)/N_{P'} Cl_K(\chi)) = \prod_P \#(Cl_{K^P}(\chi)[N_{P'}/N_P])$$

and the result follows from Solomon's Theorem.

It is not difficult to express the order of $Cl_K(\chi)$ in terms of the matrix Θ of Theorem 2.1(i). One has [1,III, sect.9, Prop.6]

$$\#Cl_K(\chi) = \#O_\chi / \left(\prod_{\psi} \psi(\det(\Theta)) \right).$$

Here ψ runs over the characters of π , and $\psi(\det(\Theta))$ indicates the value of the natural extension of ψ to an algebra homomorphism $O_\chi[\pi] \longrightarrow \overline{\mathbf{Q}}_p$ on $\det(\Theta) \in O_\chi[\pi]$.

Next we deduce a sufficient condition for the eigenspace $Cl_K(\chi)$ to be a cyclic $O_\chi[\pi]$ -module.

Theorem 2.3. *Suppose that for all primes r that are ramified in $F \subset K$ we have that $\chi(r) \neq 1$. If one of the following conditions holds:*

- $B_{1,\chi^{-1}} = pu$ for some unit $u \in O_\chi^*$;
- there exists a character $\varphi : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \overline{\mathbf{Q}}_p^*$ of order $p^k > 1$ such that $B_{1,\chi^{-1}\varphi} = (1 - \zeta_{p^k})u$ for some unit u in $O_\chi[\zeta_{p^k}]$,

then there is an isomorphism of $O_\chi[\pi]$ -modules

$$Cl_K(\chi) \cong O_\chi[\pi]/(\theta_\chi).$$

In particular, $Cl_K(\chi)$ is a cyclic $O_\chi[\pi]$ -module.

Proof. We first show that $Cl_F(\chi)$ is a cyclic O_χ -module. If $B_{1,\chi^{-1}} = pu$ for some unit $u \in O_\chi^*$, it follows from Theorem 2.2(ii) that $\#Cl_F(\chi)$ is equal to the order of the residue field $O_\chi/(p)$. Therefore $Cl_F(\chi)$ is cyclic over O_χ .

In the other case, let $E = \overline{\mathbf{Q}}^{\ker \varphi} F$ and let $P = \text{Gal}(E/F)$. Then P is cyclic and we let $F \subset E' \subset E$ be the unique subfield of E of index p . Since $\varphi \neq 1$, it follows from Theorem 2.1(ii) that the norm map $N_{E/E'} : Cl_E(\chi) \rightarrow Cl_{E'}(\chi)$ is surjective. To compute the order of the kernel of $N_{E/E'}$, we observe that

$$\text{Norm}(B_{1,\chi^{-1}\varphi}) = \text{Norm}(1 - \zeta_{p^k}) = p$$

(here the Norm is the $\mathbf{Q}_p(\zeta_{p^k})/\mathbf{Q}_p$ -norm). By Solomon's Theorem [22, Thm II,1], we conclude that $Cl_E(\chi)[N_{E/E'}]$ has the same order as the residue field $O_\chi/(p)$ of R_χ . Therefore so does $Cl_E(\chi)/(N_{E/E'})$. By Nakayama's Lemma, $Cl_E(\chi)$ is therefore cyclic over the group ring $O_\chi[P]$. It follows that $Cl_F(\chi)$ is cyclic over O_χ in this case as well.

To complete the proof, we observe that, by Theorem 2.1, $Cl_K(\chi)$ is cohomologically trivial and the π -norm map induces an O_χ -isomorphism between $Cl_F(\chi)$ and $Cl_K(\chi)$ modulo the augmentation ideal of $O_\chi[\pi]$. It follows from Nakayama's lemma that $Cl_K(\chi)$ is cyclic over $O_\chi[\pi]$. By Stickelberger's theorem there is therefore a surjection $O_\chi[\pi]/(\theta_\chi) \rightarrow Cl_K(\chi)$, which is an isomorphism because both groups have the same order by Theorem 2.2. This proves Theorem 2.3.

In the case the p -group π is cyclic of order p^e , say, we can be a little bit more explicit. We have the usual isomorphism of local rings, familiar in Iwasawa theory

$$O_\chi[\pi] \cong O_\chi[T]/((1+T)^{p^e} - 1),$$

where $1+T$ corresponds to some generator of π . The maximal ideal of this local ring is (T, p) . For $i \geq 0$, we let $\omega_i(T) = (1+T)^{p^i} - 1$.

By the Weierstrass Preparation theorem [24], every non-zero $f(T) \in O_\chi[[T]]/((1+T)^{p^e} - 1)$ is the residue class of a polynomial of the form $p^\mu u(T)h(T)$ where μ is a non-negative integer, $u(T)$ a unit and $h(T) = T^\lambda + a_{\lambda-1}T^{\lambda-1} + \dots + a_1T + a_0$ is a *Weierstrass polynomial* of degree $\lambda < p^e$. This means that $a_i \equiv 0 \pmod{p}$ for $i = 0, 1, \dots, \lambda - 1$.

Proposition 2.4. *Suppose that for all primes r that are ramified in $F \subset K$ we have that $\chi(r) \neq 1$. Suppose that the Galois group π is cyclic of order p^e and that $Cl_F(\chi)$ is a cyclic O_χ -module. If for some character ψ of π of order p , for some $\lambda < p - 1$ and for some unit $u \in O_\chi[\zeta_p]$, we have that $B_{1,\chi^{-1}\psi} = (1 - \zeta_p)^\lambda u$, then*

$$Cl_K(\chi) \cong (O_\chi/(p^e))^{\lambda-1} \times O_\chi/(p^e B_{1,\chi^{-1}})$$

as an O_χ -module.

Proof. We write $O_\chi[\pi] = O_\chi[T]/(\omega_e(T))$ as above. Since $Cl_F(\chi)$ is a cyclic O_χ -module, it follows from Theorem 2.1 that the eigenspace $Cl_K(\chi)$ is a cohomologically trivial cyclic $O_\chi[\pi]$ -module. Therefore $Cl_K(\chi) \cong O_\chi[\pi]/(p^\mu f(T))$ for some Weierstrass polynomial $f(T)$. Since $Cl_F(\chi) \cong O_\chi[\pi]/(T) \cong O_\chi/(p^\mu f(0))$, we have that $p^\mu f(0) = B_{1,\chi^{-1}}$, up to a p -adic unit. Similarly, for the subfield $F \subset E \subset K$ of degree p over F we have that $Cl_E \cong O_\chi[T]/(f(T), \omega_1(T))$. Applying Solomon's Theorem [22, Thm II,1], we find that, up to a p -adic unit, $f(1 - \zeta_p) = B_{1,\chi^{-1}\psi} = (1 - \zeta_p)^\lambda$.

