Trace forms of G-Galois algebras in virtual cohomological dimension 1 and 2

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Abstract. Let G be a finite group and let k be a field of $\mathrm{char}(k) \neq 2$. We explicitly describe the set of trace forms of G-Galois algebras over k when the virtual 2-cohomological dimension $\mathrm{vcd}_2(k)$ of k is at most 1. For fields with $\mathrm{vcd}_2(k) \leq 2$ we give a cohomological criterion for the orthogonal sum of a trace form of a G-Galois algebra with itself to be isomorphic to another such form.

Introduction.

Let k be a field of characteristic different from 2 and let G be a finite group. A *G-form over k* is a G-invariant quadratic form defined over k. An important class of G-forms consists of the *trace forms* q_L associated to G-Galois algebras L. Here a G-Galois algebra is a finite étale k-algebra that is Galois over k with group G. See section 1 and [8, 1.3] for the precise definitions. This paper is devoted to the study of these trace forms.

Our first main result is a classification of the trace forms of G-Galois algebras when the virtual 2-cohomological dimension $\mathrm{vcd}_2 k$ of k or rather of its absolute Galois group $G_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$, is at most 1. This complements an earlier result by E. Bayer-Fluckiger and J.-P. Serre [8, Th. 2.2.3]. Since it is similar but somewhat simpler to state, we formulate our result here only for fields of 2-cohomological dimension $\mathrm{cd}_2 k$ at most 1. For fields of *virtual* 2-cohomological dimension $\mathrm{vcd}_2 k$ at most 1, see Theorem 2.2.

Theorem. *Let G be a finite group and let k be a field with $\mathrm{char}(k) \neq 2$ and $\mathrm{cd}_2 k \leq 1$. Then the map*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of trace} \\ \textit{forms of } G\textit{-Galois algebras} \end{array} \right\} \quad \longrightarrow \quad \mathrm{Hom}_{\mathrm{cont}}(G_k, G/G^2),$$

that sends the trace form q_L of a G-Galois algebra L to the homomorphism given by $\sigma \mapsto \varphi_L(\sigma) \pmod{G^2}$, is a well defined bijection.

Here G^2 denotes the subgroup of G generated by the squares and the homomorphism $\varphi_L : G_k \longrightarrow G$ is the one introduced in [8, 1.3.1]. Its definition is recalled in section 1.

When $\mathrm{vcd}_2 k > 1$, it seems difficult to obtain any results without imposing strong restrictions on the group G (see [4, 8]). Therefore, rather than study trace forms directly, we consider *double* trace forms, i.e. orthogonal sums of trace forms with themselves. Our second main result is concerned with fields of virtual 2-cohomological dimension at most 2. It extends [4, Thm 3.1] in the sense that we do *not* assume that k has the strong approximation property [4, section 3].

Theorem. *Let G be a finite group and let k be a field with* $\mathrm{char}(k) \neq 2$ *and* $\mathrm{vcd}_2 k \leq 2$. *Let q_L and $q_{L'}$ be trace forms of G-Galois algebras L, L' over k. Then the orthogonal sums $q_L \oplus q_L$ and $q_{L'} \oplus q_{L'}$ are isomorphic G-forms if and only if the following hold.*

- *the G-forms q_L and $q_{L'}$ are isomorphic over all real closures of k;*
- *the cup products $\varphi_L \cup (-1)$ and $\varphi_{L'} \cup (-1)$ are equal in $H^2(G_k, G/G^2)$.*

Here (-1) denotes the cohomology class in $H^1(G_k, \mu_2) \cong k^*/k^{*2}$ corresponding to the element $-1 \in k^*/k^{*2}$.

We prove the first theorem in section 1 and its adaptation to fields of virtual 2-cohomological dimension at most 1 in section 2. We prove the second theorem in section 3.

1. 2-Cohomological dimension at most 1.

In this section we prove the first theorem of the introduction. We briefly recall some definitions and introduce the cohomology sets that we use in the proof.

Let G be a finite group and let k be a field of $\mathrm{char}(k) \neq 2$. A *G-form* is a pair (V, q) with V a finite dimensional k-vector space V equipped with linear G-action and $q : V \otimes V \longrightarrow k$ a quadratic form on V for which $q(gx, gy) = q(x, y)$ for all $x, y \in V$ and $g \in G$. Two G-forms (V, q) and (V', q') are isomorphic when there exists a G-equivariant invertible linear map $f : V \longrightarrow V'$ for which $q(x, y) = q'(f(x), f(y))$ for all $x, y \in V$.

A *G-Galois algebra over k* is a finite étale k-algebra L equipped with a right action by G that is simply transitive on the k^{sep}-points of L. The *trace form* associated to a G-Galois algebra L is the quadratic form $q_L : L \times L \longrightarrow k$ given by $q_L(x, y) = \mathrm{Tr}(xy)$ for $x, y \in L$. It is a G-form. Important examples of G-Galois algebras are fields L that are Galois over k with Galois group G. Another example is provided by the group ring $L = k[G]$ with its natural (left) G-action. This algebra admits the k-linear involution given by $[g]^* = [g^{-1}]$ for $g \in G$. It gives rise to the *unit G-form* q_0 which for $x, y \in k[G]$ is given by $q_0(x, y) = a_1$ where a_1 is the coefficient of $x^* y = \sum_{g \in G} a_g [g]$ corresponding to the neutral element $1 \in G$. Any G-Galois algebra whose trace form is isomorphic to the unit form is said to have a *self-dual normal basis* [3].

For any k-algebra R we consider the algebra $R[G]$ equipped with the R-linear involution given by $[g]^* = [g^{-1}]$ for $g \in G$. The algebraic group U_G is given by

$$U_G(R) = \{x \in R[G] : x^* x = 1\}.$$

Since $\mathrm{char}(k) \neq 2$, it is smooth. We view G itself as a finite constant (i.e. étale with trivial Galois action) algebraic group and consider the closed immersion $G \longrightarrow U_G$ given by $g \mapsto [g]$. We do the same for G/G^2. Since G/G^2 has exponent 2, the group U_{G/G^2} is constant and finite of exponent 2. This implies first of all that the closed immersion $G/G^2 \longrightarrow U_{G/G^2}$ admits a left inverse. In addition, the natural morphism $U_G \longrightarrow U_{G/G^2}$ factors through the component group U_G/U_G^0. Here U_G^0 denotes the connected component of identity of the algebraic group U_G. By [8, Prop. 2.3.2], the group U_G/U_G^0 is of exponent 2. Therefore, the morphism $G \longrightarrow U_G \longrightarrow U_G/U_G^0$ factors through G/G^2. This leads to the following commutative diagram of algebraic groups, together with the commutative diagram obtained by taking their k^{sep}-points and then their Galois cohomology. Here we write $H^1(k,H)$ for the Galois cohomology set $H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), H(k))$ associated to an algebraic group H over k.