Since $\lambda < p - 1$, this implies $\mu = 0$ and $\deg(f) = \lambda$. Since $O_\chi[T]/(f(T), \omega_e(T))$ is cohomologically trivial, we have the following exact sequence

$$0 \rightarrow O_\chi[T]/(f(T), \omega_e(T)/T) \xrightarrow{T} O_\chi[T]/(f(T), \omega_e(T)) \rightarrow O_\chi/(f(0)) \rightarrow 0.$$

We analyze the ideal $(f(T), \omega_e(T)/T)$. Consider for $0 \leq i < e$ the quotient

$$\frac{\omega_{i+1}(T)}{\omega_i(T)} = (1+T)^{p^i(p-1)} + \dots + (1+T)^{p^i} + 1.$$

Since $\lambda < p-1$ we have that $T^{p-1} \equiv Tpg(T) \pmod{f(T)}$ for some polynomial $g(T) \in O_\chi[T]$. This implies that $\omega_{i+1}/\omega_i = p + pTh(T)$ for some $h(T) \in O_\chi[T]$. Therefore

$$\frac{\omega_e(T)}{T} = \prod_{i=0}^{e-1} \frac{\omega_{i+1}}{\omega_i} \equiv p^e \cdot u(T) \pmod{f(T)}$$

where $u(T)$ is some unit in $O_\chi[T]/(\omega_e(T))$. This shows that the ideals $(f(T), \omega_e(T)/T)$ and $(f(T), p^e)$ are equal and that there is an isomorphism of O_χ -modules

$$O_\chi[T]/(f(T), \omega_e(T)/T) \cong (O_\chi/p^e O_\chi)^\lambda.$$

To complete the proof, we observe that $f(0) \in O_\chi[T]/(f(T), \omega_e(T))$ is the image of

$$\frac{f(T) - f(0)}{T} \in O_\chi[T]/(f(T), \omega_e(T)/T) = O_\chi[T]/(f(T), p^e),$$

under the multiplication by T map. Since f is monic, this implies that $f(0)$ has order p^e . Therefore $1 \in O_\chi[T]/(\omega_e(T), f(T))$ has, up to p -adic unit, order $f(0)p^e$.

This completes the proof

The following simple result often suffices to determine the structure of the p -part of the minus class group of $\mathbf{Q}(\zeta_l)$ when p divides $l-1$. Note that the proof does not rely on the theorems of Mazur-Wiles, Kolyvagin or Solomon.

Theorem III. *Let l and p be odd primes and let M be the p -part of the minus class group of $\mathbf{Q}(\zeta_l)$. If $\#M$ divides $(l-1)^2$, then M is a cyclic group.*

Proof. Let π denote the p -part of $G = \text{Gal}(\mathbf{Q}(\zeta_l)/\mathbf{Q})$; it is a cyclic group of order p^e . Let F be the fixed field of π , let χ be a character of G of order prime to p and let $M(\chi)$ be the corresponding eigenspace of M . We assume that $M(\chi) \neq 0$. Since the condition of Theorem 2.1 is satisfied for $K = \mathbf{Q}(\zeta_l)$, there is an exact sequence

$$0 \longrightarrow O_\chi[\pi]^d \xrightarrow{\Theta} O_\chi[\pi]^d \longrightarrow M(\chi) \longrightarrow 0,$$

where d is the O_χ -rank of $Cl_F(\chi)$. Let $q = p^a$ denote the number of elements in the residue field of O_χ . We write $\det(\Theta) = p^\mu u(T) f(T) \in O_\chi[\pi] \cong O_\chi[T]/(\omega_e(T))$ for some Weierstrass polynomial $f(T) = T^\lambda + a_{\lambda-1}T^{\lambda-1} + \dots + a_1T + a_0$ and some unit $u(T)$. Then $\#M(\chi) = \#O_\chi/(\prod_{\zeta^{p^e}=1} p^\mu f(\zeta - 1))$, so that

$$\#M(\chi) \geq q^{\mu p^e + \min(\lambda, p-1)e + 1}$$

and hence

$$2e \geq a(\mu p^e + \min(\lambda, p-1)e + 1).$$

Since $2e < p^e + 1$, we have $\mu = 0$. Since $M(\chi) \neq 0$, this implies that $\lambda > 0$. Moreover, since $a \cdot \min(\lambda, p-1) < 2$, we have that $\lambda = 1$ and $a = 1$ so that $O_\chi = \mathbf{Z}_p$. This shows that, up to a unit, $f(T) = \det(\Theta) = T - \beta$ for some $\beta \in p\mathbf{Z}_p$. Since d is the O_χ -rank of $Cl_F(\chi)$, any surjection $O_\chi[\pi]^d \twoheadrightarrow Cl_l(\chi)$ is an isomorphism modulo the maximal ideal \mathfrak{m} of the local ring $O_\chi[\pi]$. This implies that all entries of the matrix Θ are contained in \mathfrak{m} so that $\det(\Theta) \in \mathfrak{m}^d$.

It follows that $d = 1$, so that $M(\chi) \cong \mathbf{Z}_p[T]/((1+T)^{p^e} - 1, T - \beta) \cong \mathbf{Z}_p/p^e \beta \mathbf{Z}_p$ is a cyclic group. We conclude the proof by observing that $\#M(\chi) \geq p^{e+1}$, so that only one eigenspace $M(\chi)$ is non-trivial and hence $M = M(\chi)$.

3. The 2-part.

In this section we study the 2-part of the minus class group of a complex abelian number field K . We show that certain eigenspaces of the 2-part are cohomologically trivial Galois modules. This has consequences for their structure. Finally we prove a criterion for cyclicity of these eigenspaces as Galois modules.

Let $G = \text{Gal}(K/\mathbf{Q})$, let $\iota \in G$ denote complex conjugation and let K^+ denote the fixed field of ι . We have inclusions of idèle class groups $C_{K^+} \subset C_K$ and of idèle unit groups $U_{K^+} \subset U_K$. There is a natural map $Cl_{K^+} \rightarrow Cl_K$. We define

$$\begin{aligned} U_K^- &= U_K/U_{K^+}, \\ C_K^- &= C_K/C_{K^+}, \\ Cl_K^- &= Cl_K/\text{im } Cl_{K^+}, \\ \mu_K^- &= \mu_K \cap U_K^-. \end{aligned}$$

Note that U_K^- is isomorphic to the submodule $U_K^{1-\iota}$ of U_K . The intersection $\mu_K \cap U_K^-$ is taken inside U_K .

A diagram chase involving the exact sequence $0 \rightarrow O_K^* \rightarrow U_K \rightarrow C_K \rightarrow Cl_K \rightarrow 0$ and the analogous sequence for K^+ shows that there is an exact sequence [19]

$$0 \rightarrow \mu_K^- \rightarrow U_K^- \rightarrow C_K^- \rightarrow Cl_K^- \rightarrow 0.$$

It is important to use the definition of the minus class group Cl_K^- that we give here. Often the minus class group of an abelian number field K is defined to be the kernel of the norm map $N : Cl_K \rightarrow Cl_{K^+}$. The present definition differs at most in the 2-part. It has several advantages: as we will see below, it is easy to compute the Galois cohomology of Cl_K^- ; the results for the 2-part are very similar to the results for the odd parts. I don't know how to do the calculations using the other definition.