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & U_G \arrow[d, "s"] \\
G/G^2 \arrow[r] \arrow[dr, "t"'] & U_G/U_G^0 \arrow[d] \\
 & U_{G/G^2}
\end{tikzcd}
\qquad
\begin{tikzcd}
H^1(k,G) \arrow[r] \arrow[d] & H^1(k,U_G) \arrow[d, "s'"] \\
H^1(k,G/G^2) \arrow[r] \arrow[dr, "t'"'] & H^1(k,U_G/U_G^0) \arrow[d] \\
 & H^1(k,U_{G/G^2})
\end{tikzcd}$$

The homomorphism $\varphi_L : G_k \longrightarrow G$ associated to a G-Galois algebra L was introduced in [8, 1.3.1]. It is defined as follows. Choose a point, i.e. a k-algebra homomorphism $P : L \longrightarrow k^{\mathrm{sep}}$. Then for each $\sigma \in G_k$ there is a unique $g_\sigma \in G$ so that $\sigma^{-1}P = Pg_\sigma$. We put $\varphi_L(\sigma) = g_\sigma$. It is natural to view φ_L as a 1-cocycle $G_k \longrightarrow G$. Its class in $H^1(k,G)$ is independent of the choice of the point P. By Galois theory, $H^1(k,G)$ classifies in this way G-Galois algebras up to isomorphism. Since U_G is the group of G-equivariant automorphisms of the unit form q_0 and since every G-form becomes isomorphic to the unit form over k^{sep}, the set $H^1(k,U_G)$ classifies in a similar way G-forms up to isomorphism. The map $H^1(k,G) \longrightarrow H^1(k,U_G)$ sends a G-algebra L or, equivalently, the 1-cocycle φ_L to its trace form q_L. Therefore the image of the pointed set $H^1(k,G)$ inside $H^1(k,U_G)$ classifies isomorphism classes of trace forms of G-Galois algebras.

The following lemma plays a role in the proof of the theorem.

Lemma 1.1. *Let k be a field with $\mathrm{cd}_2(k) \leq 1$. Suppose $P \twoheadrightarrow Q$ is a surjective homomorphism between finite 2-groups. Then the induced map $H^1(k,P) \longrightarrow H^1(k,Q)$ is also surjective.*

Proof. Since the homomorphism $P \twoheadrightarrow Q$ is the composition of a number of surjective homomorphisms between 2-groups having kernels of order 2, it suffices to consider the case where $Z = \ker(P \longrightarrow Q)$ has order 2. Then Z is contained in the center of G and we have an exact sequence of pointed sets [14, I.5.7]

$$H^1(k,P) \longrightarrow H^1(k,Q) \longrightarrow H^2(k,Z).$$

Since $\mathrm{cd}_2 k \leq 1$, we have that $H^2(k,Z) = 0$ and the lemma follows.

Theorem 1.2. *Let G be a finite group and let k be a field with $\mathrm{char}(k) \neq 2$ and $\mathrm{cd}_2 k \leq 1$. Then the map*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of trace} \\ \textit{forms of } G\textit{-Galois algebras} \end{array} \right\} \xrightarrow{\ h\ } \mathrm{Hom}_{\mathrm{cont}}(G_k, G/G^2),$$

that sends the trace form q_L of a G-Galois algebra L to the homomorphism given by $\sigma \mapsto \varphi_L(\sigma) \pmod{G^2}$, is a well defined bijection.

Proof. The cohomology pointed set $H^1(k,G/G^2)$ is naturally isomorphic to the group $\mathrm{Hom}_{\mathrm{cont}}(G_k, G/G^2)$. Consider the diagrams above. Since the map t admits a left inverse, the same is true for the induced map $t' : H^1(k, G/G^2) \longrightarrow H^1(k, U_{G/G^2})$. It follows that t' is injective. Identifying the set of isomorphism classes of trace forms of G-Galois algebras with the image of $H^1(k, G)$ in $H^1(k, U_G)$, it follows that the map h is well defined. Since $\mathrm{cd}_2 k \leq 1$, we have that $H^1(k, U_G^0) = 0$, a result due to R. Steinberg. This implies that the map s' is injective [8, 2.3.2]. Since the top square of the diagram above commutes, the map h is injective and we recover [8, Thm. 2.2.3].

It remains to prove the surjectivity of h. Let $P \subset G$ be a 2-Sylow subgroup of G. Since $G^2 P = G$, the natural map $P \hookrightarrow G \longrightarrow G/G^2$ is surjective. It follows from Lemma 1.1 that $H^1(k, P)$ maps surjectively onto $H^1(k, G/G^2)$. Therefore the map $H^1(k, G) \longrightarrow H^1(k, G/G^2)$ is also surjective. This implies that h is surjective, as required.

Before stating a corollary to this theorem, we introduce some more cohomology sets. Put $n = \#G$. For any k-algebra R, the unit form is a non-degenerate R-bilinear form on the group ring $R[G]$. Left multiplication by $\alpha \in U_G(R)$ is an orthogonal map $R[G] \longrightarrow R[G]$. This implies that there is a natural morphism from U_G to the orthogonal group O_n. This morphism maps the image of the closed immersion $G \longrightarrow U_G$ to the subgroup scheme S_n of the permutation matrices. The group scheme S_n is constant with underlying group the symmetric group on n letters. The determinant morphism from O_n to the group scheme μ_2 induces the sign map $S_n \longrightarrow \mu_2$. Here μ_2 denotes the group scheme given by $\mu_2(R) = \{x \in R : x^2 = 1\}$ for any k-algebra R. It is constant because $\mathrm{char}(k) \neq 2$.