Another advantage over the usual definition is the following. It is easy to deduce the following formula for the order of Cl_K^- from the usual class number formula:

$$\#Cl_K^- = \frac{2}{[\mu_K : \mu_K^-]} \# \mu_K \prod_{\chi \text{ odd}} -\frac{1}{2} B_{1,\chi}.$$

This formula does not involve the unit index “ Q_K ” of Hasse [5, Ch.20], which is, in general, difficult to compute. This time there is the factor $2/[\mu_K : \mu_K^-]$, which is either 1 or 2, but this quantity is easy to compute; it captures, in some sense, only the easy aspects of the unit index Q_K and its calculation is precisely the content of Hasse's Satz 22 in [5].

In this section we fix a complex abelian number field K with $G = \text{Gal}(K/\mathbf{Q})$. Let π be the 2-part of G with fixed field $k = K^\pi$. We fix a non-trivial character χ of G of odd order. We denote the fixed field of K under ι by K^+ . Note that $k \subset K^+$.

Theorem 3.1. *let $P \subset \pi$ be a 2-group that does not contain ι and let $E = K^P$. Let E^+ be the fixed field of E under ι . If all primes r that ramify in $E^+ \subset K$ satisfy $\chi(r) \neq 1$, then*

- (i) $Cl_K^-(\chi)$ is a cohomologically trivial $O_\chi[P]$ -module;
- (ii) the natural map $Cl_{E^+}^-(\chi) \rightarrow Cl_K^-(\chi)^P$ is bijective and the norm map $N : Cl_K^-(\chi) \rightarrow Cl_{E^+}^-(\chi)$ is surjective.

Proof. Note that $\text{Gal}(K/E^+) \cong P \times \{1, \iota\}$. The proof follows the pattern of the proof of Theorem 2.1.

(i) It suffices to show that $\widehat{H}^q(P, Cl_K^-(\chi)) = 0$ for all $q \in \mathbf{Z}$. Consider the exact sequence

$$0 \longrightarrow \mu_K^- \longrightarrow U_K^- \longrightarrow C_K^- \longrightarrow Cl_K^- \longrightarrow 0.$$

We show that the χ -parts of the P -cohomology groups of the first three modules are trivial. Lemma 1.1 then implies that $\widehat{H}^q(P, Cl_K^-(\chi)) = 0$ for all $q \in \mathbf{Z}$.

Since χ has odd order, it acts trivially on the 2-part of μ_K^- and therefore on its P -cohomology groups. This shows that $\widehat{H}^q(P, \mu_K^-)(\chi) = 0$ for all $q \in \mathbf{Z}$. By *global* class field theory $\widehat{H}^q(P, C_K^-)$ and $\widehat{H}^q(P, C_{K^+})$ are isomorphic to $\widehat{H}^{q-2}(P, \mathbf{Z})$ and have therefore trivial G -action and, since $\chi \neq 1$, trivial χ -parts. It follows that $\widehat{H}^q(P, C_{K^-})(\chi) = 0$ for all $q \in \mathbf{Z}$.

By *local* class field theory and the fact that $\chi(r) \neq 1$ for the primes r that ramify in $E \subset K$ and $E^+ \subset K^+$ we have that $\widehat{H}^q(P, U_K^-)$ and $\widehat{H}^q(P, U_{K^+})$ have trivial χ -parts. The proofs are similar to the proof of part (i) of Theorem 2.1.

(ii) The natural map $C_E^-/N(C_K^-) \longrightarrow Cl_E^-/N(Cl_K^-)$ is surjective. We saw already in the proof of part (i) that $C_E^-/N(C_K^-) = \widehat{H}^0(P, C_K^-)$ has trivial χ -part. Therefore the norm map $N : Cl_K^-(\chi) \twoheadrightarrow Cl_E^-(\chi)$ is surjective. Note that we only used the fact that $\chi \neq 1$ to prove this.

To prove the second statement, we consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_E^- & \longrightarrow & U_E^- & \longrightarrow & C_E^- & \longrightarrow & Cl_E^- & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mu_K^{-P} & \longrightarrow & U_K^{-P} & \longrightarrow & C_K^{-P} & \longrightarrow & Cl_K^{-P} & \longrightarrow & 0 \end{array}$$

An easy diagram chase shows that the first three vertical arrows are injective and have cokernels with trivial χ -parts. By the proof of part (i), the P -cohomology groups of each of the modules μ_K^- , U_K^- , C_K^- and Cl_K^- have trivial χ -parts as well. This easily implies that the rightmost map induces an isomorphism $Cl_E^-(\chi) \xrightarrow{\sim} Cl_K^-(\chi)^P$ as required.

Theorem 3.2. *If all primes r that ramify in $k \subset K$ satisfy $\chi(r) \neq 1$, then*

(i) *there is an exact sequence*

$$0 \longrightarrow (O_\chi[\pi]/(1+\iota))^d \xrightarrow{\Theta} (O_\chi[\pi]/(1+\iota))^d \longrightarrow Cl_K^-(\chi) \longrightarrow 0;$$

(ii) *If, in addition, the prime 2 is not ramified in the field K , then*

$$\#Cl_K^-(\chi) = O_\chi / \left(\prod_{\psi} \frac{1}{2} B_{1, \chi^{-1}\psi} \right),$$

where the product runs over the odd characters ψ of G of 2-power order.

Proof. Choose $\sigma \in \pi$ so that $\langle \sigma \rangle$ is a direct summand of π containing ι . Let 2^e denote the order of σ and let P be a complement of $\langle \sigma \rangle$ in π : we have $\pi = P \times \langle \sigma \rangle$. The eigenspace $Cl_K^-(\chi)$ is a $O_\chi[\pi]$ -module on which $\iota = \sigma^{2^e-1}$ acts as -1 . Therefore $Cl_K^-(\chi)$ is a module over the ring $O_\chi[P \times \langle \sigma \rangle]/(1+\iota) \cong O_\chi[\zeta_{2^e}][P]$,

By Theorem 3.1, $Cl_K^-(\chi)$ is a cohomologically trivial P -module. Let $O_\chi[\zeta_{2^e}][P]^d \twoheadrightarrow Cl_K^-(\chi)$ be a surjective $O_\chi[\zeta_{2^e}][P]$ -homomorphism. The kernel is a cohomologically trivial torsion-free $O_\chi[\zeta_{2^e}][P]$ -module. As in the proof of Theorem 2.3, we copy the proof of [2, p.113, Thm.8] with \mathbf{Z}

replaced by the discrete valuation ring $O_\chi[\zeta_{2^e}]$. It follows that the kernel is projective and hence free over the local ring $O_\chi[\zeta_{2^e}][P]$. Since the quotient is finite, the kernel has rank d . This proves (i).

(ii) We proceed with induction with respect to the order of π . Since 2 is unramified we may apply C. Greither's Theorem [4, p.453, Thms A and B] and we see that the result holds when π is cyclic. Suppose π is not cyclic. Writing $\pi = \langle \sigma \rangle \times P$ as in part (i), the group P is not trivial. Let $\tau \in P$ be an element of order 2. The fixed fields K^τ and $K^{\tau\iota}$ of τ and $\tau\iota$ are both complex abelian number fields containing k . The set of odd characters of G is the disjoint union of the sets of odd characters of $\text{Gal}(K^\tau/\mathbf{Q})$ and $\text{Gal}(K^{\tau\iota}/\mathbf{Q})$.

By induction, the result holds for the fields K^τ and $K^{\tau\iota}$. By Theorem 3.1(i), $M = Cl_K^-(\chi)$ is cohomologically trivial, both as a $\{1, \tau\}$ -module and as a $\{1, \tau\iota\}$ -module. Moreover, by part (ii) of that theorem, $(1 + \tau)M$ and $(1 + \tau\iota)M$ are isomorphic to the χ -part of the 2-part of the minus class group of K^τ and $K^{\tau\iota}$ respectively. Since ι acts as -1 on M , it follows from the cohomological triviality of M that $\#M = \#(1 + \tau)M \cdot \#(1 - \tau)M = \#(1 + \tau)M \cdot \#(1 + \tau\iota)M$. This proves (ii).

Finally we prove a sufficient condition for the eigenspace $Cl_K^-(\chi)$ to be a cyclic $O_\chi[\pi]/(1 + \iota)$ -module.

Theorem 3.3. *Suppose that all primes r that ramify in $k \subset K$ satisfy $\chi(r) \neq 1$. If there exists an odd character φ of odd conductor and of order 2^k for which each of the following two conditions hold:*

- $\frac{1}{2}B_{1, \chi^{-1}\varphi} = (1 - \zeta_{2^k})u$ for some unit $u \in O_\chi[\zeta_{2^e}]^*$,
- $\chi(r) \neq 1$ for all primes r dividing the conductor of φ ,

then $Cl_K^-(\chi)$ is a cyclic $O_\chi[\pi]/(1 + \iota)$ -module.

Proof. Let k_φ denote the composite field $k\mathbf{Q}^{\ker \varphi}$ and let K_φ denote $K\mathbf{Q}^{\ker \varphi}$. Both fields $k_\varphi \subset K_\varphi$ are complex. Put $\pi' = \text{Gal}(K_\varphi/k)$ and $P = \text{Gal}(K_\varphi/k_\varphi)$. We have that $\iota \notin P$.

Since 2 is not ramified, it follows from Greither's Theorem that the order of $Cl_{k_\varphi}^-(\chi)$ is equal to the order of $O_\chi/(\text{Norm}(\frac{1}{2}B_{1, \chi^{-1}\varphi}))$. Here the Norm is the $O_\chi[\zeta_{2^k}]/O_\chi$ -Norm. Since $\text{Norm}(\frac{1}{2}B_{1, \chi^{-1}\varphi}) = \text{Norm}(1 - \zeta_{2^k}) = 2$, we see that the order of $Cl_{k_\varphi}^-(\chi)$ is equal to the order of the residue field of O_χ . Therefore $Cl_{k_\varphi}^-(\chi)$ is a cyclic Galois module. By Theorem 3.1, applied to $E = k_\varphi \subset K_\varphi$, the eigenspace $Cl_{K_\varphi}^-(\chi)$ is a cohomologically trivial P -module and the P -norm map induces an isomorphism between $Cl_{k_\varphi}^-(\chi)$ and $Cl_{K_\varphi}^-(\chi)$ modulo the P -augmentation ideal. Therefore another application of Nakayama's Lemma implies that $Cl_{K_\varphi}^-(\chi)$ is a cyclic $O_\chi[P]$ -module and hence a cyclic $O_\chi[\pi']/(1 + \iota)$ -module. Therefore its quotient $Cl_K^-(\chi)$ is a cyclic $O_\chi[\pi]/(1 + \iota)$ -module, as required.

If the group π is cyclic, then $O_\chi[\pi]/(1 + \iota) \cong O_\chi[\zeta_{2^e}]$ where $\#\pi = 2^e$. Since the ring $O_\chi[\zeta_{2^e}]$ is a discrete valuation ring, the structure of finite modules over $O_\chi[\pi]/(1 + \iota)$ is particularly simple.

Proposition 3.4. *Suppose that π is cyclic and that $Cl_K^-(\chi)$ is cyclic over $O_\chi[\pi]$. If $\#Cl_K^-(\chi) = 2^{ft}$, where 2^f is the order of the residue field $O_\chi/(2)$, then there is an isomorphism of $O_\chi[\zeta_{2^e}]$ -modules*

$$Cl_K^-(\chi) \cong O_\chi[\zeta_{2^e}]/((1 - \zeta_{2^e})^t)$$

and there is an isomorphism of abelian groups

$$Cl_K^-(\chi) \cong (\mathbf{Z}/2^r\mathbf{Z})^{f(2^{e-1}-s)} \times (\mathbf{Z}/2^{r+1}\mathbf{Z})^{fs}$$

where $r, s \in \mathbf{Z}$ are determined by $t = r2^{e-1} + s$ and $0 \leq s < 2^{e-1}$.

Proof. This follows from the fact that $O_\chi[\zeta_{2^e}]$ is a discrete valuation ring with uniformizing element $1 - \zeta_{2^e}$.

4. Tables.

In this section we present the proof of Theorem II. An essential ingredient is the table of class numbers h_l^- given in the appendix. We briefly explain the notation.

Let l be an odd prime. We have $l - 1 = 2^e \cdot m$ with m odd. For every divisor d of $l - 1$ which itself is divisible by 2^e we define

$$h_l^-(d) = \prod_{\text{ord}(\chi)=d} -\frac{1}{2}B_{1,\chi}$$

where the product runs over the characters $\chi : (\mathbf{Z}/l\mathbf{Z})^* \rightarrow \mathbf{C}^*$ of order d ; except when $d = l - 1$, in which case we multiply this product by l , and when $d = 2^e$, in which case we multiply it by 2. In the rare occasion when $l - 1$ is equal to 2^e , the only possible value for d is $l - 1 = 2^e$ and we put

$$h_l^-(d) = 2l \prod_{\text{ord}(\chi)=d} -\frac{1}{2}B_{1,\chi}.$$

This last case occurs only when l is a Fermat prime i.e., when $l = 3, 5, 17, 257, 65537$ or has more than 2 500 000 decimal digits.