This leads to the following commutative diagram of algebraic groups, together with the commutative diagram obtained by first taking k^{sep}-points and then Galois cohomology.

$$\begin{tikzcd}
 & G \arrow[r] \arrow[dl] \arrow[d] & U_G \arrow[d] \\
G/G^2 \arrow[dr] & S_n \arrow[r] \arrow[d] & O_n \arrow[dl, "\det"] \\
 & \mu_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H^1(k,G) \arrow[r] \arrow[dl] \arrow[d] & H^1(k,U_G) \arrow[d] \\
H^1(k,G/G^2) \arrow[dr] & H^1(k,S_n) \arrow[r] \arrow[d] & H^1(k,O_n) \arrow[dl] \\
 & H^1(k,\mu_2) &
\end{tikzcd}$$

The morphism $G \longrightarrow S_n \longrightarrow \mu_2$ associates to $g \in G$ the sign of the permutation $h \mapsto gh$ (for $h \in G$). It factors through G/G^2. The induced map $H^1(k, U_G) \longrightarrow H^1(k, \mu_2) \cong k^*/k^{*2}$ associates to a G-form in $H^1(k, U_G)$ its *determinant*.

Corollary 1.3. *Let G be a finite group and let k be a field with $\mathrm{cd}_2 k \leq 1$. Then*

(i) all trace forms of G-Galois algebras are isomorphic to the unit form q_0 or equivalently, all G-Galois algebras have a self-dual normal basis, if and only if G/G^2 is trivial or k is quadratically closed.

(ii) Suppose that there exists at least one G-Galois algebra whose trace form is not isomorphic to q_0. Then the trace forms of G-Galois algebras are characterized by their determinants if and only if the 2-Sylow subgroup of G is cyclic.

Proof. *(i)* Any continuous homomorphism $G_k \longrightarrow G/G^2$ factors through G_k/G_k^2. Therefore the group $\mathrm{Hom}_{\mathrm{cont}}(G_k, G/G^2)$ is trivial if and only if one of the groups G_k/G_k^2 or G/G^2 is. This proves the first part.

To prove the second, more interesting part, we observe that the determinant characterizes trace forms of G-Galois algebras precisely when the determinant map above is injective on the image of $H^1(k,G)$ in $H^1(k,U_G)$. By the commutative diagram above and Theorem 1.2 this happens precisely when the map $H^1(k,G/G^2) \longrightarrow H^1(k,\mu_2)$ is injective.

We first look at the map $G/G^2 \longrightarrow \mu_2$ itself. The permutation induced by an element $g \in G$ is a product of $[G : \langle g \rangle]$ disjoint cycles of length equal to the order of g. Therefore the morphism $G/G^2 \longrightarrow \mu_2$ is non-trivial or, equivalently, surjective precisely when there exists $g \in G$ of even order for which the index $[G : \langle g \rangle]$ is odd. This happens if and only the 2-Sylow subgroup of G is a non-trivial cyclic group.

Now we finish the proof. Suppose that $H^1(k,G/G^2) \longrightarrow H^1(k,\mu_2)$ is injective. Then it is non-constant and the same is true for $G/G^2 \longrightarrow \mu_2$. By the discussion above, this implies that the 2-Sylow subgroup P of G is cyclic. Conversely, suppose that P is cyclic. Since P maps onto the group G/G^2, which is non-trivial by *(i)*, we see that G/G^2 has order 2 and P is non-trivial. This implies that the homomorphism $G/G^2 \longrightarrow \mu_2$ is a bijection. Therefore the induced map $H^1(k,G/G^2) \longrightarrow H^1(k,\mu_2)$ is injective, as required.

2. Virtual 2-cohomological dimension at most 1.

The *virtual* 2-cohomological dimension $\mathrm{vcd}_2 k$ of a field k is defined as the 2-cohomological dimension of $k(\sqrt{-1})$. In the interesting case where it is strictly smaller than the cohomological 2-dimension of k we automatically have that k is formally real [14, Ch.II, Prop. 10′] and hence that $\mathrm{char}(k) = 0$. In this section we prove an analogue of Theorem 1.2 for fields with $\mathrm{vcd}_2 k \leq 1$. In order to formulate the result, we let Ω_k denote the real spectrum of k, i.e. the set of orderings on k equipped with the Harrison topology. It is a boolean topological space. In other words, Ω_k is a compact totally disconnected topological space [12, III.5]. For any finite discrete set X, we let $C(\Omega_k, X)$ denote the set of continuous or, equivalently, locally constant functions from Ω_k to X.

For every $v \in \Omega_k$ we let $\iota_v \in G_k$ denote the restriction to k^{sep} of the non-trivial automorphism of $k_v(\sqrt{-1})$ over the real closure k_v of k. The automorphism ι_v is an involution, well defined up to conjugation. For any finite group G we denote by $G_{\tilde{2}}$ the pointed set of conjugacy classes of $g \in G$ for which $g^2 = 1$. Note that $G_{\tilde{2}} = G$ when G has exponent 2. There is a natural map h_G from $H^1(k,G)$ to the set of continuous maps $C(\Omega_k, G_{\tilde{2}})$. It sends $\chi \in H^1(k,G)$ to the function that maps $v \in \Omega_k$ to $\chi(\iota_v)$.

Definition. For any homomorphism $G \longrightarrow F$ of finite groups we define the set $H^1_G(k,F)$ by insisting that the square

$$\begin{array}{ccc} H^1_G(k,F) & \longrightarrow & H^1(k,F) \\ \big\downarrow & & \big\downarrow h_F \\ C(\Omega_k, G_{\tilde{2}}) & \longrightarrow & C(\Omega_k, F_{\tilde{2}}) \end{array}$$

be Cartesian.

In other words, the set $H^1_G(k,F)$ consists of pairs (χ, f) where χ is in $H^1(k,F)$ and $f : \Omega_k \longrightarrow G_{\tilde{2}}$ is a continuous map so that for all $v \in \Omega_k$ we have that $\chi(\iota_v)$ is conjugate to $f(v)$ in F. Note that the set $H^1_G(k,F)$ depends on the homomorphism $G \to F$ rather than the groups F and G. In the sequel it will always be clear with respect to which homomorphism $G \to F$ we define $H^1_G(k, F)$. By the Cartesian property, there is a natural map $H^1(k, G) \longrightarrow H^1_G(k, F)$, induced by h_G and by the map $H^1(k, G) \longrightarrow H^1(k, F)$.

The following lemma plays a role in the proof of Theorem 2.2.

Lemma 2.1. *Let k be a field with $\mathrm{char}(k) \neq 2$ and $\mathrm{vcd}_2(k) \leq 1$. Suppose that $P \twoheadrightarrow Q$ is a surjective homomorphism between finite 2-groups. Then the natural map*

$$H^1(k, P) \longrightarrow H^1_P(k, Q)$$

is surjective.

Proof. We proceed by induction with respect to the order of $Z = \ker(P \longrightarrow Q)$. If Z is trivial, it follows from the definition that the map is an isomorphism. Assume that Z has order 2. Then Z is contained in the center of G and we have the following commutative diagram of pointed sets

$$\begin{array}{ccccccc} H^1(k, Z) & \longrightarrow & H^1(k, P) & \longrightarrow & H^1(k, Q) & \longrightarrow & H^2(k, Z) \\ \big\downarrow h_Z & & \big\downarrow h_P & & \big\downarrow h_Q & & \big\downarrow h'_Z \\ C(\Omega_k, Z_{\tilde{2}}) & \longrightarrow & C(\Omega_k, P_{\tilde{2}}) & \longrightarrow & C(\Omega_k, Q_{\tilde{2}}) & \longrightarrow & \Gamma(\mathcal{H}^2(Z)) \end{array}$$

This is the diagram that occurs in the proof of C. Scheiderer's [13, Lemma (4.6)]. For the finite groups $F = Z$, P and Q, we merely replaced the pointed set of global sections $\Gamma\mathcal{H}^1(F)$ of the Ω_k-sheaf $\mathcal{H}^1(F)$ by the pointed set $C(\Omega_k, F_{\tilde{2}})$ that is naturally isomorphic to it [13, Remarks (2.4) and (2.11)]. Up to this isomorphism Scheiderer's map h_F agrees with ours. By [13, Lemma (4.6)], the rows of the diagram are exact rows of pointed sets. Since $\mathrm{vcd}_2 k \leq 1$, it follows from [13, Thm. (3.1)] that the map h'_Z is a bijection while the map h_Z is surjective.

The proof that the map $H^1(k, P) \longrightarrow H^1_P(k, Q)$ is surjective, is a chase in this diagram of pointed sets, which we perform now. Let $\chi \in H^1(k, Q)$ and $f \in C(\Omega_k, P_{\tilde{2}})$ and suppose that their images in $C(\Omega_k, Q_{\tilde{2}})$ are both equal to g. Then g maps to 0 in $\Gamma(\mathcal{H}^2(Z))$. Since h'_Z is a bijection, this implies that the image of χ in $H^2(k, Z)$ is zero. Since the top row is exact, there exists therefore $\chi' \in H^1(k, P)$ mapping to $\chi \in H^1(k, Q)$. Let f' be the image of χ' in $C(\Omega_k, P_{\tilde{2}})$. Then both f and f' map to $g \in C(\Omega_k, Q_{\tilde{2}})$. This means that for every $v \in \Omega_k$ there is $p_v \in P$ so that $f'(w) = p_v f(w) p_v^{-1}$ for all w in an open neigborhood of v. This means that the map $h : v \mapsto p_v f(v) p_v^{-1} f'(v)^{-1}$ is a continuous map from Ω_k to $Z = Z_{\tilde{2}}$. Since the map h_Z is surjective, there is an element $\psi \in H^1(k, Z)$ mapping to h. The cocycle $\xi \in H^1(k, P)$ given by $\xi(\sigma) = \psi(\sigma)\chi'(\sigma)$ is the one we are looking for. Its images in $H^1(k, Q)$ and $C(\Omega_k, P_{\tilde{2}})$ are equal to χ and f respectively.

This completes the proof in the case the kernel Z of $P \twoheadrightarrow Q$ has order 2. When $\#Z > 2$, we choose a 2-group P' and surjections $P \twoheadrightarrow P' \twoheadrightarrow Q$ that are not isomorphisms. By induction we know that the maps $g_1 : H^1(k, P) \longrightarrow H^1_P(k, P')$ and $g_2 : H^1(k, P') \longrightarrow$

$H^1_{P'}(k,Q)$ and are surjective. Consider the following diagram

$$\begin{array}{ccccc}
H^1(k,P) & & & & \\
\Big\downarrow{\scriptstyle g_1} & & & & \\
H^1_P(k,P') & \longrightarrow & H^1(k,P') & & \\
\Big\downarrow{\scriptstyle g_3} & & \Big\downarrow{\scriptstyle g_2} & & \\
H^1_P(k,Q) & \xrightarrow{g_4} & H^1_{P'}(k,Q) & \longrightarrow & H^1(k,Q) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
C(\Omega_k, P_{\tilde{2}}) & \longrightarrow & C(\Omega_k, P'_{\tilde{2}}) & \longrightarrow & C(\Omega_k, Q_{\tilde{2}})
\end{array}$$

The maps g_3 and g_4 exist by the Cartesian property. The top square is Cartesian. Therefore, since g_2 is surjective, so is g_3. The map $g_3 g_1 : H^1(k,P) \longrightarrow H^1_P(k,Q)$ coincides with the natural one. It is surjective, because both g_1 and g_3 are. This proves the lemma.

Theorem 2.2. *Let G be a finite group and let k be a field with* $\mathrm{char}(k) \neq 2$ *and* $\mathrm{vcd}_2 k \leq 1$. *Then the map*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of trace} \\ \textit{forms of } G\textit{-Galois algebras} \end{array} \right\} \xrightarrow{h} H^1_G(k, G/G^2)$$

given by $h(q_L) = (\chi_L, f_L)$ where $\chi_L(\sigma) = \varphi_L(\sigma) \pmod{G^2}$ and $f_L(v) = \varphi_L(\iota_v)$ for all $v \in \Omega_k$, is a well defined bijection.

Proof. As we already saw in the proof of Theorem 1.2, the natural map $H^1(k,G) \longrightarrow H^1(k,G/G^2)$ factors through the image of $H^1(k,G)$ in $H^1(k,U_G)$. In other words, it induces a well defined map on the set of trace forms. In a similar way, the natural map $H^1(k,G) \longrightarrow C(\Omega_k, G_{\tilde{2}})$ factors through the image of $H^1(k,G)$ in $H^1(k,U_G)$. Indeed, if two G-Galois algebras L and L' have isomorphic trace forms $q_L \cong q_{L'}$, then the trace forms are in particular isomorphic over every real closure k_v of k. Since for any $v \in \Omega_k$, the G-Galois algebras over k_v are *determined* by their trace forms [8, Prop. 3.1.], the cocycles φ_L and $\varphi_{L'}$ are equal when restricted to $\langle \iota_v \rangle$. This means that $\varphi_L(\iota_v)$ is conjugate to $\varphi_{L'}(\iota_v)$ for every $v \in \Omega_k$ and hence that the functions f_L and $f_{L'}$ are equal.

It follows from the Cartesian property of $H^1_G(k,G/G^2)$ that the map h is well defined. The fact that h is injective is precisely the content of [4, Thm. 2.1]. It remains to show the surjectivity of h.