The numbers $h_l^-(d)$ are listed in the appendix. They are rational integers [5, 24] and they are related to the minus class number h_l^- by

$$h_l^- = \#Cl_l^- = \prod_{2^e|d|l-1} h_l^-(d).$$

In [15] D.H. Lehmer and J.M. Masley presented a table with the numbers $h_l^-(d)$ for $l \leq 509$. Of most of these numbers the complete prime factorization was given, but their table contains 22 unfactored composite numbers. These were factored by Peter Montgomery (PM), Bob Silverman (BS), Herman te Riele (HtR) and Arjen Lenstra (AL). The most laborious factorization, for $l = 467$, was performed by Arjen Lenstra, who factored a 103 digit factor of h_{467}^- into a product of two primes of 49 and 55 digits respectively. We list the various contributions in Table 4.1. By p_n we denote a prime factor of n decimal digits. The order in which the initials are given corresponds to the order of the prime factors.

Table 4.1.

l			l		
233	$p_{14} \cdot p_{29}$	PM	419	$p_{16} \cdot p_{30} \cdot p_{49}$	PM, HtR
269	$p_{16} \cdot p_{31}$	PM	433	$p_{14} \cdot p_{34}$	PM
317	$p_{25} \cdot p_{49}$	HtR	439	$p_{11} \cdot p_{21} \cdot p_{23} \cdot p_{24}$	PM, PM, PM
337	$p_{13} \cdot p_{15} \cdot p_{15}$	PM, PM	449	$p_{18} \cdot p_{84}$	PM
359	$p_{13} \cdot p_{30} \cdot p_{45}$	PM, HtR	463	$p_{18} \cdot p_{21} \cdot p_{25}$	PM, BS
379	$p_{22} \cdot p_{24}$	BS	467	$p_{19} \cdot p_{49} \cdot p_{55}$	PM, AL
383	$p_{19} \cdot p_{24} \cdot p_{46}$	PM, HtR	479	$p_{20} \cdot p_{27} \cdot p_{70}$	PM, AL
389	$p_{24} \cdot p_{60}$	AL	487	$p_{30} \cdot p_{49}$	HtR
397	$p_8 \cdot p_{26} \cdot p_{27}$	PM, BS	499	$p_{15} \cdot p_{18} \cdot p_{47}$	PM, PM
401	$p_{16} \cdot p_{18} \cdot p_{31}$	PM, PM	503	$p_{12} \cdot p_{14} \cdot p_{112}$	PM, PM
409	$p_{12} \cdot p_{52}$	PM	509	$p_{16} \cdot p_{28} \cdot p_{101}$	PM, AL

In order to prove Theorem II and, at the same time, determine the structure of Cl_l^- as an abelian group, we study the table of numbers $h_l^-(d)$ of the appendix. Clearly, if a prime p divides the class number h_l^- exactly once, the p -part of Cl_l^- is cyclic as a group and hence as a Galois module. This happens for most large prime divisors. All other cases are listed below. Tables 4.2, 4.3 and 4.4 contain the prime pairs (p, l) with $l \leq 509$ for which p^2 divides h_l^- . We discuss each table in some detail.

The class group Cl_l^- is a product of its p -parts and each p -part is a product of eigenspaces $Cl_l^-(\chi)$. The minus class group Cl_l^- is a cyclic Galois module if and only if for each prime p , each eigenspace $Cl_l^-(\chi)$ is cyclic over the local ring $O_\chi[\pi]$, where π is the p -part of $G = \text{Gal}(\mathbf{Q}(\zeta_l)/\mathbf{Q})$.

Table 4.2. Primes p not dividing $l - 1$.

l	p	d	f	$h_l(d)$	class group	
41	11	40	2	11^2	11×11	
131	3	26	3	3^3	$3 \times 3 \times 3$	
139	47	46	1	47^2	2209	Thm.2.3 with $r = 283$
		277	1	277	277	
		138	1	277	277	
149	3	4	2	3^2	3×3	
151	11	30	2	11^2	11×11	
157	157	156	1	157^2	157×157	Thm.2.2
211	281	14	1	281	281	
		70	1	281	281	
227	2939	226	1	2939^3	$2939 \times 2939 \times 2939$	Thm.2.2
241	47	16	2	47^2	47×47	
277	47	276	2	47^2	47×47	
281	11	40	2	11^2	11×11	
		41	1	41^2	1681	Thm.2.3 with $r = 83$
293	3	4	2	3^2	3×3	
313	37	24	2	37^2	37×37	
337	17	16	1	17^2	17×17	Thm.2.2
353	353	352	1	353^2	353×353	Thm.2.2
379	379	42	1	379	379	
		378	1	379	379	
397	23	132	2	23^2	23×23	
401	41	80	2	41^2	41×41	
409	5	24	2	5^2	5×5	
419	3	2	1	3^2	9	Thm.2.3 with $r = 7$
443	3	26	3	3^6	$9 \times 9 \times 9$	Thm.2.3 with $r = 7$
457	5	24	2	5^2	5×5	
467	467	466	1	467^2	467×467	Thm.2.2
479	5	2	1	5^2	25	Thm.2.3 with $r = 11$
487	7	2	1	7	7	
		6	1	7	7	
		37	1	37^2	37×37	Thm.2.2
491	3	2	1	3^2	9	Thm.2.3 with $r = 7$
		11	1	11^3	11×121	Thm.2.2, Thm.2.3 with $r = 23$
		491	1	491	491	
		490	1	491^2	491×491	Thm.2.2

In Table 4.2 we have listed all pairs (p, l) for which p is odd and p^2 divides h_l^- , but p does not divide $l - 1$. In this case the p -part π of the Galois group of $\mathbf{Q}(\zeta_l)$ over \mathbf{Q} is trivial and an eigenspace

$Cl_l(\chi)$ is cyclic as a Galois module if and only if it is a cyclic O_χ -module. It turns out that in all cases every $Cl_l(\chi)$ is cyclic as an O_χ -module.

To explain the table, we first note that in the case $l = p$, the Teichmüller eigenspace $Cl_l^-(\omega)$ is always trivial. Therefore we only have contributions for the characters $\chi \neq \omega$. Let d be a divisor of $l - 1$ for which p divides $h_l^-(d)$. Then for all characters χ of order d the ring O_χ has a residue field with p^f elements where f is the order of p modulo d . If p^f happens to be the exact power of p dividing $h_l^-(d)$, then it is clear that for exactly one character χ of order d the eigenspace $Cl_l^-(\chi)$ is isomorphic to $O_\chi/(2)$ while all others are trivial. These cases are listed without comment. In the remaining cases we apply the Theorem of Mazur and Wiles which is the case with trivial π of Theorem 2.2. If the precise power of p dividing $h_l^-(d)$ is p^{f_a} and for precisely a characters χ of order d the generalized Bernoulli number $B_{1,\chi^{-1}}$ is divisible by p , then each eigenspace $Cl_l^-(\chi)$ is either isomorphic to $O_\chi/(2)$ or is zero. In particular, each $Cl_l(\chi)$ is a cyclic Galois module. This happens in all but seven cases. In the remaining seven cases we use Theorem 2.3 and show that each eigenspace is a cyclic O_χ module by computing an additional Bernoulli number $B_{1,\chi^{-1}\varphi}$ where φ is a suitable even character of order p and conductor r .

Table 4.3. Odd primes p dividing $l - 1$.