So, suppose we are given a continuous homomorphism $\chi : G_k \longrightarrow G/G^2$ and a continuous map $f : \Omega_k \longrightarrow G_{\tilde{2}}$ so that their natural images in $C(\Omega_k, G/G^2)$ agree. In other words, such that $\chi(\iota_v) \equiv f(v) \pmod{G^2}$ for all $v \in \Omega_k$. Let P be a 2-Sylow subgroup of G. Since every element of order 2 in G is conjugate to one in P, the natural map $P_{\tilde{2}} \longrightarrow G_{\tilde{2}}$ is surjective. Let s be a section. Then sf is in $C(\Omega_k, P_{\tilde{2}})$ and its natural image in $C(\Omega_k, G_{\tilde{2}})$ is f. It follows now from Lemma 2.1 that there exists a cocycle φ in $H^1(k,P)$ whose natural images in $H^1(k,G/G^2)$ and $C(\Omega_k, P_{\tilde{2}})$ are χ and sf respectively. The image of φ

in $H^1(k,G)$ has the property that its natural images in $H^1(k,G/G^2)$ and $C(\Omega_k, G_{\tilde{2}})$ are χ and f respectively. The trace form of the corresponding G-Galois algebra maps therefore to $(\chi, f) \in H^1_G(k, G/G^2)$ as required.

3. Virtual 2-cohomological dimension at most 2.

In this section we prove the second theorem of the introduction. It deals with double trace forms over fields k of virtual 2-cohomological dimension $\mathrm{vcd}_2 k$ at most 2.

There are several questions and conjectures [2] and partial results [4, 5, 11] for multiples of trace forms. Most of the results obtained for fields of virtual cohomological dimension 2 have been proved for fields that, like number fields, together with all their finite extensions, have the strong approximation property. Moreover, the proofs of many of the existing results on the classification of hermitian forms over an involutorial algebra over a field of virtual cohomological dimension 2 require this strong approximation property [11]. Our proof avoids its use.

We recall a few basic facts. Let D be a central division algebra over a field k endowed with an orthogonal involution τ and let Ω_k be as in the previous section. For $v \in \Omega_k$ we set $D_v = D \otimes_k k_v$. We define the rank, the discriminant and the signature (denoted respectively $\mathrm{rk}(h)$, $d(h)$ and $\mathrm{sgn}(h)$) of a nondegenerate hermitian form h as in [6, 7]. In fact, the rank of h is the dimension over D of the underlying vector space V supporting h. The discriminant $d(h)$ is given by

$$d(h) = (-1)^{m(m-1)/2}\mathrm{Nrd}(M(h)) \in k^*/k^{*2},$$

where $M(h)$ is a matrix representing h, Nrd is the reduced norm and $m = \deg_k \mathrm{End}_D(V)$. We consider a refined version of the discriminant and set, as in [7],

$$\mathrm{Disc}(h) = (-1)^{m(m-1)/2}\mathrm{Nrd}(M(h)) \in k^*/{k^{*+}}^2$$

where

$$k^{*+} = \{\lambda \in k^* : \lambda >_v 0 \text{ for all } v \in \Omega_k \text{ such that } D_v \text{ is not split}\}.$$

If the field k is of virtual 2-cohomological dimension at most 2, we have $k^{*+} = \mathrm{Nrd}(D^*)$ (see [7]), so that we can consider $\mathrm{Disc}(h)$ as an element of $k^*/(\mathrm{Nrd}(D^*))^2$. These invariants are defined for hermitian forms over arbitrary central simple algebra with involution using Morita equivalence (see [6, 7]).

Remark. Let k be of virtual 2-cohomological dimension at most 2 and h a hermitian form over D such that m is divisible by 4 and $\mathrm{Disc}(h) = 1$. Then we can choose a matrix S representing h such that $\mathrm{Nrd}(S) = 1$. Indeed, if S' is a matrix representing h, then by the definition of $\mathrm{Disc}(h)$ and the condition on m,, we have $\mathrm{Nrd}(S') = \mu^2$ where $\mu = \mathrm{Nrd}(w)$ for some $w \in D^*$. Let W denote the diagonal matrix

$$W = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & w \end{pmatrix}.$$

Then the matrix $W^{-1}S'(\bar{W}^t)^{-1}$ represents h and has reduced norm 1. Here $\bar{W}$ is the matrix obtained by applying τ to the entries of W.

Let h be a nondegenerate hermitian form having even rank and trivial signature and discriminant; in this case we can consider the Clifford invariant $c(h)$ of h, taking values in $H^2(k, \mu_2)$ modulo the subgroup generated by the class of D. Furthermore, if the Clifford invariant is also trivial, we can define the Rost invariant $R(h)$ taking values in the group $H^3(k, \mathbf{Q}/\mathbf{Z}(2))$ modulo its subgroup $H^1(k, \mu_2) \cup [D]$. See [9] and [7, section 3.4] for the definition and the properties of these invariants.

We denote by $\mathrm{Spin}_{2n}(D, \tau)$, $U_{2n}(D, \tau)$ and $SU_{2n}(D, \tau)$ respectively, the spin group, the unitary and special unitary groups of the hyperbolic form $H_{2n} = \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$ over (D, τ). If $n\deg D$ is even, $\mathrm{Nrd}(H_{2n}) = 1$.

The set $H^1(k, SU_{2n}(D, \tau))$ is in one-one correspondence with the set of pairs (S, λ) where $S \in GL_{2n}(D)$ is a $2n \times 2n$ matrix such that $\bar{S} = S^t$ and $\mathrm{Nrd}(H_{2n})\lambda^2 = \mathrm{Nrd}(S)$ modulo the equivalence $(S, \lambda) \sim (S', \lambda')$ if and only if there exists $W \in GL_{2n}(D)$ such that $S' = WS\bar{W}^t$ and $\lambda' = \mathrm{Nrd}(W) \cdot \lambda$ (see [9], section 29.27, p 406). The distinguished point of this set is given by $(H_{2n}, 1)$.

Lemma 3.1. *Let $v \in \Omega_k$ be an ordering on the field k such that D_v is not split. We consider a pair (S, λ) of $H^1(k, SU_{2n}(D, \tau))$, such that the element (S) in $H^1(k, U_{2n}(D, \tau))$ is trivial in $H^1(k_v, U_{2n}(D, \tau))$. Then, (S, λ) is trivial over k_v if and only if $\lambda >_v 0$.*

Proof. Since D_v is not split, degree of D is even and $\mathrm{Nrd}(H_{2n}) = 1$. By hypothesis (S) is trivial over k_v and hence there exists a matrix W in $GL_{2n}(D_v)$ such that $WS\bar{W}^t$ is equal to H_{2n} . Thus, $\mathrm{Nrd}(W)^2\lambda^2 = 1$ and hence $\lambda = \pm\mathrm{Nrd}(W)^{-1}$. If $\lambda >_v 0$, then $\mathrm{Nrd}(W)$ being also positive, we have $\lambda = \mathrm{Nrd}(W)^{-1}$ and $(S, \lambda) \sim (H_{2n}, 1)$. Conversely, if $(S, \lambda) \sim (H_{2n}, 1)$ one may choose W such that $WS\bar{W}^t = H_{2n}$ and $\lambda\mathrm{Nrd}(W) = 1$. Thus $\lambda >_v 0$.

The principal tool in proving the main result of this section is the following theorem.

Theorem 3.2. *Let k be a field with $\mathrm{char}(k) \neq 2$ and with $\mathrm{vcd}_2 k \leq 2$. Let D/k be a central division algebra over k endowed with an orthogonal involution τ. Let (V, h) be a hermitian space over (D, τ) and let $\mathrm{sgn}(h)$ and $d(h)$ denote the total signature and the discriminant of h respectively. Suppose that the rank of h is even, $\mathrm{sgn}(h) = 0$ and that the cup product $d(h) \cup (-1)$ is also 0. Then the orthogonal sum $h \oplus h$ is hyperbolic.*

Proof. First step. Let $\mathrm{Nrd}(M(h)) = [(-1)^{m(m-1)/2}a] \in k^*/(\mathrm{Nrd}(D^*))^2$, where $M(h)$ is a matrix representing h and a belongs to k^*. We have $d(h) \cdot a \in k^{*2}$. By hypothesis, $d(h) \cup (-1)$ is trivial, so that $(a) \cup (-1) = 0$, hence $a = \mathrm{N}_{k(\sqrt{-1})/k}(b)$ for some $b \in k(\sqrt{-1})^*$. Since $\mathrm{cd}_2(k(\sqrt{-1})) \leq 2$, there exists a hermitian form h_1 over $k(\sqrt{-1})$ represented by a matrix $S = S_1 + \sqrt{-1}S_2$, where S_i's have entries from D and $\mathrm{Nrd}(S) = b^{-1}$ (cf [15]). By adding a suitable form represented by the identity matrix, we may assume without loss of generality that the rank of h_1 is $2m$. The hermitian form $\tilde{h_1} = \mathrm{Tr}_{k(\sqrt{-1})/k}(h_1)$ is represented by $\begin{pmatrix} S_1 & -S_2 \\ -S_2 & -S_1 \end{pmatrix} = M(\tilde{h_1})$, where the trace of a hermitian form is as defined in ([3], Proof of Proposition 1.2). The rank of $\tilde{h_1}$ is $4m$ and one easily verifies that $\mathrm{Disc}(\tilde{h_1}) = [\mathrm{Nrd}(M(\tilde{h_1})] = [\mathrm{N}_{k(\sqrt{-1})/k}(Nrd(S))] = [a^{-1}]$. In fact in order to check this equality, one may reduce to the case when D is split. In this case, for a rank one quadratic form $\langle \alpha + \sqrt{-1}\beta \rangle$ over $k(\sqrt{-1})$, $\alpha, \beta \in k$, $\mathrm{Tr}_{k(\sqrt{-1})/k}(h_1)$ is represented by the matrix $\begin{pmatrix} \alpha & -\beta \\ -\beta & -\alpha \end{pmatrix} = M(\tilde{h_1})$ with determinant $-\mathrm{N}_{k(\sqrt{-1})/k}(\alpha + \sqrt{-1}\beta)$. Let $\hat{h} = h \oplus \tilde{h_1}$. Since $\tilde{h_1} \oplus \tilde{h_1} = \mathrm{tr}_{k(\sqrt{-1})/k)}(h_1 \oplus h_1)$ is hyperbolic, the form $\hat{h} \oplus \hat{h}$ is Witt equivalent to $h \oplus h$. It suffices to show that $\hat{h} \oplus \hat{h}$ is hyperbolic. The form $\hat{h}$ has even rank and $\mathrm{Disc}(\hat{\mathrm{h}}) = 1$. Further since $\tilde{h_1}$ is the trace of a form over $k(\sqrt{-1})$, we have that $\mathrm{sgn}(\tilde{h_1}) = 0$ so that $\mathrm{sgn}(\hat{h}) = \mathrm{sgn}(h) + \mathrm{sgn}(\tilde{h_1}) = 0$. We replace h by $\hat{h}$ and still call it h. Let $m = \deg_k \mathrm{End}_D(V)$. Since the rank of h is even, m is not divisible by 4 only when D is split. In this case, by adding a hyperbolic plane if necessary, we assume without loss of generality that m is divisible by 4. Since $\mathrm{Disc}(h) = 1$, by the remark, we may choose a matrix S representing h such that $\mathrm{Nrd}(S) = 1$.

Second step. Let the rank of h be $2n$. Since the index of D multiplied by $2n$ is divisible by 4, $\mathrm{Nrd}(H_{2n}) = 1$. Let $\psi = (S, 1) \in H^1(k, SU_{2n}(D, \tau))$. Let $v \in \Omega_k$ be an ordering of k. We denote by ψ_v the image of ψ in $H^1(k_v, SU_{2n}(D, \tau))$. We claim that ψ_v is trivial for all $v \in \Omega_k$. To prove this claim , we first consider the case D_v split. In this case, the map

$$H^1(k_v, SU_{2n}(D, \tau)) \to H^1(k_v, U_{2n}(D, \tau))$$

is injective, the hermitian form h_v is hyperbolic and hence $\psi_v = 1$. The case D_v non–split follows from Lemma 3.1 applied to the pair $(S, 1)$. This proves the claim. Consider the exact sequence

$$1 \to \mu_2 \to \mathrm{Spin}_{2n}(D, \tau) \to SU_{2n}(D, \tau) \to 1 \tag{$*$}$$

and the induced cohomology long exact sequence of pointed sets

$$\ldots \to H^1(k, \mathrm{Spin}_{2n}(D, \tau)) \to H^1(k, SU_{2n}(D, \tau)) \xrightarrow{\delta} H^2(k, \mu_2) \to \ldots \tag{$**$}$$