ℓ	p	d	h_0, h_1, \dots	group	
31	3	2	3, 3	9	
71	7	2	7, 7	49	
101	5	4	5, 25, 25	25×125	Prop.2.4, $\lambda = 2$
131	5	2	5, 5	25	
137	17	8	17, 17	289	
139	3	2	3, 3	9	
157	13	12	13, 13	169	
181	5	4	25, 5	125	Prop.2.4, $\lambda = 1$
199	3	2	9, 3, 3	81	
211	3	2	3, 3	9	
	7	6	7, 7	49	
283	3	2	3, 3	9	
307	3	2	3, 3, 3	27	
331	3	2	3, 9	3×9	Thm.2.3, $\theta = T^2 - 15T + 3$
	3	10	81, 81	$9 \times 9 \times 9 \times 9$	Prop.2.4, $\lambda = 1$
337	7	16	49, 49	49×49	Prop.2.4, $\lambda = 1$
367	3	2	9, 3	27	Prop.2.4, $\lambda = 1$
379	3	2	3, 3, 3, 3	81	
409	17	8	17, 17	289	
421	5	4	25, 5	125	Prop.2.4, $\lambda = 1$
439	3	2	3, 27	9×9	Thm.2.3, $\theta = T^2 - 3T - 3$
461	5	4	25, 25	5×125	Thm.2.3 with $r = 11$; Prop.2.4, $\lambda = 2$
463	7	2	7, 7	49	
	7	6	7, 7	49	
499	3	2	3, 3	9	

In Table 4.3 we have listed all pairs (p, l) with $p \neq 2$ dividing $l - 1$. We'll see below that in this case the class number h_l^- is automatically divisible by p^2 , so that Table 4.3 actually contains all pairs (p, l) for which p divides $\gcd(h_l^-, l - 1)$. In order to explain the contents of the table, we fix p and l and we let p^e be the exact power of p dividing $l - 1$.

If d and d' are two divisors of $l - 1$ that only differ by a power of p , then $B_{1,\varphi^{-1}} \equiv B_{1,\varphi'^{-1}} \pmod{(1 - \zeta_{p^e})}$ for all characters φ of order d and φ' of order d' . Therefore, as Lehmer observed [14,

Thm.5], either both $h_l^-(d)$ and $h_l^-(d')$ are divisible by p or none is. For this reason we have ordered the class numbers as follows: for each divisor d of $l-1$ which is itself not divisible by p but for which $h_l^-(d)$ is divisible by p , we list, for $i = 0, 1, \dots, e$ the p -part h_i of $h_l^-(dp^i)$. By Lehmer's observation, each h_i is divisible by p . We note in passing that this implies that h_l^- is divisible by p^2 .

For each character χ of order d the residue field of O_χ has order p^f where f is the order of p modulo d . In all but one case either $h_0 = p^f$ or $h_1 = p^f$. In the latter case we have that, up to a unit, $B_{1,\chi^{-1}\psi} = 1 - \zeta_p$ for the characters ψ of conductor l and order p . In either case Theorem 2.3 applies and we see that $Cl_l(\chi)$ is cyclic over $O_\chi[\pi]$. The only exception is $l = 461$ with $p = 5$. In this case $h_0 = h_1 = 25$. In this case we have applied Theorem 2.3 with φ a character of order 5 and conductor 11. It turns out that in this exceptional case $Cl_l(\chi)$ is a cyclic $O_\chi[\pi]$ -module as well.

In most cases we can apply Theorem III and conclude that the eigenspace is a cyclic group. These cases are listed without comment. In the cases $(l, p) = (101, 5)$, $(337, 7)$, $(461, 5)$ and $(331, 3)$ (the latter for $d = 10$) an application of Prop.2.4 immediately gives the structure of $Cl_l(\chi)$. Finally, in the cases $(l, p) = (439, 3)$ and $(331, 3)$ (the latter for $d = 2$) we have explicitly computed the Stickelberger element θ and applied Theorem 2.3 directly.

Table 4.4. $p = 2$.

l	d	$\text{ord}(\chi)$	2^e	f	$h_l^-(d)$	2-class group	r
29	28	7	4	3	8	$2 \times 2 \times 2$	
113	112	7	16	3	8	$2 \times 2 \times 2$	
163	6	3	2	2	4	2×2	
197	28	7	4	3	8	$2 \times 2 \times 2$	
239	14	7	2	3	8^2	$4 \times 4 \times 4$	3
277	12	3	4	2	4^2	$2 \times 2 \times 2 \times 2$	3
311	62	31	2	5	32^2	$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$	
337	336	21	16	6	64	$2 \times 2 \times 2 \times 2 \times 2 \times 2$	
349	12	3	4	2	4^2	$2 \times 2 \times 2 \times 2$	7
373	124	31	4	5	32	$2 \times 2 \times 2 \times 2 \times 2$	
397	12	3	4	2	4^3	$4 \times 4 \times 2 \times 2$	3
421	60	15	4	4	16	$2 \times 2 \times 2 \times 2$	
463	14	7	2	3	8	$2 \times 2 \times 2$	
491	14	7	2	3	8^2	$2 \times 2 \times 2 \times 2 \times 2 \times 2$	

Finally we discuss the contents of Table 4.4. Let χ be a character of $(\mathbf{Z}/l\mathbf{Z})^*$ of odd order. The 2-part of Cl_l^- is a module over $O_\chi[\pi]/(1 + \iota) \cong O_\chi[\zeta_{2^e}]$. Here 2^e is the exact power of 2 dividing $l-1$. It is well known that $Cl_l^-(\chi)$ is trivial when $\chi = 1$. This implies that the prime $p = 2$ never divides h_l^- with multiplicity 1. Therefore Table 4.4 actually contains all primes $l \leq 509$ for which h_l^- is even.

It turns out that $Cl_l^-(\chi)$ is in all cases a cyclic Galois module. This follows from several applications of Theorem 3.3. In all but 4 cases we have that $\prod_\psi \frac{1}{2} B_{1,\chi^{-1}\psi} = 2u$ for some unit $u \in O_\chi$. Here the product runs over the odd characters ψ of 2-power order and conductor l . In this case $Cl_l^-(\chi) \cong O_\chi/(2)$ which is a vector space of dimension f over \mathbf{F}_2 . Here f is the degree of $\mathbf{F}_2(\zeta_d)$ over \mathbf{F}_2 and d is the order of χ . In the remaining cases we applied Theorem 3.3 with an odd quadratic character φ of conductor r . Here $r \equiv 3 \pmod{4}$ is a prime for which $\chi(r) \neq 1$.

The structure of $Cl_l^-(\chi)$ then follows easily from Theorem 3.4.