(see [9, 31.41]). We denote by $\delta(\psi) \in H^2(k, \mu_2)$ the image of ψ under the connecting map. Then $c(h) = [\delta(\psi)] \in H^2(k, \mu_2)/([D])$. As $\delta(\psi)$ is locally trivial for all $v \in \Omega_k$, it is a (-1)-torsion element of $H^2(k, \mu_2)$ (see [1, Satz 2]) and hence $(-1) \cup (\delta(\psi))$ is zero since $H^3(k, \mu_2)$ is (-1)-torsion free. In view of the exact sequence (see for instance, [7], Section 2)

$$H^2(k(\sqrt{-1})) \quad \overset{cores}{\longrightarrow} \quad H^2(k, \mu_2) \quad \overset{\cup(-1)}{\longrightarrow} \quad H^3(k, \mu_2)\,,$$

there exists an element $\eta \in H^2(k(\sqrt{-1}), \mu_2)$ such that $\mathrm{cores}_{k(\sqrt{-1})/k}(\eta) = \delta(\psi)$. We can write $\eta = \sum_i (a_i) \cup (b_i)$ for some elements $a_i \in k^*$ and $b_i \in k(\sqrt{-1})^*$ (see [7, Prop. 2.8]). For each i, let (V_i, h_i) be a hermitian space over $D \otimes_k k(\sqrt{-1})$ such that $d(h_i) = b_i$ and the rank of V_i as a $D \otimes_k k(\sqrt{-1})$-module is even. Let h_i be represented by a matrix $M(h_i)$ with $\mathrm{Nrd} M(h_i) = b_i$. Let $\tilde{h_i}$ be the form given by $\tilde{h_i} = \langle 1, -a_i \rangle \otimes h_i$. We have $d(\tilde{h_i}) = 1$ and $\mathrm{Disc}(\mathrm{Tr}_{k(\sqrt{-1})/k}(\tilde{h_i}) = 1$ (cf proof of Step 1). We claim that the Clifford invariant $c(h_i)$ of h_i is $(a_i) \cup (b_i) \in H^2(k(\sqrt{-1}), \mu_2)/[D]$. Since $c(\tilde{h_i})$ is defined in $H^2(k(\sqrt{-1}), \mu_2)/[D]$, it suffices to check this equality after splitting D where it is clear. Let $\tilde{h} = \mathrm{Tr}_{k(\sqrt{-1})/k}(\oplus_i \tilde{h}_i)$. Then, $\mathrm{Disc}(\tilde{h}) = 1$. Further, $c(\tilde{h}) = \mathrm{cores} \sum_i c(\tilde{h}_i) = c(h)$. To check this, it is enough to verify this equality when D is split. In this case this is a consequence of the following commutative diagram:

$$\begin{array}{ccc} I^2(k(\sqrt{-1})) & \xrightarrow{c} & H^2(k(\sqrt{-1}), \mu_2) \\ {\scriptstyle Tr}\downarrow & & \downarrow{\scriptstyle cores} \\ I^2(k) & \xrightarrow{c} & H^2(k, \mu_2) \end{array}$$

The commutativity is easily verified on generators of $I^2(k(\sqrt{-1}))$ of the form $\langle\langle a, b\rangle\rangle$ where $a \in k^*$ and $b \in k((\sqrt{-1}))$. Hence, if we replace h by the form $h \oplus \tilde{h}$, we get a form having still even dimension, trivial signature and Discriminant, and having also trivial Clifford invariant. In other words, we have that $c(h \oplus \tilde{h}) = 0 \in H^2(k, \mu_2)/[D]$. As in step 1, we observe that when $h \oplus \tilde{h} \oplus h \oplus \tilde{h}$ is hyperbolic, so is $h \oplus h$. Therefore we can work from now on with this new form $h \oplus \tilde{h}$ and, for simplicity, we still call it h. The integer m associated to h is clearly divisible by 4.

Third step. Let h be the form obtained in step 2, let S be a matrix representing h, such that $\mathrm{Nrd}(S) = 1$, and let $\psi = (S, 1) \in H^1(k, SU_{2n}(D, \tau))$ be the associated element. We still have $\psi_v = 1$ for all $v \in \Omega_k$.

Again we consider the image $\delta(\psi) \in H^2(k, \mu_2)$ and we remark that $[\delta(\psi)] = [c(h)] = 0 \in H^2(k, \mu_2)/[D]$ so that either $\delta(\psi) = 0$, or $\delta(\psi) = [D]$. In the first case, when $\delta(\psi) = 0$, by the exact sequence (**), there exist $\zeta \in H^1(k, \mathrm{Spin}_{2n}(D, \tau))$ mapping to $\psi \in H^1(k, SU_{2n}(D, \tau))$. Suppose now $\delta(\psi) = [D] \neq 0$. We first remark that D is locally split. In fact, for all $v \in \Omega_k$ we have $\psi_v = 1$ and so $\delta(\psi_v) = 0 = D_v$; thus D_v is split for all $v \in \Omega_k$. Consider the exact sequence

$$1 \to SU_{2n}(D, \tau) \to U_{2n}(D, \tau) \to \mu_2 \to 1$$

and the induced long exact sequence

$$SU_{2n}(D, \tau)(k) \to U_{2n}(D, \tau)(k) \to \{\pm 1\} \to H^1(k, SU_{2n}(D, \tau)) \to H^1(k, U_{2n}(D, \tau)) \to \dots$$

In general, we have two elements $\psi,\ \psi' \in H^1(k, SU_{2n}(D, \tau))$ in the preimage of $[h] \in H^1(k, U_{2n}(D, \tau))$, with $\delta(\psi) = \delta(\psi') + [D]$ (see [6]). Since $\delta(\psi) = [D]$, we get $\delta(\psi') = 0$ and we can replace ψ by ψ'. We note that $(\psi')_v$ is also trivial for all $v \in \Omega_k$. This is because of the fact that, for all $v \in \Omega_k$, D_v is split and $(\psi)_v$ and $(\psi')_v$ are equal since they are both preimages of $[h]$ in $H^1(k_v, SU_{2n}(D, \tau))$ which injects into $H^1(k_v, U_{2n}(D, \tau))$; then, $(\psi)_v = (\psi')_v = 1$. Thus we replace (ψ) by (ψ') and call it still ψ which has the property that $(\psi)_v = 0$ for all $v \in \Omega_k$ and $\delta(\psi) = 0$. We unify the two cases calling ζ the preimage in $H^1(k, \mathrm{Spin}_{2n}(D, \tau))$ of ψ.