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Appendix.

l	d	$h_l^-(d)$
3	2	1
5	4	1
7	2	1
	6	1
11	2	1
	10	1
13	4	1
	12	1
17	16	1
19	2	1
	6	1
	18	1
23	2	3
	22	1
29	4	1
	28	2^3
31	2	3
	6	3
	10	1
	30	1
37	4	1
	12	1
	36	37
41	8	1
	40	11^2
43	2	1
	6	1
	14	1
	42	211
47	2	5
	46	139
53	4	1
	52	4889
59	2	3
	58	$59 \cdot 233$
61	4	1
	12	1
	20	41
	60	1861
67	2	1
	6	1
	22	67
	66	12739
71	2	7
	10	1
	14	7

l	d	$h_l^-(d)$
	70	79241
73	8	89
	24	1
	72	134353
79	2	5
	6	1
	26	53
	78	377911
83	2	3
	82	279405653
89	8	113
	88	118401449
97	32	3457
	96	$577 \cdot 206209$
101	4	5
	20	5^2
	100	$5^2 \cdot 101 \cdot 601 \cdot 18701$
103	2	5
	6	1
	34	1021
	102	$103 \cdot 17247691$
107	2	3
	106	$743 \cdot 9859 \cdot 2886593$
109	4	17
	12	1
	36	1009
	108	9431866153
113	16	17
	112	$2^3 \cdot 11853470598257$
127	2	5
	6	13
	14	43
	18	3079
	42	547
	126	$883 \cdot 626599$
131	2	5
	10	5
	26	$3^3 \cdot 53$
	130	$131 \cdot 1301 \cdot 4673706701$
137	8	17
	136	$17 \cdot 47737 \cdot 46890540621121$
139	2	3
	6	3
	46	$47^2 \cdot 277$
	138	$277 \cdot 967 \cdot 1188961909$
149	4	3^2

l	d	$h_l^-(d)$
151	148	$149 \cdot 5129663383200408 / 05461$
	2	7
	6	1
	10	281
	30	11^2
157	50	25951
	150	$1207501 \cdot 312885301$
	4	5
	12	13
156	52	3148601
	156	$13 \cdot 157^2 \cdot 1093 \cdot 1873 \cdot 4 / 18861$
	2	1
163	6	2^2
	18	181
	54	365473
167	162	$23167 \cdot 441845817162679$
	2	11
	166	$499 \cdot 5123189985484229 / 035947419$
173	4	5
	172	$20297 \cdot 231169 \cdot 725717 / 29362851870621$
179	2	5
	178	$1069 \cdot 144586673923349 / 48286764635121$
181	4	5^2
	12	37
	20	$5 \cdot 41$
	36	2521
	60	$61 \cdot 1321$
191	180	5488435782589277701
	2	13
	10	11
	38	51263
190	190	$612771091 \cdot 3673395066 / 9733713761$
	64	192026280449
193	192	$6529 \cdot 15361 \cdot 29761 \cdot 91 / 969 \cdot 10369729$
	4	5
197	28	$2^3 \cdot 1877$
	196	$7841 \cdot 939830268487086 / 6656225611549$
199	2	3^2
	6	3

l	d	$h_l^-(d)$
	18	$3 \cdot 19$
	22	727
	66	25645093
	198	$207293548177 \cdot 31681904128 / 39$
211	2	3
	6	$3 \cdot 7$
	10	41
	14	281
	30	181
	42	$7 \cdot 421$
210	70	$71 \cdot 281 \cdot 12251$
	210	$1051 \cdot 113981701 \cdot 4343510221$
	2	7
	6	43
223	74	17909933575379
	222	$11757537731851 \cdot 342480448 / 3726447$
	2	5
227	226	$2939^3 \cdot 1692824021974901 \cdot 13444015915122722869$
	4	17
229	12	13
	76	$705053 \cdot 47824141$
	228	$457 \cdot 7753 \cdot 41415390332169 / 2666991589$
	8	1433
232	232	$233 \cdot 79933937980769 \cdot 13046 / 008204119903320572430489$
	2	$3 \cdot 5$
239	14	2^6
	34	511123
	238	$14136487 \cdot 123373184789 \cdot 2 / 2497399987891136953079$
	16	47^2
241	48	2359873
	80	$15601 \cdot 126767281$
	240	$13921 \cdot 518123008737871423 / 891201$
	2	7
251	10	11
	50	348270001
	250	$9631365977251 \cdot 3696311145 / 67755437243663626501$
	257	$257 \cdot 20738946049 \cdot 1022997 / 74456391196156129869818 / 3419037149697$

l	d	$h_l^-(d)$
263	2	13
	262	$263 \cdot 787 \cdot 385927 \cdot 418759100955678867328189444629948074260186283$
269	4	13
	268	$40170973189 \cdot 8625962877077617 \cdot 8297860832320483544484903227261$
271	2	11
	6	1
	10	31
	18	37
	30	1201
	54	751928131
	90	$21961 \cdot 7288651$
	270	$271 \cdot 811 \cdot 1621 \cdot 15391 \cdot 20238391 \cdot 666587726641$
277	4	17
	12	2^4
	92	$89977 \cdot 1371353 \cdot 30697273$
	276	$47^2 \cdot 829 \cdot 4873333 \cdot 1776834909244716811072486129$
281	8	17
	40	$11^2 \cdot 41^2 \cdot 401$
	56	64523056921
	280	$3235961 \cdot 977343139976233968569461075411406081$
283	2	3
	6	3
	94	$2064523 \cdot 39341481709417$
	282	$283 \cdot 5484646647490654799157896194266098076673$
293	4	3^2
	292	$293 \cdot 38901409 \cdot 52561753 \cdot 354041533 \cdot 19844792749 \cdot 702405569982494626097 / 54079833$
307	2	3
	6	3
	18	$3 \cdot 37$
	34	$137 \cdot 443 \cdot 1429$
	102	$307 \cdot 10191268178209$
	306	$613 \cdot 919 \cdot 512412441029648479897766391339165893563$
311	2	19
	10	41
	62	$2^{10} \cdot 9918966461$
	310	$311 \cdot 856882084088129553550988747251311805392434897275868681$
313	8	233
	24	37^2
	104	$65386361 \cdot 30358065621833$
317	312	$155288017 \cdot 82941207961 \cdot 986685963782009603919680953$
	4	13
	316	$1438031130902847137607233 \cdot 8097705990409820600574529770502809400397 / 943027841$
331	2	3
	6	3^2
	10	3^4

l	d	$h_l^-(d)$
337	22	$23 \cdot 67$
	30	$3^4 \cdot 61$
	66	17406850561
	110	476506973241784667381
	330	$270271 \cdot 221475181712309125848473872740271$
347	16	$7^2 \cdot 17^2 \cdot 353$
	48	238321
	112	$7^2 \cdot 894469355265098929$
349	336	$2^6 \cdot 3246769 \cdot 3622267546801 \cdot 110537863229809 \cdot 225164259907777$
	2	5
353	346	$347 \cdot 1954086942666238828259012186195350500935086726556960834433397 /$ $/220152315402574339617$
	4	5
	12	$2^4 \cdot 13$
355	116	$421081 \cdot 943429 \cdot 2021708236660033$
	348	$2089 \cdot 17749 \cdot 29247661 \cdot 16684629796320170064136004281782850431997$
	32	$6113 \cdot 9473$
359	352	$353^2 \cdot 281249 \cdot 1380611233 \cdot 3001891553 \cdot 394388386054183213731974638871 /$ $/81225470103134619777$
	2	19
367	358	$5862361010431 \cdot 813287316389858595758239885873 \cdot 58922190801687625383 /$ $/9609863906122210269152723$
	2	3^2
373	6	3
	122	$733 \cdot 268738874461290742168853881$
	366	$39163 \cdot 127480330983805586375654833118494134773442493271686377913$
	4	5
379	12	61
	124	$2^5 \cdot 1117 \cdot 6218451821 \cdot 1699148567515153$
	372	$1489 \cdot 191953 \cdot 124204598699794021789479401683826456140588477617076789$
	2	3
	6	$3 \cdot 13$
383	14	1499
	18	$3 \cdot 991$
	42	$379 \cdot 547$
	54	$3 \cdot 29997973$
	126	$127 \cdot 757 \cdot 9199 \cdot 154412119$
	378	$379 \cdot 1087873417 \cdot 3111358344381146608939 \cdot 214670345683920446286163$
	2	17
	382	$300032351 \cdot 3000702226373096449 \cdot 290945169106342852317343 \cdot 250644232 /$ $/2771948099181404130620436761970705901$
389	4	41
	388	$389 \cdot 1553 \cdot 4847366257 \cdot 128029167243805465177973 \cdot 1027742679263367083 /$ $/43655333188809496622747915533012083866597$
397	4	13
	12	2^6
	36	$109 \cdot 4861$

l	d	$h_l^-(d)$
401	44	23910808769
	132	$23^2 \cdot 132189553 \cdot 1917436489$
	396	$9901 \cdot 14141557 \cdot 28894150148400351045400753 \cdot 241092554399010330726544957$
	16	64849
	80	$41^2 \cdot 476056112401$
409	400	$401 \cdot 462972001 \cdot 3692494801 \cdot 2106370412068801 \cdot 166771329637484801 \cdot 348925 /$ $/0662765811145388290782801$
	8	$5^2 \cdot 17$
	24	$73 \cdot 1321$
419	136	$17 \cdot 122181721 \cdot 7960379881 \cdot 29097077764969$
	408	$409 \cdot 725945254273 \cdot 6183699722087375941883228469840272721633145678440121$
	2	3^2
	22	647747
	38	$1103 \cdot 5410099$
421	418	$2719452561369347 \cdot 440305024994584776198045120721 \cdot 38089642480704298751 /$ $/25494615628571625716516342483$
	4	5^2
	12	37
	20	$5 \cdot 2521$
	28	$29 \cdot 39509$
	60	$2^4 \cdot 22064701$
	84	$70309 \cdot 46085341$
431	140	$409781 \cdot 16521541 \cdot 672896721281$
	420	$421 \cdot 39901 \cdot 3455761 \cdot 57979541174101 \cdot 2655579516751331409910861$
	2	$3 \cdot 7$
	10	$11 \cdot 701$
	86	$676649 \cdot 2709472364809333$
433	430	$14621 \cdot 7970051 \cdot 112225988494992246639243672859450218083129490012657313 /$ $/823968596573192207124531$
	16	842353
	48	4727329
	144	$3457 \cdot 3021564742348701537217$
439	432	$433 \cdot 12097 \cdot 21601 \cdot 47521 \cdot 1403137 \cdot 102550753 \cdot 96686549358769 \cdot 64340730822 /$ $/61985367563988399449713$
	2	$3 \cdot 5$
	6	3^3
	146	$293 \cdot 527207 \cdot 7171667 \cdot 50898521 \cdot 327151064937209$
443	438	$40139516617 \cdot 607057872831881225737 \cdot 15343765387604391577783 \cdot 7611086694 /$ $/50601851817037$
	2	5
	26	$3^6 \cdot 79 \cdot 157$
	34	367926037
449	442	$12377 \cdot 2099059 \cdot 309860291076943369037303413323285158985313526398152831 /$ $/008871913595050372353059812436688273929$
	64	500402969557121
	448	$168449 \cdot 226736972834339969 \cdot 772865886177933052632667046915246737827100 /$ $/790144773744195236265619879496879953539649$

l	d	$h_l^-(d)$
457	8	41
	24	$5^2 \cdot 577$
	152	$1217 \cdot 43777 \cdot 23353152677443223648257268496337$
	456	$63841 \cdot 28668613681009535839148397954381101468353560199403645535773916736 / 6347873193$
461	4	5^2
	20	$5^2 \cdot 661$
	92	$461 \cdot 463413261346674397069$
	460	$161461 \cdot 3702458172193117785898149655903648058852928086226081699845637442 / 0371674719539068279993529581$
463	2	7
	6	7
	14	$2^3 \cdot 7 \cdot 29$
	22	$89 \cdot 1123$
	42	$7 \cdot 631 \cdot 673$
	66	$4423 \cdot 33642841$
	154	$463 \cdot 664064207818594609257539327251$
	462	$8779 \cdot 604417477499456083 \cdot 334167173856936895861 \cdot 1451125083064477390379041$
467	2	7
	466	$467^2 \cdot 7842513546558078253 \cdot 154987811800520892460672570209646897293261969 / 1231 \cdot 4511882445351575687067360009368178199225508063847112361$
479	2	5^2
	478	$48757 \cdot 62141 \cdot 2560169 \cdot 26756241308309805857 \cdot 177581990178050932739148007 \cdot 3939232521558670638697337486372397962981765904709957802472308181004309$
487	2	7
	6	7
	18	37^2
	54	$919 \cdot 2647 \cdot 10909$
	162	$105792786991 \cdot 1355141213869532941$
	486	$58321 \cdot 105290443 \cdot 294594702996402697646390639203 \cdot 90058027084074393088174 / 14913576150427261734980259$
491	2	3^2
	10	11^3
	14	$2^6 \cdot 29$
	70	1262296191031
	98	$491 \cdot 101566319 \cdot 2311247713517$
490	$491^2 \cdot 8489251 \cdot 17841391 \cdot 74468731 \cdot 18022473215169065702224279183302091210 / 994749548801576948376558921841$	
499	2	3
	6	3
	166	$167 \cdot 8170189 \cdot 4568950377354424102616078873671968013$
498	$628477 \cdot 2498605441 \cdot 476526575352703 \cdot 125184090531384337 \cdot 2313122953817705 / 5312162275545594472697442144611$	
503	2	$3 \cdot 7$
	502	$15061 \cdot 182337132259 \cdot 67961871500791 \cdot 142639305944396395662911180592353348 / 4420318131081450920505530106099684339754321688566291891565574466073368 / 455407$

l	d	$h_l^-(d)$
509	4 508	13 1102305661663669 · 3595837345204924707130453993 · 285986765137386082677131/ /2108749623279941544025506130156144149865490359669858574049275462019230/ /8152597