Let now $\theta = R(\zeta) \in H^3(k, \mathbf{Z}/4)$, be the Rost invariant of ζ (see [7]). Since $\psi_v = 1$ in $H^1(k_v, SU_{2n}(D, \tau))$ for all $v \in \Omega_k$, we have $R(\zeta_v) = (u_v) \cup ([D])$ with $u_v \in k_v^*/k_v^{*2}$. Hence

$2\theta_v = 0$ for all $v \in \Omega_k$. Since the map

$$H^3(k, \mathbf{Z}/4) \to \prod_{v \in \Omega_k} H^3(k_v, \mathbf{Z}/4)$$

is injective (see [7]), we have $2\theta = 0$. Hence $R(h \oplus h) = [2\theta] = 0$ in $H^3(k, \mathbf{Z}/4)/(k^*) \cup ([D])$.

Finally, $h \oplus h$ has even dimension, trivial Discriminant, trivial Clifford invariant and trivial Rost invariant, then it is hyperbolic by [7, Thm. 7.3]. That completes the proof of the theorem.

Theorem 3.3. *Let k be a field with $\mathrm{char}(k) \neq 2$ and $\mathrm{vcd}_2(G_k) \leq 2$. Let G be a finite group and let L and L' be two G-Galois algebras. Then, the G-forms $[2]q_L$ and $[2]q_{L'}$ are isomorphic if and only if the following two conditions hold.*

(i) we have that $\varphi_L \cup (-1) = \varphi_{L'} \cup (-1)$ in $H^2(k, G/G^2)$;

(ii) the G-forms q_L and $q_{L'}$ are isomorphic over k_v for all $v \in \Omega_k$.

Proof. The conditions *(i)* and *(ii)* are necessary by [5]. We now prove the sufficiency of the conditions. In view of [4] where $cd_2(k) \leq 2$ case has been proved, we may assume that k admits orderings and $\mathrm{char}(k) = 0$. The group algebra $k[G]$ has a decomposition

$$k[G] = B_1 \times \cdots \times B_r \times C_1 \times C_1^{op} \times \cdots \times C_s \times C_s^{op}$$

where B_i and C_j are simple. The involution on $k[G]$ preserves the factors B_i and switches the components of $C_j \times C_j^{op}$. Let E_i be the centre of B_i and $F_i \subset E_i$ the fixed field for the involution. Let $u_L, u_{L'}$ denote the classes representing q_L and $q_{L'}$ respectively in $H^1(k, U_G)$ and $\overline{u}_L, \overline{u}_{L'}$ their images under the composite map

$$H^1(k, U_G) \to H^1(k, R_{F_i/k}(U_{B_i})) \to H^1(F_i, U_{B_i}).$$

Following the proof of ([5], Theorem 2.2), it suffices to show the hermitian forms $[2]\overline{u}_L$ and $[2]\overline{u}_{L'}$ are isometric. In the cases when $(B_i, *)$ is unitary or symplectic, this can be seen following the proof of ([4], Theorem 3.1) since strong approximation property hypothesis is not used in these cases. Suppose $(U_{B_i}, *)$ is orthogonal. Then $h = \overline{u}_L \oplus -\overline{u}_{L'}$ is a hermitian form over $(B_i, *)$ defined on a rank two free module over B_i. Further if $a = d(h)$, the first hypothesis gives $(a) \cup (-1) = 0 \in H^2(k, \mu_2)$. Moreover, the "local" hypothesis $q_{L_v} \cong q_{L'_v}$ for all $v \in \Omega_k$, gives us that $\mathrm{sgn}(h) = 0$.

Let $B_i = M_l(D_i)$ where D_i is a central division algebra over F_i and let the involution $*$ on B_i be adjoint to a hermitian form over (D_i, τ_i) for some orthogonal involution τ_i on D_i. Then under Morita equivalence, the form h corresponds to a hermitian form h_i over (D_i, τ_i) of rank $2l$ with the same discriminant and signature as h. Hence by Theorem 3.2, $h_i \oplus h_i$ is hyperbolic and hence $h \oplus h$ is hyperbolic.

Bibliography

[1] Arason, J.K.: Primideale im graduierten Wittring und im mod 2 Cohomologiering, Math. Z. **145** (1975) 139–143.

[2] Bayer-Fluckiger, E.: Galois cohomology and the trace form, *Jahresber. DMV* **96** (1994) 35–55.

[3] Bayer-Fluckiger, E. and Lenstra, H.W.: Forms in odd degree extensions and self dual normal bases, *Amer. J. Math.* **112** (1990) 359–373.

[4] Bayer-Fluckiger, E. and Monsurrò, M.: Doubling trace forms, *St. Petersburg Math J.* **11** (2000) 405–417.

[5] Bayer-Fluckiger, E. and Morales J.: Multiples of trace for ms in number fields, *AMS Proc. Symposia Pure Math.* **58.2** (1995) 73–81.

[6] Bayer-Fluckiger, E. and Parimala, R.: Galois cohomology of linear algebraic groups over fields of cohomological dimension ≤ 2, *Invent. Math.* **122** (1995) 195–229.

[7] Bayer-Fluckiger, E. and Parimala, R.: Classical groups and the Hasse principle, *Ann. of Math.* **147** (1998) 1–43.

[8] Bayer-Fluckiger, E. and Serre, J.-P.: Torsions quadratiques et bases normales autoduales, *Amer. J. Math.* **116** (1994) 1–64.

[9] Knus, M., Merkurjev, A., Rost, M. and Tignol, J.-P.: *Book of involutions*, American Mathematical Society, Colloquium Publications **44** (1998).

[10] Monsurrò, M.: *Formes trace de G-algèbres galoisiennes*, Thèse présenté à l'UFR des Sciences et Techniques de l'Université de Franche-Comté 1999.

[11] Monsurrò, M.: Trace forms times Pfister forms, preprint 2002.

[12] Scharlau, W.: *Quadratic and Hermitian forms*, Grundlehren Math. Wissenschaften **270**, Springer-Verlag, Berlin 1975.

[13] Scheiderer, C.: Hasse principles and approximation theorems for homogeneous spaces over fields of virtual cohomological dimension one, *Invent. Math.* **125** (1996) 307–365.

[14] Serre, J.-P.: *Cohomologie Galoisienne*, cinquième édition, révisée et complétée, Lecture Notes in Math. **5**, Springer-Verlag, New York 1994.

[15] Yanchevskii, V.I.: Simple algebras with involution and unitary groups, *Math. Sbornik* **22** (1974) 372–384.